

# ABOUT POSITIVE TRIGONOMETRIC POLYNOMIALS AND 1-D DISCRETE PHASE RETRIEVAL PROBLEM

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## ABSTRACT

We reconsider the discrete form of the one dimensional phase retrieval problem from the point of view of magnitude input data. It has been previously mentioned that input magnitude of DFT should satisfy certain conditions in order to provide the correct solution. These requirements ask for the corresponding trigonometric polynomial to be positive definite. Alternatively, an arbitrary set of DFT magnitude may not provide a correct solution. In this paper we study whether this may be a reason for iterative methods to stagnate. We present a sequence of steps in order to obtain a correct solution to the one dimensional phase retrieval problem. Experimental results are also provided.

**Index Terms**— Signal reconstruction, Discrete Fourier Transform, phase retrieval.

## 1. INTRODUCTION

Signal reconstruction from Fourier transform magnitude has been called phase retrieval [1]. The term comes from the fact that the Fourier phase is not known and the signal should be reconstructed. Although certain constraints may be added according to application, the main one dimensional discrete phase retrieval (1-D DPhR) problem can be stated as follows:

Let  $x(n)$  be a discrete signal of length  $N$  and let  $X(k)$  be its  $N$ -point Discrete Fourier Transform (DFT). Given knowledge that  $x(n)$  has  $M$ -point support, and given the values of the DFT magnitudes  $|X(k)|$ ,  $k = 0, 1, \dots, N - 1$ , determine  $x(n)$  or equivalently  $X(k)$  [2].

In order to find the solution of 1-D DPhR problem, the most common approaches are iterative transform algorithms [3], which alternate between time and frequency domains. This type of algorithms can implement very easily time-domain constraints like compactness of the support. It has been observed that these algorithms fail to converge to a solution as they usually stagnate [4]. Another way to find a solution to 1-D DPhR problem is finding the zeros of  $z$ -transform of autocorrelation. This may provide the minimum-phase solution, however this method behaves poorly numerically and can be recommended only for rather short length [5]. Alternative for solving the 1-D DPhR problem are Hilbert transform

[6], computation of cepstral coefficients [7] or solving linear systems of equations [2].

Previously we have considered 1-D DPhR problem from the point of view of magnitude input data [8]. We have claimed that magnitude input data, which actually are the modulus of DFT  $|X(k)|$ , should satisfy certain requirements in order to provide a correct solution. To this end the Fejér-Riesz Theorem guarantees us that 1-D DPhR problem has always a solution if the corresponding 1-D DPhR trigonometric polynomial is positive definite. However, an arbitrary set of magnitudes does not provide always a positive definite trigonometric polynomial, consequently the 1-D DPhR problem may not have a correct solution.

The goal of this paper is to analyze whether the previous requirements may be one of the reasons for 1-D DPhR iterative methods to stagnate. Another target is to present a sequence of steps in order to obtain a correct solution to the 1-D DPhR problem.

The paper is organized as follows. First we mention some support requirements of 1-D DPhR problem (Section 2). A short description of the iterative algorithms for solving the 1-D DPhR problem is presented in Section 3. The Fejér-Riesz Theorem and its consequences are recalled in Section 4. Steps for a correct solution of 1-D DPhR problem are specified in Section 5. Experimental results are discussed in Section 6. Finally conclusions are also delivered.

## 2. SUPPORT REQUIREMENTS

The 1-D DPhR problem deals with sequences having finite length. Let us consider the signal  $x(n)$  having  $M$ -point support. Since its autocorrelation  $r(n)$  has the support  $[-(M - 1), M - 1]$ , sampling of the Fourier transform squared magnitude at  $\omega_k = \frac{2\pi k}{N}$  with

$$N \geq 2M - 1, \quad (1)$$

will be sufficient to extract autocorrelation from circular autocorrelation  $\tilde{r}(n)$ , without time-domain aliasing.

If the support of  $x(n)$  does not satisfy (1), the 1-D DPhR problem is an ill-posed problem. Indeed, the set of squares of the DFT magnitudes is the DFT of the circular autocorrelation

$\tilde{r}(n)$  of  $x(n)$ . If  $x(n)$  has  $M$ -point support, its autocorrelation  $r(n)$  has  $2M - 1$  length. If (1) is not satisfied, then  $\tilde{r}(n)$  will be corrupted because of time-aliasing, and  $r(n)$  cannot be recovered from  $\tilde{r}(n)$ .

We can also assume that  $N$  is odd.

