COMPUTATIONALLY EFFICIENT IAA-BASED ESTIMATION OF THE FUNDAMENTAL FREQUENCY

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ABSTRACT

Optimal linearly constrained minimum variance (LCMV) filtering methods have recently been applied to fundamental frequency estimation. Like many other fundamental frequency estimators, these methods are constructed using an estimate of the inverse data covariance matrix. The required matrix inverse is typically formed using the sample covariance matrix via data partitioning, although this is well-known to adversely affect the spectral resolution. In this paper, we propose a fast implementation of a novel optimal filtering method that utilizes the LCMV principle in conjunction with the iterative adaptive approach (IAA). The IAA formulation enables an accurate covariance matrix estimate from a single snapshot, i.e., without data partitioning, but the improvement comes at a notable computational cost. Exploiting the estimator's inherently low displacement rank of the necessary products of Toeplitz-like matrices, we form a computationally efficient implementation, reducing the required computational complexity with several orders of magnitude. The experimental results show that the performance of the proposed method is comparable or better than that of other competing methods in terms of spectral resolution.

Index Terms— Fundamental frequency estimation, optimal filtering, data adaptive estimators, efficient algorithms

1. INTRODUCTION

There exists a multitude of signal processing applications in which the fundamental frequency is an essential parameter including, for instance, parametric coding of audio and speech, automatic music transcription, musical genre classification, tuning of musical instruments, and separation and enhancement of audio and speech sources. Due to the importance of knowing the fundamental frequency, numerous of approaches and methods have been proposed for estimating this parameter (see, e.g., [1–5] and the references therein). Typically, most such estimators utilize an estimate of the sample covariance matrix or its inverse, both commonly being formed by partitioning the available measurement into sub-vectors and forming the outer-product covariance matrix estimate. As is well-known, this approach adversely affects the achievable spectral resolution, and there is therefore an interest in developing methods that achieve a higher spectral resolution. In [6], Jensen, Christensen and Jensen examined alternative of forming the covariance matrix estimate using the iterative adaptive approach (IAA) presented in [7]. The IAA estimate is a non-parametric, data-dependent, spectral estimate that does not require a data partitioning of the measurements. The estimate is instead formed iteratively, alternatingly by estimating the spectral amplitudes of the measurement as well as the covariance matrix formed from this amplitude spectrum. Generally, the IAA-based estimation techniques are able to provide accurate estimates even when only a few samples are available. However, the improved performance comes at a cost of a considerable computational complexity. In this work, we improve on the earlier IAA-based pitch estimation algorithm presented in [6]. As the resulting estimate will suffer from a high computational complexity, we here introduce a computationally efficient implementation of the resulting algorithm, extending recent work on efficient IAA implementations [8, 9].

This work was supported in part by the Swedish Research Council and Carl Trygger's foundation.

2. FUNDAMENTAL FREQUENCY ESTIMATION

As audio and voiced speech signals are quasi-periodic, one may well model such signals as (see, e.g., [5])

$$x(n) = \sum_{l=1}^{L} \alpha_l e^{jl\omega_0 n} + w(n), \tag{1}$$

for $n = 0, \dots, N-1$, where L is the number of harmonics, $\alpha_l = A_l e^{j\phi_l}$, with $A_l > 0$ and ϕ_l denoting the real-valued

 - ISSN 2076-1465

amplitude and the phase of the lth harmonic, ω_0 is the fundamental frequency, and $w(n)$ is a complex-valued noise process. For simplicity, we assume that the model order, L, is known, noting that it may be obtained using a model order estimator [10, 11], or by forming the model order and fundamental frequency estimation jointly, reminiscent to the ideas presented in [5]. The problem of interest is thus estimating ω_0 from (1), without making any strong assumptions on the statistics of the noise process. This may, for instance, be done using the optimal filtering method introduced in [12], being based on an optimal harmonic LCMV (hLCMV) filter. Consider M time-reversed samples from (1) in vector format

$$\mathbf{x}_M(n) = \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-M+1) \end{bmatrix}^T, \quad (2)$$

