DISTORTION BOUNDS ON ANYTIME SOURCE TRANSMISSION USING UEP CHANNEL CODING

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ABSTRACT

In this paper, we study the design of causal anytime codes for transmission over symmetric discrete memoryless channel. In our earlier work [1], we proposed an anytime transmission scheme which is based on unequal error protection using Luby transform codes (UEP-LT) and sequential belief propagation (BP) decoding. In this paper we extend our previous result by providing an analysis on the proposed scheme. In particular, an upperbound on the end-to-end distortion of the anytime transmission scheme is derived.

1. INTRODUCTION

The study of reliable transmission of real-time information of a dynamic source over a noisy rate-limited communication channel has received much attention in recent years since it is a desired quality in several emerging applications. These applications may include state estimation in wireless sensor networks (WSN), online multimedia streaming on the Internet, etc. For example, consider a production line whose action is regulated by a remote controller. Especially, in many control applications the system performance is typically very sensitive to delay in action. In these cases it is particularly interesting to apply transmission schemes for which the reliability is improved with time.

The traditional approach to deal with channel imperfection such as packet loss, delay, data-rate limitations, and etc, when a dynamic source is considered, is dynamic programming (DP) [2]. In many cases, the dynamic programming suffers from a serious problem which is called the curse of dimensionality, i.e., the computations become intractable when the number of states and decisions increases. Although approximate dynamic programming (ADP) [3, 4] can relax the problem to obtain some feasible solutions, it does not provide a unique solution to the original DP problem, and the computational complexity is still high in many cases.

In contrast with the traditional approach, Sahai in [5] proposed the fundamental concept of anytime information transmission. There are two main features that distinguish traditional communication theory from the anytime concept. The anytime transmitter has only access to a part of the source message at anytime, and the anytime receiver can use information in current channel output, as well as in previous channel outputs. As shown by [5], for a scalar linear system $x(t+1) = ax(t)+v(t)$, $a > 1$, where $x(t)$ and $v(t)$ denote the plant state and the process noise, respectively, there exists an estimate of the state $x(t)$, referred to as $\hat{x}(t)$, such that the distortion over a noisy channel is always bounded.

In [6], bounds on anytime error exponent have been determined, and a time-sharing anytime channel code for a discrete memoryless channel (DMC) with perfect feedback is proposed. As a matter of fact, in many practical cases, a perfect feedback from the decoder to the remote transmitter is not possible because of diverse channel limitations. In [7], the distortion convergence rates for certain anytime coding schemes have been derived based on unequal error protection (UEP) property and assuming no channel feedback is available at the transmitter. In applications such as controlling unstable plants, exploiting UEP principles in the anytime design is crucial since not only delay in action is of great importance, but also some parts of information bits need to be treated differently.

In the literature, rateless codes have been shown to have advantages in UEP applications [8, 9]. Rateless codes are a class of random sparse channel codes in which the encoder produces an unlimited number of symbols such that the decoder can recover the source symbols from a sufficiently large subset of channel outputs. Luby Transform (LT) codes [10] are rateless error control codes suitable for erasure channels. From [11], we also know that LT and Raptor codes [12] (concatenation of LT codes with high-rate LDPC outer encoder/decoder) perform very well on symmetric noisy channels such as binary symmetric channel (BSC) and binary-input additive white Gaussian noise (BIAWGN) channel.

This paper is a continuous work of our previous paper [1], in which we proposed a causal anytime transmission scheme for communication over symmetric discrete memoryless channel, based on UEP-LT codes and sequential belief propagation decoding. In the current paper, we provide an analytic analysis of our earlier proposed scheme. In particular, an upperbound on the end-to-end distortion is derived, and will be compared with existing results.

2. PROBLEM STATEMENT AND PERFORMANCE CRITERION

In this section, we will describe the system model and specify the information pattern of each building block.

We shall assume that a scalar random variable $x_t \in \mathcal{X}$ is drawn according to a known distribution at time t where $\mathcal{X} \subseteq R$. Between the source and the destination there is a binary symmetric channel with bit cross-over probability ϵ. The inputs and outputs of the channel are denoted respectively by $y_t \in \{0, 1\}$ and $z_t \in \{0, 1\}$. The conditional probability of the channel is time-invariant, i.e., $\Pr(z_t|y_t) = \Pr(z|y)$.