Indeed, there is large class of applications where the DFT magnitudes are measured by sampling the magnitude of the Fourier transform of an analog real value signal. In such case we use to have an even symmetry of the magnitude. If zero frequency is also measured, it follows that we have an odd number of DFT magnitudes.

Moreover, suppose that  $N$  is even. Then the DFT for  $k = N/2$  corresponds actually to  $\omega = \pi$ . We can consider that the component corresponding to  $\omega = \pi$  is divided in two equal half components for  $\omega = \pi$  and  $\omega = -\pi$ . Consequently what we get actually is an odd number of DFT samples.

For these reasons, in Section 4, we shall consider that  $N = 2M - 1$ , otherwise the sequence  $x(n)$  will be zero-padded correspondingly.

### 3. THE ITERATIVE ALGORITHM FOR 1-D DPHR PROBLEM

The iterative transform algorithm or 1-D DPhR problem is a standard iterative technique in which the estimate of  $x(n)$  is improved in each iteration [9]. The  $N$ -point discrete Fourier transform (DFT) of  $x(n)$  will be denoted as  $X(k) = |X(k)|e^{j\angle X(k)}$  [10]. The iterative technique to reconstruct the sequence  $x(n)$  from the  $N$  samples of its magnitude  $|X(k)|$ ,  $k = 0, 1, \dots, N - 1$  is described as follows.

1. We begin with  $\angle \tilde{X}_1(k)$ , an initial guess of the unknown DFT phase and form the first estimate,  $X_1(k)$ , of  $X(k)$  using the specified magnitude function, i.e.

$$X_1(k) = |X(k)|e^{j\angle \tilde{X}_1(k)}.$$

Computing the inverse DFT of  $X_1(k)$  provides the first estimate,  $x_1(n)$ , of  $x(n)$ . Since an  $N$ -point DFT is used,  $x_1(n)$  is an  $N$ -point sequence which is, in general, nonzero for  $M \leq n \leq N - 1$ .

2. From  $x_1(n)$ , another sequence  $\tilde{x}_2(n)$  is defined by

$$\tilde{x}_2(n) = \begin{cases} x_1(n), & 0 \leq n \leq M - 1, \\ 0, & M \leq n \leq N - 1. \end{cases}$$

3. The phase  $\angle \tilde{X}_2(k)$  of the  $N$ -point DFT of  $\tilde{x}_2(n)$  is then considered as a new estimate of  $\angle X(k)$  and a new estimate of  $X(k)$  is formed by

$$X_2(k) = |X(k)|e^{j\angle \tilde{X}_2(k)}.$$

From this, a new estimate  $x_2(n)$  is obtained from the inverse DFT of  $X_2(k)$ .

Repetitive application of Steps 2 and 3 defines the standard iterative algorithm.

In this iterative procedure, an error function that is non increasing (proof can be found in [10]) is the mean-square difference between the known magnitude and the estimate on each iteration, i.e.,

$$E_p = \frac{1}{N} \sum_{k=0}^{N-1} \left| |X(k)| - |\tilde{X}_p(k)| \right|^2.$$

Since  $E_p$  is non increasing and has a lower bound of zero, it must converge to a limit point, which may be zero or a positive nonzero number.

### 4. POSITIVE TRIGONOMETRIC POLYNOMIALS AND 1-D DPHR PROBLEM

For the beginning we shall recall a result on trigonometric polynomials, then we shall relate it to 1-D DPhR problem.

**Theorem 1 (Fejér and Riesz)** *If*

$$Y(z) = \sum_{n=-M}^M y(n)z^{-n} \quad \text{and} \quad Y(e^{j\omega}) \geq 0,$$

*then there is*

$$X(z) = \sum_{n=0}^M x(n)z^{-n}$$

*such that*

$$Y(e^{j\omega}) = |X(e^{j\omega})|^2.$$

Going back to 1-D DPhR problem, let us remind that the Fourier transform magnitude is the Fourier transform of the autocorrelation:

$$S(\omega) \equiv |X(e^{j\omega})|^2 = \mathcal{F}\{r(n)\}. \quad (2)$$

On the other hand, for a  $M$ -point support sequence  $x(n)$ , the sequence and its circular autocorrelation  $\tilde{r}(n)$  can be written via  $(2M - 1)$ -point DFT as follows :

$$\begin{aligned} X(k) &= \sum_{n=0}^{M-1} x(n)e^{-j\frac{2\pi kn}{2M-1}}, \\ |X(k)|^2 &= \sum_{n=0}^{2M-1} \tilde{r}(n)e^{-j\frac{2\pi kn}{2M-1}}. \end{aligned} \quad (3)$$

