A NEW TOOL FOR MULTIDIMENSIONAL LOW-RANK STAP FILTER: CROSS HOSVDS

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ABSTRACT

Space Time Adaptive Processing (STAP) is a two-dimensional adaptive filtering technique which uses jointly temporal and spatial dimensions to suppress disturbance. Disturbance contains both the clutter arriving from signal backscattering of the ground and the thermal noise. In practical cases, the STAP clutter can be considered to have a low rank structure, allowing to derive a low rank vector STAP filter, based on the projector onto the clutter subspace. In order to process new STAP applications (MIMO STAP, polarimetric STAP . . .) and keeping the multidimensional structure, we propose in this paper a new low-rank tensor STAP filter based on a generalization of the Higher Order Singular Value Decomposition (HOSVD): the Cross-HOSVDs. This decomposition uses at the same time the simple (like polarimetric) and the combined information (for example spatio-temporal). We apply the filter on polarimetric STAP and compute the SNR Loss with Monte-Carlo simulations.

1. INTRODUCTION

Space Time Adaptive Processing (STAP) is technique used in airborne phased array radar to detect moving target embedded in an interference background such as jamming or strong clutter [1]. While conventional radars are capable of detecting targets both in the time domain related to target range and in the frequency domain related to target velocity, STAP uses an additional domain (space) related to the target angular localization. The consequence is a two-dimensional adaptive filtering technique which uses jointly temporal and spatial dimensions to suppress disturbances and to improve target detection. The disturbance contains both the clutter arriving from signal backscattering of the ground and the thermal noise resulting from the sensors noise. From the Brennan's rule formula [2], the STAP clutter can be considered to have a low rank structure. Using this assumption, a low rank vector STAP filter is derived [3, 4] based on the projector onto the subspace orthogonal to the clutter.

The STAP data are collected as a data cube, but they are usually folded as vectors to use the space-time information. With new STAP applications like MIMO STAP or polarimetric STAP [5], the generalization of the classic filters to multidimensional configurations arises. A possible solution consists in keeping the multidimensional structure and in extending the classic filters with multilinear algebra [6, 7] as it is done in [8]. Using the low-rank structure of the clutter, the HOSVD [9] (High Order Singular Value Decomposition) seems natural to find the tensor projector of the clutter subspace. However, we concluded in [10], that this decomposition is not appropriated because the spatio-temporal dimension is not considered (only the spatial and temporal dimensions separately are seen by HOSVD). In this paper, we propose a set of decompositions the cross-HOSVDs, a generalization of the HOSVD which can use single (e.g., spatial or temporal) information and combined (like spatio-temporal) information. Based on these new decompositions, we propose a new multidimensional Low-rank STAP filter, which remove the clutter while keeping the target response even if it is located near the clutter. We apply our new filter on polarimetric STAP and we show the interest of our approach by computing the SNR Loss with Monte-Carlo simulations.

The following convention is adopted: scalars are denoted as italic letters, vectors as lower-case bold-face letters, matrices as bold-face capitals, and tensors are written as bold-face calligraphic letters. We use the superscripts ^{H}, for Hermitian transposition and *, for complex conjugation.

2. SOME MULTILINEAR ALGEBRA TOOLS

This section contains the main multilinear algebra tools used in this paper. Let $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$, two 3-dimensional tensors and let $a_{i_1 i_2 i_3}$, $b_{i_1 i_2 i_3}$ their elements. We will use the following operators; for more details, especially the case of n-order tensors, we refer the reader to [6, 9].

2.1. Unfolding

Let us start with 2 unfolding operators, which arrange the elements of a tensor in a matrix or a vector:

- vector: vec transforms a tensor $\boldsymbol{\mathcal{A}}$ into a vector, $vec(\boldsymbol{\mathcal{A}}) \in \mathbb{C}^{I_1 I_2 I_3}$. vec^{-1} is the inverse operator.

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- matrix: this operator transforms the tensor $\mathcal{A}$ into a matrix. For example, $[\mathcal{A}]_1 \in \mathbb{C}^{I_1 \times I_2 I_3}$ and $[\mathcal{A}]_{1,2} \in \mathbb{C}^{I_1 I_2 \times I_3}$.
- square matrix: this operator transforms the tensor $\mathcal{R} \in \mathbb{C}^{I_1 \times I_2 \times I_3 \times I_1 \times I_2 \times I_3}$ into a square matrix, $SqMat(\mathcal{R}) \in \mathbb{C}^{I_1 I_2 I_3 \times I_1 I_2 I_3}$. $SqMat^{-1}$ is the inverse operator.