In our anytime transmission scheme, the two main functional units, source coding and channel coding are considered separately. Throughout the paper, we employ truncated binary expansion as the source coding scheme to map x_t into a sequence of bits $(b_1, \ldots, b_{j_t})$, where j_t denotes the first j_t ($j_t \geq j_{t-1}$) bits available at the anytime encoder at time t. As will be clear later, the binary expansion fits well into the anytime framework that provides anytime reliability. In our problem formulation, we use the notation $\mathbf{x}_a^b = \{x_a, \ldots, x_b\}$ which denotes the evolution of a discrete-time signal $x(t)$ from $t = a$ to $t = b$. The binary expansion is

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defined by a map $\mathcal{E}^s_{j_t}:\mathcal{X}\mapsto\mathcal{Y}^{j_t}$, where $\mathcal{Y}^{j_t}$ takes values from $\mathbf{b}_1^{j_t}=\{0,1\}^{j_t}$. In fact, the choice of j_t depends on the rate of source coding, i.e., if the source generates a bit, b_j, every $1/R$ time units for a given source rate $R>0$, then $j_t=\lceil Rt\rceil$. Without loss of generality, we assume that the source rate is $R=1$. The anytime channel encoder is described by a map $\mathcal{E}^c_t:\mathcal{Y}^{j_t}\mapsto\mathcal{Y}^t$ which outputs a bit at each time according to the function $E^c(\mathbf{b}_1^{j_t})=y_t$.

At the remote receiver, the channel decoder, specified by a map $\mathcal{D}^c_t:\mathcal{Z}^t\mapsto\mathcal{Y}^{j_t}$, is allowed to exploit the information from the current, as well as the previous, received symbols, for the purpose of estimation, i.e., $\hat{\mathbf{b}}_1^{j_t}=D^c_t(\mathbf{z}_1^t)$. Finally, the source decoder can be written as the map $\mathcal{D}^s_{j_t}:\mathcal{Y}^{j_t}\mapsto\mathcal{X}$. The source decoder outputs the reconstructed value $\hat{x}_t$.

In this paper, we consider a scalar linear plant with the system equation

$$x(t+1)=ax(t)+v(t)\quad,\quad t=1,\ldots,T, \tag{1}$$

where $x(t)\in\mathcal{R}$ is the system state at time instant t, and $v(t)\in\mathcal{R}$ is the white process noise causes from inappropriate modeling. This open–loop system is unstable if $a>1$ whose value is known at both the transmitter and the receiver. The system is triggered by the initial state $x(0)$ which has the uniform probability density function (pdf) on $[0,1)$.

As in [7], we consider that the variance of the process noise is much smaller than that of the initial state, hence the noise is negligible and the only source of uncertainty in the system is $x(0)$. Therefore, the system equation can be rewritten as $x(t)=a^t x(0)$, $t=1,...,T$. Following the notation of the anytime information pattern, at $t=0$, the initial state is first quantized into j_t bits, and thereafter, a binary codeword, before it is sent over a binary symmetric channel. As a matter of fact, at each time instant t, a new channel coded bit representing the value of $x(0)$, referred to as $y_t\in\{0,1\}$, is transmitted. At the receiver, the channel decoder obtains a noisy version of y_t, referred to as $z_t\in\{0,1\}$, and starts providing estimates of the source value based on the current and previous received bits, i.e., $\mathbf{z}_1^t$. The reconstructed value $\hat{x}(0|t)$ is then given by applying the inverse binary expansion on the decoded bits.