Let  $\hat{r}(n)$  be defined by folding  $\tilde{r}(n)$ :

$$\hat{r}(n) = \begin{cases} \tilde{r}(0), & n = 0 \\ \tilde{r}(n), & n = 1, 2, \dots, M - 1 \\ \tilde{r}(n + 2M - 1), & n = -1, -2, \dots, -(M - 1) \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

If time aliasing has been avoided, then  $\hat{r}(n) = r(n)$ , and we have

$$|X(k)|^2 = \sum_{k=-(M-1)}^{M-1} r(n)e^{-j\frac{2\pi kn}{M-1}}. \quad (5)$$

In view of (2), (3) and (5), the problem of finding  $x(n)$  from  $r(n)$  is actually whether the trigonometric polynomial  $\hat{S}(\omega) = \mathcal{F}\{\hat{r}(n)\}$  is a nonnegative trigonometric polynomial. In such case the Fejér-Riesz Theorem guarantees us that the 1-D DPhR problem has always a correct solution.

### Example 1

Let  $N = 5$  and

$$|X(k)| = \begin{cases} 2, & k = 0; \\ 1 & k = 1, 2, 3, 4. \end{cases} \quad (6)$$

Then

$$\tilde{r}(n) = \text{DFT}^{-1}\{|X(k)|^2\} = \begin{cases} 1.6, & n = 0; \\ 0.6 & n = 1, 2, 3, 4. \end{cases}$$

Taking into account the periodicity and the symmetry properties, we have

$$\hat{r}(n) = \begin{cases} 1.6, & n = 0; \\ 0.6 & n = \pm 1, \pm 2. \end{cases}$$

Its Fourier transform

$$\hat{S}(\omega) = \mathcal{F}\{\hat{r}(n)\} = 1.6 + 1.2 \cos \omega + 1.2 \cos 2\omega = 2.4 \cos^2 \omega + 1.2 \cos \omega + 0.4.$$

Since  $1.2^2 - 4 \cdot 2.4 \cdot 0.4 = -2.4 < 0$ , then  $\hat{S}(\omega) > 0$  for all  $\omega$  and the 1-D DPhR problem has a correct solution. Indeed,

$$\hat{S}(\omega) = |1.0736 + 0.3675e^{-j\omega} + 0.5589e^{-j2\omega}|^2$$

and

$$x(n) = \begin{cases} 1.0736 & n = 0; \\ 0.3675 & n = 1; \\ 0.5589 & n = 2; \\ 0 & \text{otherwise} \end{cases}$$

is a solution of 1-D DPhR problem when the input DFT magnitude data are given by (6).

Alternatively, not any selection of magnitude can provide a positive definite trigonometric polynomial [8].

### Example 2

Let again  $N = 5$  and

$$|X(k)| = \begin{cases} 3, & k = 0; \\ 1 & k = 1, 2, 3, 4. \end{cases}$$

Then

$$\tilde{r}(n) = \text{DFT}^{-1}\{|X(k)|^2\} = \begin{cases} 2.6, & n = 0; \\ 1.6 & n = 1, 2, 3, 4. \end{cases}$$

Taking into account the periodicity and the symmetry properties, we have

$$\hat{r}(n) = \begin{cases} 2.6, & n = 0; \\ 1.6 & n = \pm 1, \pm 2. \end{cases}$$

Its Fourier transform

$$\hat{S}(\omega) = \mathcal{F}\{\hat{r}(n)\} = 2.6 + 3.2 \cos \omega + 3.2 \cos 2\omega$$

is not always positive as  $\hat{S}(\pi/2) < 0$ . Since  $\hat{S}(\omega)$  is not positive definite, it may be not written as a square of modulus of a trigonometric polynomial. In such situation an attempt to solve correctly 1-D DPhR problem may be unsuccessful.

## 5. TOWARDS A CORRECT SOLUTION TO 1-D DPHR PROBLEM

For our purposes it is important to find the conditions for a certain set of magnitude input data to provide a correct solution to 1-D DPhR problem. One way is to compute  $\tilde{r}(n)$  using inverse DFT of  $|X(k)|^2$ , then to fold it in order to get  $\hat{r}(n)$ . Finally we have to verify whether  $\hat{S}(\omega)$  is positive for all  $\omega$ .

To this end we can consider  $\hat{S}(z) \equiv \mathcal{Z}\{\hat{r}(n)\}$  and we have to verify whether  $\hat{S}(z)$  is positive for all  $z$  with  $|z| = 1$ . To test the unit circle positivity of polynomials is possible and a procedure is known for decades [11].