2.2. Products

- scalar product : $< \mathcal{A}, \mathcal{B} >= \sum_{i_1} \sum_{i_2} \sum_{i_3} b^*_{i_1 i_2 i_3} a_{i_1 i_2 i_3} = vec(\mathcal{B})^H vec(\mathcal{A})$
- n-mode product : $\mathbf{E} \in \mathbb{C}^{J_2 \times I_2}$ $(\mathcal{A} \times_2 \mathbf{E})_{i_1 j_2 i_3} = \sum_{i_2} a_{i_1 i_2 i_3} e_{j_2 i_2}$
- outer product : $\mathcal{E} = \mathcal{A} \circ \mathcal{B} \in \mathbb{C}^{I_1 \times \ldots \times I_N \times J_1 \times \ldots \times J_N}$ avec $e_{i_1 \ldots i_N j_1 \ldots j_M} = a_{i_1 \ldots i_N} . b_{j_1 \ldots j_M}$

2.3. HOSVD

Definition One of the extension of the SVD to the tensor case is given by the HOSVD [9]. A tensor $\mathcal{A}$ can be decomposed as follows:

$$\mathcal{A} = \mathcal{K} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \mathbf{U}^{(3)}, \tag{1}$$

where $\forall n$, $\mathbf{U}^{(n)} \in \mathbb{C}^{I_n \times I_n}$ is an orthonormal matrix and where $\mathcal{K} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ is the core tensor, which satisfies the all-orthogonality conditions [9]. The matrix $\mathbf{U}^{(n)}$ is given by the left singular matrix of the n-dimension unfolding tensor, $[\mathcal{A}]_n$.

Low-rank approximation Let us introduce $\mathcal{A} = \mathcal{A}_c + \mathcal{A}_0$ where $\mathcal{A}_c$ is a (r_1, r_2, r_3) low rank tensor and where $r_k = rank([\mathcal{A}_c]_k) < I_k$, for $k = 1, 2, 3$. An approximation of $\mathcal{A}_0$ is given by [7, 8]:

$$\mathcal{A}_0 \approx \mathcal{A} \times_1 \mathbf{U}_0^{(1)} \mathbf{U}_0^{(1)H} \times_2 \mathbf{U}_0^{(2)} \mathbf{U}_0^{(2)H} \times_3 \mathbf{U}_0^{(3)} \mathbf{U}_0^{(3)H}, \tag{2}$$

with $\mathbf{U}_0^{(n)} = [\mathbf{u}_{r_n+1}^{(n)} \ldots \mathbf{u}_{I_n}^{(n)}]$. The truncation is a correct approximation in most cases, but sometimes the use of an alternating least squares algorithm is necessary for an optimal result [7].

2.4. A new tensor product

The HOSVD mainly relies on the n-mode product. This implies that all dimensions are separately taken into account. But, in several multidimensional applications, the dimensions can be strongly linked and in such a case, it is essential to consider "cross" information (e.g. the spatio-temporal information for the problem under study). Then, in order to generalize the HOSVD in the following of the paper, we have to extend the n-mode product to multiply a tensor with a matrix along several dimensions. Let us introduce $\mathbf{D} \in \mathbb{C}^{I_1 I_2 \times I_1 I_2}$ a square matrix. For example, the $1,2$-mode product, denoted $\times_{1,2}$, is defined by:

$$\mathcal{A} = \mathcal{B} \times_{1,2} \mathbf{D} \iff [\mathcal{A}]_{1,2} = \mathbf{D}[\mathcal{B}]_{1,2}, \tag{3}$$

$$(\mathcal{A})_{i_1 i_2 i_3} = \sum_n \sum_m a_{nmi_3} d_{(i_1+(i_2-1)I_1)(n+(m-1)I_1)} \tag{4}$$

2.5. Covariance Tensor

- Let $\mathcal{V} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ a random 3-order tensor
- $\mathcal{R} \in \mathbb{C}^{I_1 \times I_2 \times I_3 \times I_1 \times I_2 \times I_3}$ the Covariance Tensor is defined as :

$$\mathcal{R} = \mathbb{E}[\mathcal{V} \circ \mathcal{V}^*] \tag{5}$$

- Let $\mathcal{Z}_k \in \mathbb{C}^{I_1 \times I_2 \times I_3}$, K snapshots of $\mathcal{V}$
- By analogy with the Sample Covariance Matrix (SCM), $\hat{\mathcal{R}} \in \mathbb{C}^{I_1 \times I_2 \times I_3 \times I_1 \times I_2 \times I_3}$, the Sample Covariance Tensor (SCT) is defined as :

$$\hat{\mathcal{R}} = \frac{1}{K} \sum_{k=1}^{K} \mathcal{Z}_k \circ \mathcal{Z}_k^* \tag{6}$$