We evaluate the system performance using end-to-end mean-squared error (MSE_{e2e}), defined as

$$\text{MSE}_{e2e}(t)\triangleq\mathbb{E}[(x(t)-\hat{x}(t))^2]=a^{2t}\mathbb{E}[(x(0)-\hat{x}(0|t))^2], \tag{2}$$

where the last equality follows from $\hat{x}(t)=a^t\hat{x}(0|t)$, that is to say, in the absence of the process noise. The goal is to design a channel encoder/decoder pair in order to estimate $x(0)$ precisely out of the received data up to time t so that the MSE decays fast enough, i.e., $\lim_{t\to\infty}\text{MSE}_{e2e}(t)=0$. Thus, a^{2t} in (2) will be dominated by $\Delta_t^2\triangleq\mathbb{E}[(x(0)-\hat{x}(0|t))^2]$ as time increases.

3. ANYTIME CHANNEL CODING

In this section, we briefly present the proposed channel coding scheme. First, we describe the anytime repetition coding in Section 3.1. Thereafter, we describe the anytime LT encoding and decoding algorithms in Section 3.2 and Section 3.3, respectively.

3.1 Anytime UEP-repetition coding

As a part of the channel code, an anytime repetition code is used, which is a special block repetition code. More specifically, how frequently an information bit is coded is determined by its importance level. The first bit b_1 repeats most often since it is the most significant bit. For example, if the bit stream $\mathbf{b}_1^t$ is generated up to time t, the channel encoder might pick up a bit b_t according to the sequence $\{b_1;b_1b_2;b_1b_2b_3;\ldots\}$, where the position of the bit b_t is specified by the time instant t. The decoder receives $\{\hat{b}_1\hat{b}_1\hat{b}_2\ldots\}$, and employs majority logic decoding (MLD) so as to decide $\hat{b}_i$=0 $(1\le i\le t)$ (or, 1) if it repeats more than $\hat{b}_i$=1 (or, 0) in the previous estimates. The decoder should also be able to decode if the number of received zeros or ones is equal. To handle this, it decides $\hat{b}_i$=1 or 0 using a Bernoulli trial.

The anytime repetition coding strategy, despite its relatively low computational complexity, can provide anytime reliability at the expense of zero rate and increasing delay with time. In the next section, we bring up the idea of anytime rateless scheme which is suitable for applications of finite rate and delay.

3.2 Anytime UEP-LT encoding

In this section, we describe the encoding procedure which is an adaptation of the expanding window fountain (EWF) codes [9] fitting into the anytime transmission scheme. Let the information bits (input bits) denoted by b and the encoded bits (output bits) denoted by y at time t=T be allocated to r overlapping information windows and r individual encoding windows. More specifically, at time t, the j^{th} $(j$=$1,\ldots,r)$ information window contains a portion of the input bits $\mathbf{b}_1^t$ whose length will be referred to as K_j $(K_j>K_{j-1})$. Therefore, there are r levels of importance where each contains K_j–K_{j-1} bits.

Associated with the j^{th} information window, there exists a degree distribution $\Omega_j(x)$ according to which the bits in the corresponding window are chosen to be XOR'ed and generate an encoded bit y_t which is located in the j^{th} encoding window whose length is denoted by T_j. We emphasize here that the last information window does not necessarily contain all the information bits, $\mathbf{b}_1^T$, however, the encoding windows expand up to $t=T$, i.e., $\sum_{j=1}^{r}T_j$=T. It should be noted that K_j is a function of t as well, but for the ease of notation, we denote it by K_j.

Now, let $\Omega_m(x)$=$\sum_{d_m=1}^{D_m}\Omega_{d_m}x^{d_m}$ denote the output bit degree distribution corresponding to the $m^{th}(1\le m\le j)$ encoding window, where Ω_{d_m} represents the probabilities of choosing the corresponding number of bits from the m^{th} information window. We define the output edge degree distribution as $\omega_m(x)$=$\sum_{d_m=1}^{D_m}\omega_{d_m}x^{d_m-1}$=$\Omega'_m(x)/\Omega'_m(1)$, where $\beta_m=\Omega'_m(1)$ is the average degree of an output bit in the m^{th} encoding window. Similar degree distribution can be specified for the input bits as well. For this purpose, we analyze the input degree distribution of each level of importance individually. Let $\Lambda_m(x)$=$\sum_{n_m}\Lambda_{n_m}x^{n_m}$ be the input degree distribution of the $m^{th}(1\le m\le j)$ level of importance. The coefficients can be determined from the characteristics of output degree distribution in the following way. Let $I_m(x)$=$\sum_{n_m}I_{n_m}x^{n_m}$ denote the input degree distribution of the input bits in the m^{th} information window induced only by the edges connected to the output bits in the m^{th} encoding window. Therefore, $\Lambda_{n_m}=\prod_{i=m}^{j}I_{n_i}$, where only the input bits of the m^{th} level are considered. As shown in [11], $I_m(x)\approx e^{\alpha_m(x-1)}$ where α_m=$I'_m(1)$=$\beta_m T_m/K_m$. Thus, input bit degree distribution of the m^{th} level of importance at $t\in T_j$ can be written as