To proceed we recall some properties of the polynomials which are nonnegative on the unit circle [5]:

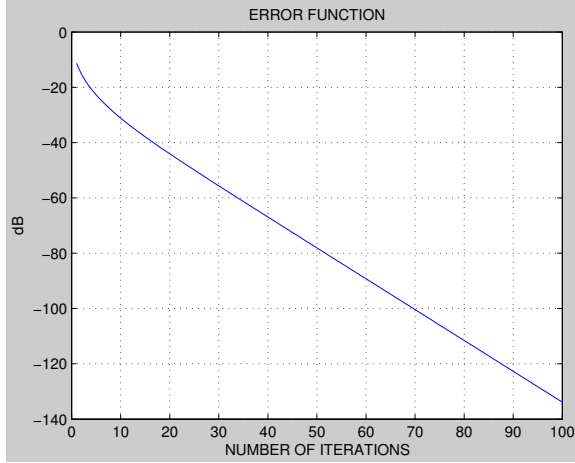
1. Positive polynomials on unit circle are either symmetric or complex Hermitian or a positive constant;
2. Positive polynomials on unit circle contain two types of zeros [5]:
  - zeros that are not on the unit circle, but come in pairs;
  - zeros that are on the unit circle and have even multiplicity.

Taking into account all above mentioned results, to obtain a correct solution to 1-D DPhR problem we may proceed as follows:

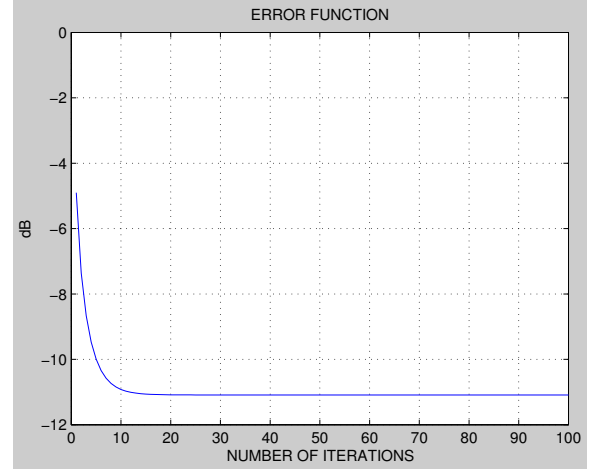
1. Evaluate the circular autocorrelation

$$\tilde{r}(n) = \text{DFT}_N^{-1}\{|X(k)|\}$$

2. Evaluate the folded circular autocorrelation  $\hat{r}(n)$  using (4);



**Fig. 1.** The variation of  $E_p$  for Example 1.



**Fig. 2.** The variation of  $E_p$  for Example 2.

3. Evaluate  $S(z) = \mathcal{Z}\{\hat{r}(n)\}$ ;
4. Verify whether all zeros on unit circle of  $S(z)$  have even multiplicity;
  - (a) If YES, we can proceed with any method for solving 1-D DPhR problem;
  - (b) If NOT, the 1-D DPhR problem may not have a correct solution.

When 4(a) is true, then the zeros must occur in conjugate pairs. From every pair, one can select a zero to form a solution of 1-D DPhR problem. This procedure is called the direct method or by finding the zeros of  $z$ -transform.

## 6. EXPERIMENTAL RESULTS

First we have implemented the same algorithm for a set of DFT input magnitude data which provides a positive trigonometric polynomial (Example 1). We run the iterative phase-retrieval procedure and we computed the mean-square difference. The variation of  $E_p$  is shown in Figure 1. It can be seen that the error function is fast converging towards zero and finally we get the solution of 1-D DPhR problem mentioned in Example 1.

Then we have implemented the iterative phase-retrieval algorithm for the input DFT magnitudes given in Example 2 and we have proceeded as before. The mean-square difference between the known magnitude and the estimate on each iteration decreases and finally stagnates (Figure 2). In this situation we cannot obtain a correct solution of 1-D DPhR problem.

We can speculate that such situation happens whenever the input DFT magnitudes do not provide a positive trigonometric polynomial. However, one can find a set of DFT magnitudes which does not provide a positive trigonometric poly-

nomial and for which the iterative phase-retrieval algorithm converges in a similar manner as in the case shown in Figure 2. Indeed, in such case the iterative phase retrieval algorithm finds a correct solution to 1-D DPhR problem, but only the DFT samples  $|X(k)|$  are correct; the overall Fourier transform, since  $\hat{S}(\omega) \neq S(\omega) \geq 0$ .