for $n = M-1, \ldots, N-1$, with $(\cdot)^T$ denoting the transpose, and introduce the FIR filter $\mathbf{h} = \begin{bmatrix} h(0) & \cdots & h(M-1) \end{bmatrix}^H$, from which the output is given by $y(n) = \mathbf{h}^H \mathbf{x}_M(n)$, where $(\cdot)^H$ denotes the conjugate transpose. The output power of the filter is then

$$\mathrm{E}\{|y(n)|^2\} = \mathbf{h}^H \mathbf{R} \mathbf{h}, \quad (3)$$

where $\mathbf{R} = \mathrm{E}\{\mathbf{x}_M(n)\mathbf{x}_M^H(n)\}$, with $E\{\cdot\}$ denoting the statistical expectation. The optimal filter response is found using the LCMV principle, such that the filter is designed to have a unit gain at the harmonic frequencies while having maximum noise suppression, for $l = 1, \ldots, L$,

$$\min_{\mathbf{h}} \mathbf{h}^H \mathbf{R} \mathbf{h} \quad \text{subj. to } \mathbf{h}^H \mathbf{z}_M(l\omega_0) = 1, \quad (4)$$

where $\mathbf{z}_M(\omega) = \begin{bmatrix} 1 & e^{-j\omega_0} & \cdots & e^{-j(M-1)\omega_0} \end{bmatrix}^T$, implying that [5]

$$\hat{\mathbf{h}} = \mathbf{R}^{-1}\mathbf{Z}_M \left(\mathbf{Z}_M^H \mathbf{R}^{-1} \mathbf{Z}_M\right)^{-1} \mathbf{1}, \quad (5)$$

with $\mathbf{Z}_M = \begin{bmatrix} \mathbf{z}_M(\omega_0) & \cdots & \mathbf{z}_M(L\omega_0) \end{bmatrix}$. An estimate of the fundamental frequency may thus be found by inserting (5) into (3) and maximizing the output power, yielding

$$\hat{\omega}_0 = \arg\max_{\omega_0 \in \Omega_0} \mathbf{1}^T \left(\mathbf{Z}_M^H \mathbf{R}^{-1} \mathbf{Z}_M\right)^{-1} \mathbf{1}, \quad (6)$$

where Ω_0 is a set of candidate fundamental frequencies. The covariance matrix $\mathbf{R}$ is generally unknown, and is commonly replaced by the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{N-M+1} \sum_{n=M-1}^{N-1} \mathbf{x}_M(n)\mathbf{x}_M^H(n), \quad (7)$$

where, to ensure that $\hat{\mathbf{R}}$ is invertible, the length of the subvectors, $\mathbf{x}_M(n)$, are restricted to $M < \frac{N}{2}+1$, thereby limiting the spectral resolution to be on the order of $1/M$ [11]. A direct implementation of the estimator requires roughly

$$\mathcal{C}^{\text{Cov}} \approx M^3 + M^2\bar{N} + \bar{F}(ML^2 + LM^2 + L^3) \quad (8)$$

operations, where $\bar{N} \triangleq N - M + 1$ and $\bar{F} \triangleq F/L$, with $F = |\Omega_0|$ being the size of the uniformly spaced grid of frequencies on the unit circle where the search for the optimum ω_0 is conducted. Typically, $F \gg N$ and due to the nature of the problem, the search is limited up to $\bar{F}$. We proceed to recall the IAA-based covariance matrix estimate, which is then used in conjunction with the above optimal filtering method. However, it should be stressed that this covariance matrix estimate could similarly be used in conjunction with other covariance based fundamental frequency estimators, thereby offering a similar improved spectral resolution. Following the usual IAA notation, let

$$\mathbf{x}_N = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^T. \quad (9)$$