$$\Lambda_m(x)=\exp\left[\left(\sum_{i=m}^{j-1}\alpha_i+\beta_j t/K_j\right)(x-1)\right]. \tag{3}$$

In a similar fashion, we introduce the input edge degree distribution of the m^{th} level as $\iota_m(x)$=$\sum_{n_m}\iota_{n_m}x^{n_m-1}$. It can be verified (see [11] for details) that $\iota_m(x)$ has asymptotically the same distribution as $\Lambda_m(x)$, therefore, $\Lambda'_m(1)= \iota'_m(1) = \sum_{i=m}^{j-1} \alpha_i + \beta_j t/K_j$.

3.3 Anytime UEP-LT decoding

Regarding the decoding procedure, the belief propagation (BP) algorithm is recognized as one of the successful algorithms in decoding graphical codes approaching Shannon capacity. It is an iterative process such that at each iteration input and output bits exchange messages, containing log-likelihood ratio (LLR). Let $B^{(l)}$ ($Y^{(l)}$) denote the messages passed from an input (output) bit b (y) to an output (input) bit at the l^{th} iteration of BP algorithm which are related as

$$\begin{aligned} \tanh(Y^{(l)}/2) &= \tanh(q_t/2) \prod_{adj(b)} \tanh(B^{(l)}/2) \\ B^{(l+1)} &= \sum_{adj(y)} Y^{(l)}, \end{aligned} \tag{4}$$

where $q_t \triangleq \ln[\Pr(y_t = 0|z_t)/\Pr(y_t = 1|z_t)]$ denotes channel LLR at time t, and in the case of BSC, $q_t = \ln\left(\frac{1-\epsilon}{\epsilon}\right)(-1)^{z_t}$. Moreover, $adj(b)$ ($adj(y)$) represents the output (input) bits adjacent to the input (output) bit b (y). At the last iteration, the LLR of an input bit, which is obtained by $\sum_y Y^{(l)}$, is a measure to decide if the decoded bit is 1 or 0, where the sum is over all output bits adjacent to the input bit. The anytime decoding sequentially applies the BP algorithm based on the assumption that the previous received bits $\mathbf{z}_1^{t-1}$ are available at the decoder.

To simplify the analysis of the BP algorithm, one can approximate the probability density of messages passed at each iteration by simple functions, instead of tracking the true densities. Commonly in practice, e.g., [11], $B^{(l)}$ and $Y^{(l)}$ are asymptotically approximated by Gaussian variables by which we only need to determine the mean and the variance.

4. PERFORMANCE ANALYSIS

In this section, we present the the performance analyses of the anytime transmission schemes described in Section 3.

Theorem 1 *The source distortion at $t \in T_j$, subjected to the binary expansion of Section 2, is given by*

$$\Delta_s^2(t) = \frac{1}{3}\left(\frac{1}{4}\right)^{j_t}. \tag{5}$$

Theorem 2 *Given a BSC and the anytime UEP-LT codes of Section 3.2 and Section 3.3, the channel distortion $\Delta^2_{c,lt}(t)$ at $t \in T_j$ is upper bounded by*

$$\Delta^2_{c,lt}(t) \leq \frac{1}{2}\sum_{i=1}^{j}\sum_{k=1+K_{i-1}}^{K_i} 2^{-2k} \exp\left(\sum_{m=i}^{j} \alpha_m(t)\left(\exp\left(-\mathbb{E}[Y^{(l)}_{m,i}]/4\right)-1\right)\right), \tag{6}$$