2.6. Cross HOSVDs

Definition Let $\mathcal{H} \in \mathbb{C}^{I_1, I_2, \ldots, I_P}$, a P-order tensor. We denote $\mathbb{A} = \{I_1, \ldots, I_P\}$ the set of the dimensions and $\mathbb{A}_l$ a subset of $\mathbb{A}$ (for example $\{I_1, I_2\}$). We called Cross-HOSVD, every decompositions which satisfies:

$$\begin{aligned} \mathcal{H} = \mathcal{K}_{\mathbb{A}_1/\ldots/\mathbb{A}_L} \times_{\mathbb{A}_1} \mathbf{U}^{(\mathbb{A}_1)} \ldots \times_{\mathbb{A}_L} \mathbf{U}^{(\mathbb{A}_L)}, L \leq P, \\ \mathbb{A}_1 \cup \ldots \cup \mathbb{A}_L = \mathbb{A}, \\ \mathbb{A}_1 \cap \ldots \cap \mathbb{A}_L = \emptyset, \end{aligned} \tag{7}$$

where $\mathbf{U}^{(\mathbb{A}_l)}$ is the left singular matrix given by the SVD of $[\mathcal{H}]_{\mathbb{A}_l}$. The properties of those decompositions will be discuss in a future paper.

This decomposition allows to combine dimensions in order to enhance the subspace estimation (according to the application). For clarity we consider the case of $\mathcal{R}$, a 6-order covariance tensor which will be used for the derivation of our STAP filter. This tensor may be decomposed as follows:

$$\begin{aligned} \mathcal{R} &= \mathcal{G}_{1/2/3} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \times_3 \mathbf{U}^{(3)} \\ &\quad \times_4 \mathbf{U}^{(1)H} \times_5 \mathbf{U}^{(2)H} \times_6 \mathbf{U}^{(3)H} \\ &= \mathcal{G}_{1,2/3} \times_{1,2} \mathbf{U}^{(1,2)} \times_3 \mathbf{U}^{(3)} \times_{4,5} \mathbf{U}^{(1,2)H} \times_6 \mathbf{U}^{(3)H} \\ &= \mathcal{G}_{1/2,3} \times_1 \mathbf{U}^{(1)} \times_{2,3} \mathbf{U}^{(2,3)} \times_4 \mathbf{U}^{(1)H} \times_{5,6} \mathbf{U}^{(2,3)H} \\ &= \mathcal{G}_{1,3/2} \times_{1,3} \mathbf{U}^{(1,3)} \times_2 \mathbf{U}^{(2)} \times_{4,6} \mathbf{U}^{(1,3)H} \times_5 \mathbf{U}^{(2)H} \\ &= \mathcal{G}_{1,2,3} \times_{1,2,3} \mathbf{U}^{(1,2,3)} \times_{4,5,6} \mathbf{U}^{(1,2,3)H} \end{aligned} \tag{8}$$

The covariance tensor has a symetric structure which explains the symmetry of the decompositions. We see the classic HOSVD is included in cross-HOSVDs. We do not mention all the decompositions which satisfy the condition, but we focus on those which are useful to build a projector onto the clutter.

Low-rank approximation Let us introduce $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_0$ where $\mathcal{H}_c$ is a low rank tensor. We denote $r_{\mathbb{A}_l} = rank([\mathcal{H}_c]_{\mathbb{A}_l})$. By analogy with the HOSVD case, we assume an approximation of $\mathcal{H}_0$ is given by:

$$\mathcal{H}_0 \approx \mathcal{H} \times_{\mathbb{A}_1} \mathbf{U}_0^{(\mathbb{A}_1)} \mathbf{U}_0^{(\mathbb{A}_1)H} \ldots \times_{\mathbb{A}_L} \mathbf{U}_0^{(\mathbb{A}_L)} \mathbf{U}_0^{(\mathbb{A}_L)H} \tag{9}$$

with $\mathbf{U}_0^{(\mathbb{A}_l)} = [\mathbf{u}_{r_{\mathbb{A}_l}+1}^{(\mathbb{A}_l)} \ldots \mathbf{u}_{size(\mathbf{U}^{(\mathbb{A}_l)})}^{(\mathbb{A}_l)}]$. We assume, in this paper, the truncation is a correct approximation. The low-rank approximation will also be studied in a future paper.

3. STAP

This section is devoted to the STAP application. First we sum up the low rank vector STAP model and main results. Then, we propose a tensor approach to process multidimensional and especially polarimetric STAP data and we will show the analogy with the vectorial approach.