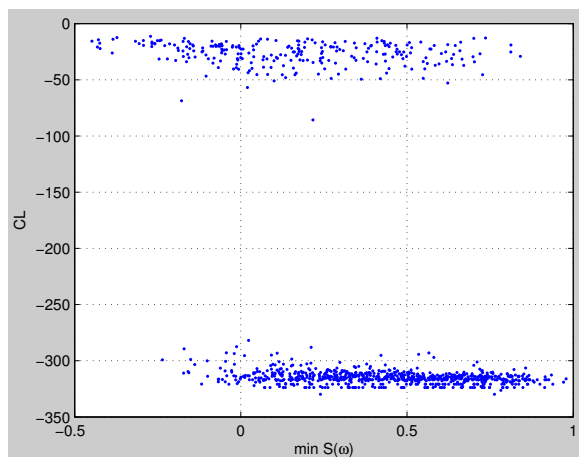
Moreover, one can find also set of DFT magnitudes which does provide a positive trigonometric polynomial and for which the iterative phase-retrieval algorithm converges in a similar manner as in the case shown in Figure 1, i.e.  $E_p$  stagnates. It follows that even the Fejér-Riesz Theorem guarantees us that the 1-D DPhR problem has always a correct solution, it is not guaranteed that the solution can be found using the iterative phase-retrieval algorithm.

We just note that such examples are not seldom. We have generated uniform random DFT input data and we have verified whether they provide a positive definite trigonometric polynomial. At the same time we have verified whether the iterative phase-retrieval algorithm converges or stagnates.

A first set of outcomes are graphically presented in Figure 3. The length of DFT magnitude input data was 5 and we have considered 1000 random input different data. The energy of autocorrelation sequences has been normalized. We have computed the convergence level (CL) in dB of  $E_p$  after  $10^5$  iterations and the minimum of  $\hat{S}(\omega)$  and we get:

- 715 DFT data with  $CL < -250$  and  $\hat{S}(\omega) \geq 0$ ;
- 32 DFT data with  $CL < -250$  and  $\hat{S}(\omega) < 0$ ;
- 168 DFT data with  $CL > -100$  and  $\hat{S}(\omega) \geq 0$ ;
- 85 DFT data with  $CL > -100$  and  $\hat{S}(\omega) < 0$ .

A second set of outcomes are shown in Figure 3. The length of DFT magnitude input data was 15 and we have only two large sets:



**Fig. 3.** Convergence level (CL) in dB of  $E_p$  after  $10^5$  iterations and the minimum of  $\hat{S}(\omega)$  for 1000 random DFT input magnitude data of 5 length.

- 380 DFT data with  $CL < -250$  and  $\hat{S}(\omega) \geq 0$ ;
- 620 DFT data with  $CL > -100$  and  $\hat{S}(\omega) \geq 0$ ;

## 7. CONCLUSIONS

In this paper we have analyzed whether the positivity requirements of the corresponding trigonometric polynomial may be one of the reasons for 1-D DPhR iterative methods to stagnate.

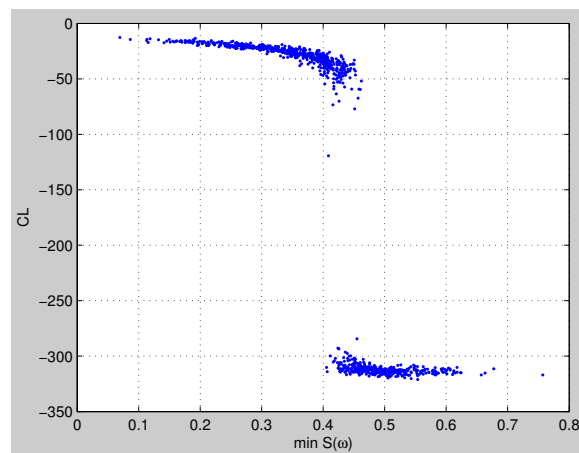
We can conclude that even the Fejér-Riesz Theorem guarantees us that the 1-D DPhR problem has always a correct solution, it is not guaranteed that the solution can be found using the iterative phase-retrieval algorithm. Thus the iterative procedure may stagnate even for cases when the direct method provides correct solution to the 1-D DPhR problem.

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**Fig. 4.** Convergence level (CL) in dB of  $E_p$  after  $10^5$  iterations and the minimum of  $\hat{S}(\omega)$  for 1000 random DFT input magnitude data of 15 length.

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