Then, the IAA estimate is formed by iteratively estimating the complex amplitudes, $\alpha(\omega_k)$, and the corresponding covariance matrix, $\tilde{\mathbf{R}}$, until practical convergence, as (see [7, 13] for further details)

$$\hat{\alpha}_k = \frac{\mathbf{z}_N^T(\omega_k)\tilde{\mathbf{R}}^{-1}\mathbf{x}_N}{\mathbf{z}_N^T(\omega_k)\tilde{\mathbf{R}}^{-1}\mathbf{z}_N^*(\omega_k)} \quad (10)$$

$$\tilde{\mathbf{R}} = \sum_{k=0}^{K-1} |\hat{\alpha}_k|^2 \, \mathbf{z}_N^*(\omega_k)\mathbf{z}_N^T(\omega_k) \quad (11)$$

for $k = 0, 1, \ldots, K-1$, with $\tilde{\mathbf{R}}$ initialized to the identity matrix, $\mathbf{I}_K$, implying that the complex amplitudes are initialized using the FFT of the sample vector. Typically, 10-15 iterations are sufficient for convergence [7]. The expression in (10) can be seen as a filtering operation if one rewrites it as

$$\hat{\alpha}_k = \mathbf{h}_{\text{IAA}}^H \mathbf{x}_N, \quad (12)$$

where $\mathbf{h}_{\text{IAA}} = \tilde{\mathbf{R}}^{-1}\mathbf{z}_N^*(\omega_k)[\mathbf{z}_N^T(\omega_k)\tilde{\mathbf{R}}^{-1}\mathbf{z}_N^*(\omega_k)]^{-1}$. We form the IAA-based optimal LCMV filter (IAA-LCMV) as

$$\hat{\mathbf{h}}_{\text{IAA-LCMV}} = \tilde{\mathbf{R}}^{-1}\mathbf{Z}_N^* \left(\mathbf{Z}_N^T \tilde{\mathbf{R}}^{-1} \mathbf{Z}_N^*\right)^{-1} \mathbf{1}, \quad (13)$$

where $\mathbf{Z}_N$ are defined analogously to $\mathbf{Z}_M$. That is, one combines the filter design in (5) with the IAA covariance matrix estimate, as obtained after convergence has been achieved. Combining (12) and (13), an estimate of the expected estimated output power is obtained as

$$\mathrm{E}\{\hat{P}_{\text{IAA-LCMV}}\} = \mathbf{1}^T \left(\mathbf{Z}_N^T \tilde{\mathbf{R}}^{-1} \mathbf{Z}_N^*\right)^{-1} \mathbf{1} \quad (14)$$

$$= \mathbf{1}^T \boldsymbol{\mathcal{G}}^{-1} \mathbf{1}, \quad (15)$$

where $\boldsymbol{\mathcal{G}} \triangleq \mathbf{Z}_N^T \tilde{\mathbf{R}}^{-1} \mathbf{Z}_N^*$.

The fundamental frequency is then estimated by maximizing the output power, i.e.,

$$\hat{\omega}_0 = \arg\max_{\omega_0 \in \Omega_0} \mathrm{E}\{\hat{P}_{\text{IAA-LCMV}}\}. \quad (16)$$

A direct implementation of (16) requires

$$\mathcal{C}^{\text{IAA}} \approx m(N^3 + 3N^2K) + \bar{F}(N^2L + L^2N + L^3) \quad (17)$$

operations, where K denotes the size of the grid of frequencies utilized in the IAA implementation, and m is the number of IAA iterations, with, usually, $K \leq F$.

3. EFFICIENT IMPLEMENTATION

Alternatively, an efficient implementation of (6) and (16) may be formed by means of a implementation using Gohberg-Semencul (GS) factorizations of the involved matrices. Consider a Hermitian matrix $\mathbf{P} \in \mathcal{C}^{N\times N}$, and define the lower shifting matrix as

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}^T & 0 \\ \mathbf{I}_{N-1} & \mathbf{0} \end{bmatrix}. \quad (18)$$