3.1. Classic STAP filter

We assume that the target is present in one cell, p, while the other cells contain only noise. Typically, the radar receiver consists in an array of N antenna elements processing M pulses in a coherent processing interval. We have K observations, corresponding to the K cells. The target is completely characterized by its speed v, and its angular position θ. Usually the data,$\mathbf{x}$, $\mathbf{x}_k$, are processed as NM vectors. The problem of detecting a complex known signal $\mathbf{s}$ corrupted by an additive disturbance $\mathbf{d}$ in an observation $\mathbf{x}$ can be formulated as follows:

$$\mathbf{x} = \alpha\mathbf{s} + \mathbf{d} \quad , \tag{10}$$

$$\mathbf{x}_k = \mathbf{d}_k \quad k \in [1, K], \tag{11}$$

where $\mathbf{s}$ is a complex *steering vector* depending on v and θ [1] and α the target attenuation term. We assume that the $\mathbf{d}_k$'s and $\mathbf{d}$ are independent and share the same statistical distribution. Moreover $\mathbf{d}_k$, $\mathbf{d}$ is decomposed as the sum of a low-rank Gaussian clutter (according to the Brennan's rule [2]), $\mathbf{c}_k$, and a white Gaussian noise $\mathbf{n}_k$:

$$\mathbf{d} = \mathbf{c} + \mathbf{n} \tag{12}$$

$$\mathbf{d}_k = \mathbf{c}_k + \mathbf{n}_k, \tag{13}$$

where $\mathbf{n}, \mathbf{n}_k \sim \mathcal{CN}(0, \sigma^2\mathbf{I}_{MN})$ and where $\mathbf{c}, \mathbf{c}_k \sim \mathcal{CN}(0, \mathbf{R}_c)$. Consequently, $\mathbf{d}, \mathbf{d}_k \sim \mathcal{CN}(0, \mathbf{R})$, where $\mathbf{R} = \mathbf{R}_c + \sigma^2\mathbf{I}_{MN}$. Usually $\mathbf{R}$ is unknown and has to be estimated. We denote $\hat{\mathbf{R}}$, its estimation. Using the equations (16), (17) and (13), the low-rank vector STAP filter is derived in 3 steps. First, $\hat{\mathbf{U}}_0$, the orthogonal clutter subspace is estimated: $\hat{\mathbf{U}}_0 = [\mathbf{u}_{r+1} \ldots \mathbf{u}_{NM}]$, where $\mathbf{u}_i$ are the $(NM - r)$-last eigenvectors of $\hat{\mathbf{R}}$ (r is the rank of $\mathbf{R}$, obtained with the Brennan's rule). Then the clutter is removed:

$$\begin{aligned} \hat{\mathbf{U}}_0^H\mathbf{x} &= \alpha\hat{\mathbf{U}}_0^H\mathbf{s} + \hat{\mathbf{U}}_0^H\mathbf{c} + \hat{\mathbf{U}}_0^H\mathbf{n} \\ &= \alpha\hat{\mathbf{U}}_0^H\mathbf{s} + \hat{\mathbf{U}}_0^H\mathbf{n} \end{aligned} \tag{14}$$

Finally the low-rank vector STAP filter is classically given by [3, 4] :

$$\hat{\mathbf{w}}_{lr} = \hat{\mathbf{U}}_0\hat{\mathbf{U}}_0^H\mathbf{s}, \tag{15}$$

3.2. Tensor Model

In this section, we assume the data are processed as 3-order tensor, denoted $\mathfrak{X}_k \in \mathbb{C}^{N\times M\times P}$ (where $P = 3$ in the case of polarimetric STAP) but all results can easily be extended to others N-dimensional STAP. The problem consists in detecting a complex known signal $\mathfrak{S}$ corrupted by an additive disturbance $\mathfrak{D}$ in an observation $\mathfrak{X}$:

$$\mathfrak{X} = \alpha\mathfrak{S} + \mathfrak{D} \tag{16}$$

$$\mathfrak{X}_k = \mathfrak{D}_k \quad k \in [1, K], \tag{17}$$

where $\mathfrak{S}$ is a complex *steering tensor* depending on v and θ [1]. We assume that the $\mathfrak{D}_k$'s are independent and share the same statistical distribution. Moreover, we assume that $\mathfrak{D}_k$ is decomposed as the sum of a low-rank Gaussian clutter (this assumption will be verified in the next section), $\mathfrak{C}_k$, and a white Gaussian noise $\mathfrak{N}_k$: $\mathfrak{D}_k = \mathfrak{C}_k + \mathfrak{N}_k$, where $\mathfrak{N}_k \sim \mathcal{CN}(0, \sigma^2\mathfrak{I})$ and where $\mathfrak{C}_k \sim \mathcal{CN}(0, \mathfrak{R})$. Consequently, $\mathfrak{D}_k \sim \mathcal{CN}(0, \mathfrak{R})$, where $\mathfrak{R} = \mathfrak{R}_c + \sigma^2\mathfrak{I}_{MN}$. $\mathfrak{R}$ is unknown but can be estimated by the SCT, $\hat{\mathfrak{R}}$.