where,

$$\begin{aligned} \alpha_m(t) &= \begin{cases} \beta_m T_m/K_m, & m = i, \cdots, j-1 \\ \beta_j t/K_j, & m = j \end{cases} \\ \mathbb{E}\left[Y^{(l)}_{m,i}\right] &= \sum_{d_m} \omega_{d_m} \xi_{d_m}(\mu^{i,(l)}) \\ \mu^{i,(l+1)} &= \iota'_m(1)\mathbb{E}[Y^{(l)}_{m,i}] \\ \xi_{d_m}(\mu^{i,(l)}) &= 2\mathbb{E}\left[\tanh^{-1}\left(\tanh(q/2)\prod_{m=1}^{d_m-1}\tanh(U_m/2)\right)\right]. \end{aligned} \tag{7}$$

where U_m's ($m = 1, ..., d_m - 1$), representing the message $B^{(l)}$, are symmetric Gaussian distributed iid random variables, following $\mathcal{N}\left(\mu^{i,(l)}, 2\mu^{i,(l)}\right)$.

Corollary 1 *Given a BSC and the anytime UEP-LT code of Section 3.2 and Section 3.3, the end-to-end distortion at $t \in T_j$ is upper bounded by*

$$\Delta_{lt}(t) \leq \Delta_s(t) + \Delta_{c,lt}(t), \tag{8}$$

where $\Delta_s(t)$ and $\Delta_{c,lt}(t)$ are defined in (5) *and* (6)*, respectively.*

Theorem 3 *Given a BSC and the anytime repetition code of Section 3.1, the end-to-end distortion is upper bounded by*

$$\Delta_{re}(t) \leq \Delta_s(t) + \Delta_{c,re}(t), \tag{9}$$

where

$$\begin{aligned} \Delta_s^2(t) &= \frac{1}{3}\left(\frac{1}{4}\right)^{j_t}, \\ \Delta^2_{c,re}(t) &= \sum_{\substack{k=1 \\ k:odd}}^{K} 2^{-2k} I_\epsilon\left(\frac{K-k+2}{2}, \frac{K-k+2}{2}\right) \\ &+ \sum_{\substack{k=1 \\ k:even}}^{K} 2^{-2k}\left[I_\epsilon\left(\frac{K-k+1}{2}, \frac{K-k+1}{2}\right)\right. \\ &\left. + \frac{1}{2}\binom{K-k+1}{\frac{K-k+1}{2}}\left(\epsilon-\epsilon^2\right)^{\frac{K-k+1}{2}}\right], \end{aligned} \tag{10}$$

where $I_\epsilon(a, b)$ is the regularized incomplete beta function defined as

$$I_\epsilon(a,b) \triangleq \sum_{j=a}^{a+b-1}\binom{a+b-1}{j}\epsilon^j(1-\epsilon)^{a+b-1-j}.$$

5. PROOF

This section is devoted to the proofs of theorems in Section 4.

Proof (Theorem 1) Let $x_s(t)$ denote the quantized value of x at time t. Hence, the source distortion $\Delta_s^2(t)$ can be written as

$$\begin{aligned} \Delta_s^2(t) &\triangleq \mathbb{E}\left[(x - x_s(t))^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^{\infty} b_k 2^{-k} - \sum_{k=1}^{j_t} b_k 2^{-k}\right)^2\right] \\ &= \mathbb{E}\left[\sum_{k=j_t+1}^{\infty}\sum_{l=j_t+1}^{\infty} b_k b_l 2^{-(k+l)}\right] = \sum_k\sum_l 2^{-(k+l)}\mathbb{E}[b_k b_l]. \end{aligned} \tag{11}$$