Clearly, $(\mathbf{D})^N = \mathbf{0}$. Then, the displacement of $\mathbf{P}$ wrt $\mathbf{D}$ and $\mathbf{D}^T$ is defined as $\nabla_{\mathbf{D},\mathbf{D}^T}\mathbf{P} \triangleq \mathbf{P} - \mathbf{D}\mathbf{P}\mathbf{D}^T$.Suppose that there exist integers ρ and $\sigma_i \in \{-1, 1\}$, $i = 1, \ldots, \rho$, such that (see also [14]) $\nabla_{\mathbf{D},\mathbf{D}^T}\mathbf{P} = \sum_{i=1}^{\rho}\sigma_i\mathbf{t}_i\mathbf{t}_i^H$with $\mathbf{t}_i$ denoting the so-called generator vectors. Then, the GS factorization of $\mathbf{P}$ may be expressed as $\mathbf{P} = \sum_{i=1}^{\rho}\sigma_i\mathcal{L}(\mathbf{D},\mathbf{t}_i)\mathcal{L}^H(\mathbf{D},\mathbf{t}_i)$, where $\mathcal{L}(\mathbf{D},\mathbf{b})$ denotes a Krylov matrix of the form

$$\mathcal{L}(\mathbf{D},\mathbf{b}) = \begin{bmatrix}\mathbf{b} & \mathbf{D}\mathbf{b} & \mathbf{D}^2\mathbf{b} & \cdots & \mathbf{D}^{N-1}\mathbf{b}\end{bmatrix}.$$

While this decomposition can be used to perform computationally demanding tasks such as matrix-vector multiplication in an efficient way, it does not provide an efficient way of computing the matrix itself when only its displacement representation is available. However, as

$$\mathbf{P} - \mathbf{D}\mathbf{P}\mathbf{D}^T = \sum_{i=1}^{\rho}\sigma_i\mathbf{t}_i\mathbf{t}_i^H, \quad (19)$$

multiplying both sides of (19) by $\mathbf{e}_{j+1}$ and noting that $\mathbf{D}^T\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{D}^T\mathbf{e}_{j+1} = \mathbf{e}_j$, where $\mathbf{e}_j$ denotes a $N \times 1$ vector with a one at the jth entry and zeros elsewhere, implies that $\mathbf{P}$ may be recovered column-wise as

$$\mathbf{p}_j = \begin{cases} \displaystyle\sum_{i=1}^{\rho}\sigma_i\mathbf{t}_i\mathbf{t}_i^H\mathbf{e}_j, & j = 1 \\ \mathbf{D}\mathbf{p}_{j-1} + \displaystyle\sum_{i=1}^{\rho}\sigma_i\mathbf{t}_i\mathbf{t}_i^H\mathbf{e}_j & j > 1 \end{cases}, \quad (20)$$

for $j = 1, 2, \ldots, N$, with $\mathbf{p}_j$ denoting the jth column of $\mathbf{P}$. Estimating $\mathbf{P}$ in this way will require roughly ρN^2 operations. The coefficients of the trigonometric polynomial $\varphi(\omega) \triangleq \mathbf{z}^H(\omega)\mathbf{P}\mathbf{z}(\omega)$ can then be formed at a cost of approximately $\rho N\log_2 N$ using the method detailed in [15]. However, to form the coefficients of the trigonometric polynomials

$$\psi(\omega) \triangleq \mathbf{z}^H(l_1\omega)\mathbf{P}\mathbf{z}(l_2\omega) \quad (21)$$

for l_1 and $l_2 \in \mathcal{Z}$, one needs to consider the augmented frequency vectors

$$\mathbf{z}_k(\omega) = \mathbf{S}_{l_k}\begin{bmatrix}\mathbf{z}^T(l_k\omega) & \times\end{bmatrix}^T \quad (22)$$

for $k = 1$ or 2, where $\mathbf{S}_{l_k}$ is the selection matrix with zeros and ones indicating the presence or absence of a harmonic component, respectively, $\mathbf{S}_{l_k}\mathbf{S}_{l_k}^T = \mathbf{I}_{l_kN}$, and $\times$ denotes terms of no relevance. Using (22), (21) may be written as

$$\psi(\omega) = \mathbf{z}_1^H(\omega)\bar{\mathbf{P}}\mathbf{z}_2(\omega) = \sum_{\kappa=-l_1M+1}^{l_2M-1}\bar{c}_\kappa e^{-j\omega} \quad (23)$$