3.3. Low-rank tensor STAP filter

We propose to derive a low-rank tensor STAP filter in 3 steps. First we decompose $\hat{\mathfrak{R}}$ with the Cross-HOSVD to estimate the clutter subspace. Then, using the low-rank approximation, a projector, which remove the clutter, is calculated. At last the corresponding filter is proposed.

- By analogy to the vector case, where the filter is obtained from the SVD of the data, we first propose to use the cross HOSVD on the estimate covariance matrix $\hat{\mathfrak{R}}$:

$$\begin{aligned} \hat{\mathfrak{R}} = \; & \mathfrak{K}_{\mathbb{A}_1/\ldots/\mathbb{A}_{2L}} \times_{\mathbb{A}_1} \hat{\mathbf{U}}^{(\mathbb{A}_1)} \ldots \times_{\mathbb{A}_L} \hat{\mathbf{U}}^{(\mathbb{A}_L)} \\ & \times_{\mathbb{A}_{L+1}} \hat{\mathbf{U}}^{(\mathbb{A}_1)H} \ldots \times_{\mathbb{A}_{2L})} \hat{\mathbf{U}}^{(\mathbb{A}_L)H} \end{aligned} \tag{18}$$

- In order to apply the low-rank approximation, $r_{\mathbb{A}_1}, \ldots, r_{\mathbb{A}_L}$ ($r_{\mathbb{A}_l} = rank([\mathfrak{R}]_{\mathbb{A}_l})$) have to be estimated.
- The clutter $\mathfrak{C}$ can be removed by the following tensor projector:

$$\begin{aligned} \mathfrak{X}'_{(\mathbb{A}_1/\ldots/\mathbb{A}_L)} &= \mathfrak{X} \times_{\mathbb{A}_1} \hat{\mathbf{U}}_0^{(\mathbb{A}_1)}\hat{\mathbf{U}}_0^{(\mathbb{A}_1)H} \ldots \times_{\mathbb{A}_L} \hat{\mathbf{U}}_0^{(\mathbb{A}_L)}\hat{\mathbf{U}}_0^{(\mathbb{A}_L)H} \\ &= \mathfrak{S}'_{(\mathbb{A}_1/\ldots/\mathbb{A}_L)} + \mathfrak{N}'_{(\mathbb{A}_1/\ldots/\mathbb{A}_L)}, \end{aligned} \tag{19}$$

- Using the projector given by (19), the problem is then to detect a complex tensor signal $\mathfrak{S}'$ disturbed by a white gaussian noise $\mathfrak{N}'$. The optimal filter in this case is known [11]:

$$\begin{aligned} \hat{\mathfrak{W}}_{lr(\mathbb{A}_1/\ldots/\mathbb{A}_L)} &= \mathfrak{S} \times_{\mathbb{A}_1} \hat{\mathbf{U}}_0^{(\mathbb{A}_1)}\hat{\mathbf{U}}_0^{(\mathbb{A}_1)H} \ldots \times_{\mathbb{A}_L} \hat{\mathbf{U}}_0^{(\mathbb{A}_L)}\hat{\mathbf{U}}_0^{(\mathbb{A}_L)H} \\ &= \mathfrak{S}'_{(\mathbb{A}_1/\ldots/\mathbb{A}_L)} \end{aligned} \tag{20}$$

  We see the analogy with the vector case of Eq. (15). The filter output is given by:

$$y = | < \hat{\mathfrak{W}}_{lr(\mathbb{A}_1/\ldots/\mathbb{A}_L)}, \mathfrak{X}'_{(\mathbb{A}_1/\ldots/\mathbb{A}_L)} > | \tag{21}$$

4. APPLICATION TO POLARIMETRIC STAP

4.1. Model

This section contains the model of the polarimetric STAP. We first propose a vectorial model and then we show its extension to the tensor case. This model is adapted from [5], we assume the antenna elements can emit and receive in 3 polarimatric channels (HH, VV, HV). The polarimetric STAP data are built, concatenating the 3 polarimetric channels. The polarimetric steering vector, $\mathbf{s}(\theta, v)$ is built as follows:

$$\mathbf{s}(\theta, v) = \begin{pmatrix} \mathbf{s}_{HH}(\theta, v) \\ \alpha_{VV}\mathbf{s}_{HH}(\theta, v) \\ \alpha_{VH}\mathbf{s}_{HH}(\theta, v) \end{pmatrix}, \tag{22}$$

where $\mathbf{s}_{HH}(\theta, v)$ is the 2D classic steering vector [1] and α_{VV}, α_{VH} two complex coefficients. We assume the polarimetric properties of the target are known.