Since b_k's are iid distributed as Bernoulli(1/2), $\mathbb{E}[b_k b_l]$ can be obtained as

$$\mathbb{E}[b_k b_l] = \frac{1}{4} + \frac{1}{4}\delta(k-l), \tag{12}$$

where $\delta(k-l)$ is the delta-Dirac function defined as $\delta(k-l) = 1$ if and only if $k = l$, otherwise zero. Plugging (12) into (11) and applying the sum of geometric series, it yields

$$\Delta_s^2(t) = \frac{4}{3}\left(\frac{1}{4}\right)^{j_t+1}. \tag{13}$$

Proof (Theorem 2) Since the channel distortion of each level of importance varies by forming each encoding window as time continues, we consider the anytime transmission happens at the interval T_j, i.e., $t \in T_j$, then the channel distortion can be written as

$$\begin{aligned}\Delta^2_{c,lt}(t) &\triangleq \mathbb{E}\left[(x_s(t)-\hat{x}(t))^2\right] \\ &= \mathbb{E}\left[\left(\sum_{k=1}^{K_j} b_k 2^{-k} - \sum_{k=1}^{K_j} \hat{b}_k(t)2^{-k}\right)^2\right] \\ &= \sum_k 2^{-2k}\mathbb{E}[(b_k-\hat{b}_k(t))^2] \\ &= \sum_{k=1}^{K_j} 2^{-2k}\Pr(\hat{b}_k(t)=1|b_k=0) \\ &= \sum_{i=1}^{j}\sum_{k=1+K_{i-1}}^{K_i} 2^{-2k}P_{e_i}(t),\end{aligned} \quad (14)$$

where the last equality follows by defining $\Pr(\hat{b}_k(t)=1|b_k=0)$ as $P_e(t)$. Furthermore, $P_{e_i}(t)$ denotes bit error probability of the input bits in the i^{th} level of importance. Assuming the Gaussian approximation and all-zero codeword sent, $P_{e_i}(t)$ can be obtained by (15), where $Y_{j,i}$ is the message sent from output bits in the j^{th} encoding window to the input bits of the i^{th} level. Also, $I^i_{n_j}$ is the probability that an input bit (in the i^{th} class) connected only to the j^{th} encoding window is of degree n_j regardless of the other edges. Applying the Chernoff bound $Q(x)\leq\frac{1}{2}\exp(-x^2/2)$ to (15), yields

$$\begin{aligned}P_{e_i}(t) &\leq \frac{1}{2}\prod_{m=i}^{j}\sum_{n_m} I^i_{n_m}\exp\left(-n_m\mathbb{E}[Y^{(l)}_{m,i}]/4\right) \\ &= \frac{1}{2}\prod_{m=i}^{j} I^i_m\left(\exp\left(-\mathbb{E}[Y^{(l)}_{m,i}]/4\right)\right).\end{aligned} \quad (16)$$

By following (3), (16) is equivalent to

$$P_{e_i}(t) \leq \frac{1}{2}\exp\left(\sum_{m=i}^{j}\alpha_m(t)\left(\exp\left(-\mathbb{E}[Y^{(l)}_{m,i}]/4\right)-1\right)\right), \quad (17)$$

where the average degree of the input bits in the m^{th} window regardless of other edges can be derived as

$$\alpha_m(t) = \begin{cases}\beta_m T_m/K_m, & m=i,\cdots,j-1 \\ \beta_j t/K_j, & m=j\end{cases} \quad (18)$$

Now, the only concern is to determine $\mathbb{E}[Y^{(l)}_{m,i}]$. For this purpose, we follow the semi-Gaussian approximation [11] in order to obtain the updating rules. Based on the semi-Gaussian assumption, the messages sent from input bits of the i^{th} class at round l of the BP algorithm are symmetric Gaussian variables with mean $\mu^{i,(l)}$, and variance $2\mu^{i,(l)}$. From the second equation of (4), the relation of $\mu^{i,(l)}$ and $\mathbb{E}[Y^{(l)}_{m,i}]$ $(i \leq m \leq j)$ can be shown to be

$$\mu^{i,(l+1)} = \sum_{n_m}\iota_{n_m}(n_m-1)\mathbb{E}[Y^{(l)}_{m,i}] = \iota'_m(1)\mathbb{E}[Y^{(l)}_{m,i}], \quad (19)$$

where $\iota'_m(1)=\sum_{m=i}^{j-1}\beta_m T_m/K_m+\beta_j t/K_j$ is asymptotically the average degree of an input bit in the i^{th} class by considering all output neighbors connected to it.