The covariance matrix of the clutter, $\mathbf{R}_{pc} \in \mathbb{C}^{3MN \times 3MN}$ is given by:

$$\mathbf{R_{pc}} = \begin{pmatrix} \mathbf{R_c} & \rho\sqrt{\gamma_{VV}}\,\mathbf{R_c} & \mathbf{0} \\ \rho^*\sqrt{\gamma_{VV}}\,\mathbf{R_c} & \gamma_{VV}\,\mathbf{R_c} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma_{VH}\,\mathbf{R_c} \end{pmatrix}, \quad (23)$$

where $\mathbf{R_c} \in \mathbb{C}^{MN \times MN}$ is the covariance matrix of the HH channel *clutter*, build as the 2D classic clutter [1]. γ_{VV}, γ_{VV} and γ_{VV} are three coefficients relative to the nature of the ground and ρ is the correlation coefficient between HH and VV. The low-rank structure of the clutter is kept.

Using the vectorial model and the inverse unfolding given in 2.1, $\mathcal{X}, \mathcal{S}, \mathcal{N} \in \mathbb{C}^{M \times N \times 3}$ and $\mathcal{R} \in \mathbb{C}^{M \times N \times 3 \times M \times N \times 3}$ are given by:

$$\mathcal{S}(\theta, v) = vec^{-1}(\mathbf{s}(\theta, v)), \quad (24)$$

$$\mathcal{R} = SqMat^{-1}(\mathbf{R}_{pc}). \quad (25)$$

We are in the case of the previous section, we can apply the results derived, especially the low-rank tensor filter.

4.2. Polarimetric Low rank STAP Filter

We focus on three low-rank filter: $\hat{\mathcal{W}}_{lr(1,2,3)}$, $\hat{\mathcal{W}}_{lr(1/2/3)}$ and $\hat{\mathcal{W}}_{lr(1,2/3)}$. $\hat{\mathcal{W}}_{lr(1,2,3)}$ is the same filter as the polarimetric vector low rank filter with one deficient rank, $r_{1,2,3}$. $\hat{\mathcal{W}}_{lr(1/2/3)}$ (which are based on the classic HOSVD) has 3 ranks to study, r_1, r_2 and r_3. The rank r_1, r_2 are deficient if $\beta \neq 1$ ($\beta = \frac{2V}{F_r d}$ where V is the platform speed, F_r the PRF and d the distance between 2 sensors). r_3 is deficient and depends of the correlation coefficient ρ between the polarimetric channels. $\hat{\mathcal{W}}_{lr(12,3)}$ has two ranks to study: $r_{1,2}$ and r_3. $r_{1,2}$ is the same as the 2D low rank vector case and can be calculated by the Brennan's rule [2].

4.3. SNR loss

In order to evaluate the performance of our filter, we study the SNR Loss defined as follows:

$$\rho_{loss} = \frac{SNR_{out}}{SNR_{max}}, \quad (26)$$

where SNR_{out} is the SNR at the output of the LR tensor STAP filter and SNR_{max} the SNR at the output of the optimal filter ($\mathcal{W}_{opt}$). The Signal to Noise Ratio at the filter output SNR_{out} is:

$$SNR_{out} = \frac{|<\hat{\mathcal{W}}_{lr}, \mathcal{S}'>|^2}{\mathbf{E}[|<\hat{\mathcal{W}}_{lr}, \mathcal{N}>|^2]} = \frac{vec(\mathcal{S}')^H vec(\mathcal{S}')}{vec(\mathcal{S}')^H SqMat(\mathcal{R}) vec(\mathcal{S}')} \quad (27)$$

The SNR_{out} is maximum when $\mathcal{W} = \mathcal{W}_{opt} = SqMat(\mathcal{R})^{-1} vec(\mathcal{S})$.

After some steps, the SNR loss is finally equal to:

$$\rho_{loss} = \frac{|(vec(\mathcal{S}')^H vec(\mathcal{S}'))|^2}{vec(\mathcal{S}')^H SqMat(\mathcal{R}) vec(\mathcal{S}') vec(\mathcal{S})^H SqMat(\mathcal{R})^{-1} vec(\mathcal{S})} \quad (28)$$

4.4. Simulations

Parameters The simulations are performed with $\theta = 0°$ ans $v = 10\ m.s^{-1}$ for the target, a case where the classic 2D STAP is known to be inefficient. The RADAR receiver is characterized by : $M = N = 8$, $f_0 = 450\ MHz$, the platform speed, $V = 100\ m.s^{-1}$. We perform simulations for two cases: $\rho = 1$ and $\rho = 0.5$. There are no deficient rank in the spatial and temporal dimensions, r_1 and r_2 ($\beta = 1$). The rank of the space-time dimension $r_{1,2}$ is equal to the result of the Brennan's rule [2], i.e 15 in our simulations. r_3 is equal to 2 for $\rho = 1$ and 3 for $\rho = 0.5$. $r_{1,2,3}$ is equal to $2r_{1,2}$ for $\rho = 1$ and $3r_{1,2}$ for $\rho = 0.5$. K, the number of secondary data, is equal to $2r_{1,2,3}$ ($K = 60$ for $\rho = 1$ and $K = 90$ for $\rho = 0.5$). SNR losses are performed from (28) with $N_{rea} = 1000$ samples for each value of K.

Results Figure 1 (respectively figure 2) shows results of several STAP filters for $\rho = 1$ (respectively for $\rho = 0.5$). As $v = 10\ m.s^{-1}$, the target will be very close of the clutter. We notice, on the top left of figures 1 and 2, that the target can not be detected by the 2D classical STAP filter given by (15). For $\rho = 1$ the filter $\hat{\mathcal{W}}_{lr(1,2,3)}$ which are similar to the polarimetric vector STAP shows the polarimetric information allows to enhance the target detection but many false alarms remains. The filter $\hat{\mathcal{W}}_{lr(1/2/3)}$ strongly reduces the false alarms by exploiting the polarimetric information and performs the best result. The filter $\hat{\mathcal{W}}_{lr(1,2/3)}$ is not efficient in this case. For $\rho = 0.5$, the filter $\hat{\mathcal{W}}_{lr(1,2,3)}$ also shows the interest of polarization. The filter $\hat{\mathcal{W}}_{lr(1/2/3)}$ is not shown because the polarimetric dimension can not be reduced ($r_3 = 3$). In this case, $\hat{\mathcal{W}}_{lr(1,2/3)}$ performs the most efficient result. We notice that the most efficient filter depends of ρ. Moreover, these results depends on the number of secondary data, K. In order to study this dependence, we perform the SNR loss in function of K. Figure 3 shows the SNR Losses for each filter. For $\hat{\mathcal{W}}_{lr(1,2,3)}$ we have the same results as the classic 2D low-rank STAP, the SNR loss is equal to $-3dB$ when $K = 2r_{1,2,3}$. The other filters need less data to be efficient. We notice that the performances of the filters are strongly linked to the number of secondary data: for $K < 2r_{1,2,3}$, $\hat{\mathcal{W}}_{lr(1,2,3)}$ is less efficient than the other filters but becomes equivalent or better when $K > 2r_{1,2,3}$. The SNR losses confirm the result concerning the most efficient filter for each value of ρ. Finally, we notice the interest of our approach, which can be adapted to the data structures and/or parameters.

5. CONCLUSION

In this paper, we introduced a new multilinear tool: the cross-HOSVDs. This new decomposition is characterized by the use of combined and/or single information. This tool leads to a set of low-rank tensor STAP filters. The first preliminary results on the polarimetric STAP illustrate the interest of these multilinear filters. These results are confirmed by the study of the SNR losses. We concluded that the choice of the decomposition is important for the results: the most efficient solu-

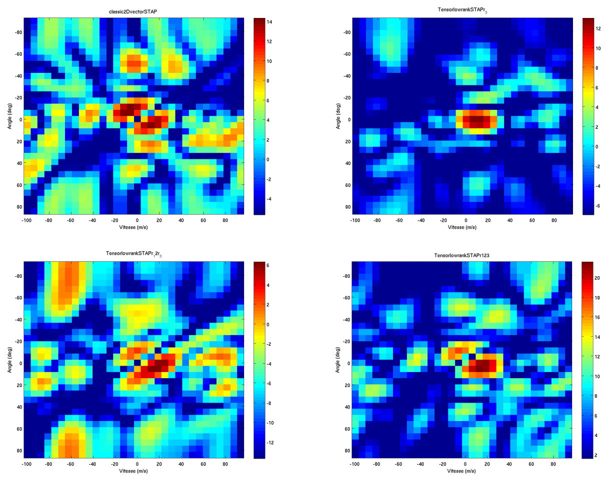

Fig. 1. Filter results for $\rho = 1$, Target located at position ($\theta = 0°$,$v = 10m.s^{-1}$). Top left: Classic 2D STAP. Top right: tensor low-rank STAP $\hat{\mathbf{w}}_{lr(1/2/3)}$ (given by the HOSVD). Bottom left: $\hat{\mathbf{w}}_{lr(1,2/3)}$, which combine polarimetric and spatio-temporal information. Bottom right: $\hat{\mathbf{w}}_{lr(1,2,3)}$, which are equivalent to the polarimetric vector STAP.