The messages passed from an output bit of window m are assumed to be Gaussian, and their expected values can be determined as

$$\begin{aligned}\xi_{d_m}(\mu^{i,(l)}) &\triangleq \mathbb{E}\left[Y^{(l)}_{m,i}|deg(y)=d_m\right] \\ &= 2\mathbb{E}\left[\operatorname{atanh}\left(\tanh(q/2)\prod_{m=1}^{d_m-1}\tanh(U_m/2)\right)\right],\end{aligned} \quad (20)$$

where U_m's $(m=1,...,d_m-1)$ are iid random variables distributed by $\mathcal{N}\left(\mu^{i,(l)},2\mu^{i,(l)}\right)$. Furthermore, the channel LLR is independent of U_i's. Therefore,

$$\mathbb{E}\left[Y^{(l)}_{m,i}\right] = \sum_{d_m}\omega_{d_m}\xi_{d_m}(\mu^{i,(l)}). \quad (21)$$

Starting from $\mu^{i,(0)}=0$, then $\mathbb{E}\left[Y^{(0)}_{m,i}\right]=q\omega_{d_1}$, according to (20); and after a sufficient number of iterations, the mean values are obtained. It is worth mentioning that the degree one of the degree distribution should not be zero, otherwise, the mean at each iteration becomes zero and the decoding algorithm does not start.

Plugging the results into (14), the upper bound for channel distortion is given by

$$\Delta^2_{c,lt}(t) \leq \frac{1}{2}\sum_{i=1}^{j}\sum_k 2^{-2k}\exp\left(\sum_{m=i}^{j}\alpha_m(t)\left(\exp\left(-\mathbb{E}[Y^{(l)}_{m,i}]/4\right)-1\right)\right). \quad (22)$$

Proof (Corollary 1) The end-to-end distortion can be written as

$$\begin{aligned}\Delta^2_{lt}(t) &= \mathbb{E}\left[(x-x_s(t)+x_s(t)-\hat{x}(t))^2\right] \\ &\overset{(a)}{\leq} \left(\mathbb{E}^{1/2}[(x-x_s(t))^2]+\mathbb{E}^{1/2}[(x_s-\hat{x}(t))^2]\right)^2 \\ &= \left(\Delta_s(t)+\Delta_{c,lt}(t)\right)^2,\end{aligned} \quad (23)$$

where (a) follows from the Minkowski inequality.

Proof (Theorem 3) The source distortion of the anytime repetition coding follows from the same proof as that of the anytime UEP-LT scheme. Note that according to the repetition strategy, K bits are used at time t in which it can be verified that $t=\frac{K(K+1)}{2}$. In order to analyze the channel distortion, the probability of error for the bits that are repeated odd number of times is considered individually compared with those which are repeated even number of times. Therefore, if we assume that at time t, K is an odd number, then it can be easily verified that

$$\begin{aligned}P_{e_{k,odd}} &= \sum_{j=\frac{k+1}{2}+1}^{n}\binom{k}{j}\epsilon^j(1-\epsilon)^{k-j} \\ P_{e_{k,even}} &= \sum_{j=\frac{k}{2}+1}^{n}\binom{k}{j}\epsilon^j(1-\epsilon)^{k-j}+\frac{1}{2}\binom{k}{\frac{k}{2}}\epsilon^{k/2}(1-\epsilon)^{k/2}.\end{aligned} \quad (24)$$

The second term in $P_{e_{k,even}}$ is due to the error in which the number of zeros is equal to the number of ones. Applying (3) and (5) to the results and using the Minkowski inequality as in (23), the proof completes.

6. NUMERICAL RESUTS

In this section, we quantify the performance of the anytime transmission schemes in terms of the end-to-end distortion and compare them with simulation results.

$$P_{e_i}(t)=\Pr\Big(\sum Y^{(l)}<0\Big)=\sum_{n_i,\cdots n_j}\Pr\left(n_iY_{i,i}^{(l)}+\cdots+n_jY_{j,i}^{(l)}|deg(b\in\text{window i})=n_i,\cdots,deg(b\in\text{window j})=n_j\right)$$

$$\cdot\Pr\left(deg(b\in\text{window i})=n_i,\cdots,deg(b\in\text{window j})=n_j\right)=\sum_{n_i}\cdots\sum_{n_j}I_{n_i}^i\cdots I_{n_j}^iQ\left(\sqrt{\left(n_i\mathbb{E}[Y_{i,i}^{(l)}]+\cdots+n_j\mathbb{E}[Y_{j,i}^{(l)}]\right)/2}\right), \quad (15)$$

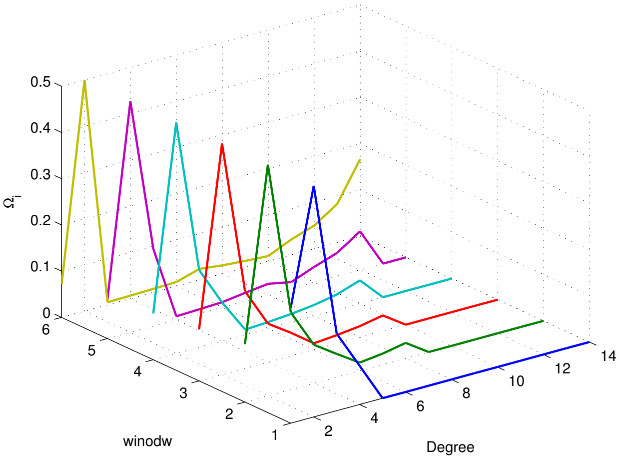

Figure 1: Degree distributions associated with the windows

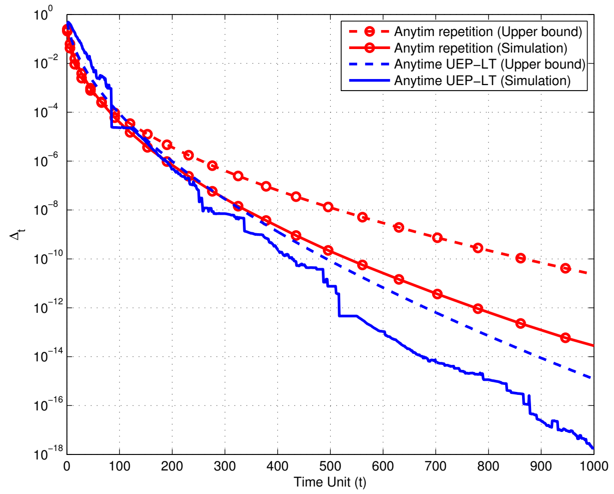

Figure 2: Comparison of simulated and analytic results of the anytime schemes

The number of information and encoding windows for the anytime UEP-LT codes is set to six windows with the lengths $\mathbf{K}$ = [10,12,30,40,50,60] and $\mathbf{T}$ = [60,65,180,250,270,300], respectively. Furthermore, the degree distributions are optimized for the BSC with cross-over probability $\epsilon=0.11$ using the approach in [1] which yields the degree distributions shown in Figure 1. The maximum order of the windows' degree distributions is set to $\mathbf{D}$=[4,8,9,10,12,14]. For the anytime UEP-LT codes, $\xi_{d_m}(\mu^i)$ in (7) is sampled using Monte-Carlo simulations by setting $\mu\in(0,16]$ and the step-size as $\Delta\mu=0.01$. The number of BP iterations is also set to 50 rounds. The analytic and simulated results are reported in Figure 2. It is worth pointing out that at lower time horizons, the asymptotic Gaussian assumption used in the proof of Theorem 2 does not hold. However, as time increases the upper bound provides the true bound on the distortion of the anytime UEP-LT scheme.

7. CONCLUSION

In this paper, we have derived analytical upper bounds for the end-to-end distortion of two anytime transmission schemes. These strategies are based on repetition channel coding with sequential MLD algorithm, and UEP-LT channel coding using sequential BP decoding. In addition, the bounds have been compared with simulations in order to verify the accuracy of the analytical results.

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