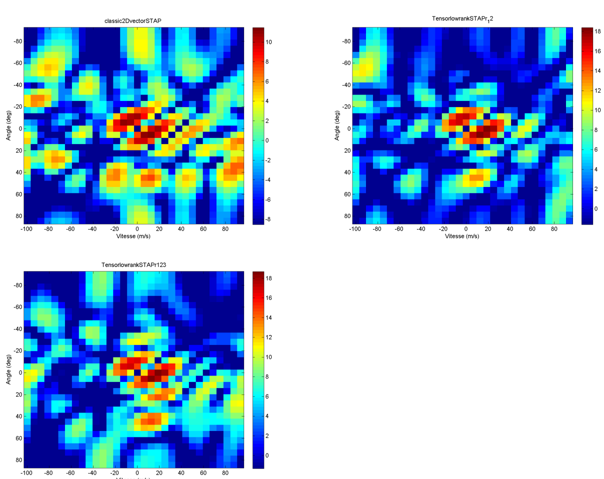

Fig. 2. Filter results for $\rho = 0.5$, Target located at position ($\theta = 0°$,$v = 10m.s^{-1}$). Top left: Classic 2D STAP. Top right: tensor low-rank STAP $\hat{\mathbf{w}}_{lr(1,2/3)}$, which combine polarimetric and spatio-temporal information. Bottom left: $\hat{\mathbf{w}}_{lr(1,2,3)}$, which are equivalent to the polarimetric vector STAP.

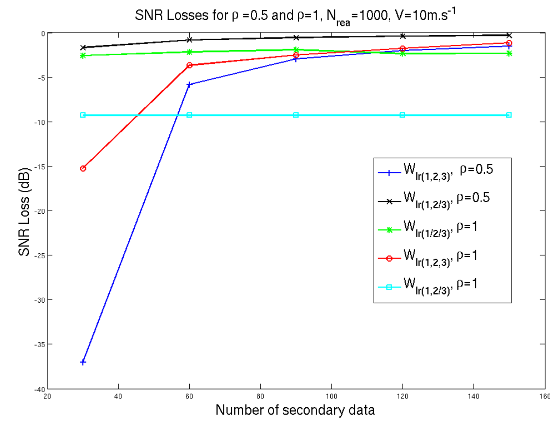

Fig. 3. SNR Losses in function of number of secondary data.

tion depends of the parameters (the polarimtric correlation in our case). In order to confirm the interest of our approach, we have to investigate this new multilinear decomposition: especially the properties of the core tensor, the validity of the low rank approximation (comparing to an alternating least squares algorithm, ...). We have also to derive a tool to choose the best decomposition, in function of a criterion.

6. REFERENCES

[1] J. Ward, "Space-time adaptive processing for airborne radar," Tech. Rep., Lincoln Lab., MIT, Lexington, Mass., USA, December 1994.

[2] L. E. Brennan and F.M. Staudaher, "Subclutter visibility demonstration," Tech. Rep., RL-TR-92-21, Adaptive Sensors Incorporated, March 1992.

[3] I. Kirsteins and D. Tufts, "Adaptive detection using a low rank approximation to a data matrix," *IEEE Trans. on Aero. and Elec. Syst.*, vol. 30, pp. 55 – 67, 1994.

[4] A. Haimovich, "Asymptotic distribution of the conditional signal-to-noise ratio in an eigenanalysis-based adaptive array," *IEEE Trans. on Aero. and Elec. Syst.*, vol. 33, pp. 988 – 997, 1997.

[5] G. Showman, W. Melvin, and M. Belenkii, "Performance evaluation of two polarimetric stap architectures," in *Proc. of the IEEE Int. Radar Conf.*, 2003, pp. 59–65.

[6] T. Kolda and B. Bader, "Tensor decompositions and applications," *SIAM Review*, vol. 51, pp. 455 – 500, 2009.

[7] L. De Lathauwer, B. De Moor, and J. Vandewalle, "On the best rank-1 and rank-$(r_1, r_2, ..., r_n)$ approximation and applications of higher-order tensors," *SIAM J. Matrix Anal. Apl.*, vol. 21, no. 4, pp. 1324–1342, 2000.

[8] M. Haardt, F. Roemer, and G. Del Galdo, "Higher-order svd-based subspace estimation to improve the parameter estimation accuracy in multidimensionnal harmonic retrieval problems," *IEEE Trans. on Proc. Sig. Proc.*, vol. 56, no. 7, pp. 3198–3213, July 2008.

[9] L. De Lathauwer, B. De Moor, and J. Vandewalle, "A multilinear singular value decomposition," *SIAM J. Matrix Anal. Apl.*, vol. 24, no. 4, pp. 1253–1278, 2000.

[10] M. Boizard, G. Ginolhac, F. Pascal, and P. Forster, "Low rank tensor stap filter based on multilinear svd," in *Proceedings of SAM*, 2012.

[11] M. Boizard, G. Ginolhac, F. Pascal, and P. Forster, "Developpement de filtres stap tensoriels appliques des donnees multidimensionnelles," in *Proc. of the GRETSI Conf.*, 2011.