where $\bar{\mathbf{P}} \triangleq \mathbf{S}_{l_1}^T\mathbf{P}\mathbf{S}_{l_2}$ is an expanded rectangular matrix of size $(l_1N) \times (l_2N)$. Thus, the coefficients $\hat{c}_\kappa$ can be computed by summing all elements upon the kth diagonal of $\bar{\mathbf{P}}$. In practice, there is no need to form $\bar{\mathbf{P}}$, as one can easily show that it may be computed recursively as

$$\bar{\mathbf{C}}_{i+1} = \bar{\mathbf{C}}_i + \begin{bmatrix}\mathbf{0}_{(l_2-i)N} \\ (\mathbf{S}_{l_1}\mathbf{p}_{i+1}) \\ \mathbf{0}_{l_2i}\end{bmatrix} \quad (24)$$

for $i = 0, 1, \ldots, N-1$ where $\bar{\mathbf{C}} \triangleq \begin{bmatrix}\bar{c}_{-l_1N+1} & \cdots & \bar{c}_{l_2N-1}\end{bmatrix}^T$, at a cost of no more than N^2 operations. The case when $\tilde{\psi}(\omega) = \mathbf{z}_1^T(\omega)\bar{\mathbf{P}}\mathbf{z}_2^*(\omega)$ is treated similarly.

Using (20) and (24), we proceed to develop efficient implementations for (6) and (16), noting that the displacement representation of the inverse covariance estimates (7) and (11) can be computed using the fast schemes presented in [15] and [8], at cost of $4.5M^2 + 1.5N\log_2(N)$ and N^2 operations, respectively. The inverse covariance matrices are subsequently computed from their displacement representation using (20), at a cost of $2M^2$ and N^2, respectively. Using (24), the coefficients of the trigonometric polynomials involved in (6) can then be formed at a cost of $0.5L^2M^2$ operations, noting that due to the Hermitian symmetry, only half of the polynomials actually have to be estimated. Evaluating these on a uniformly spaced grid of frequencies using the Fast Fourier Transform (FFT) can be done at a cost of approximately $0.25L^2F\log_2 F$, or $0.25L^2F\log_2(F/L)$ if using FFT algorithms comprising output pruning. Finally, one may compute (6) at a cost of $L^3\bar{F} = L^2F$ operations, implying that the overall computational cost of the proposed approach is approximately

$$\begin{aligned}\mathcal{C}^{F,\text{Cov}} \approx\ & 6.5M^2 + 1.5N\log_2(N) + 0.5M^2L^2 + \\ & F\left[0.25\log_2(F/L) + 1\right]L^2.\end{aligned}$$

Using similar arguments, (16) is implemented at a cost of

$$\begin{aligned}\mathcal{C}^{F,\text{IAA}} \approx\ & m\left\{N^2 + 5N\log_2(2N) + 1.5K\log 2(K)\right\} + \\ & \left\{0.5N^2 + F\left[0.25\log_2(F/L) + 1\right]\right\}L^2\end{aligned}$$

operations, with m denoting the number of IAA iterations, and where (10) and (11) are implemented using the fast schemes of [8, 9]. The computational reduction achieved by the proposed implementation is illustrated in Fig. 1.

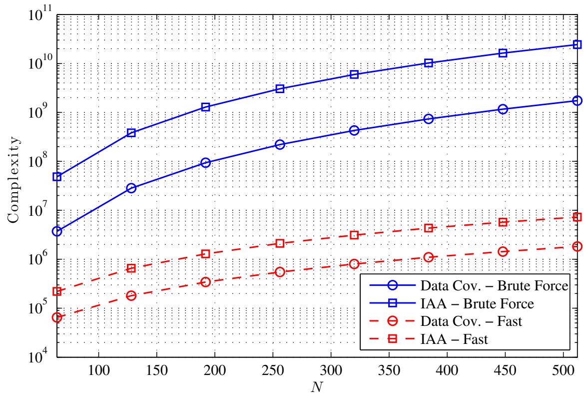

Fig. 1: Computational complexity of the hLCMV fundamental frequency estimation algorithms using the data covariance approach where $M = N/2 + 1$, the IAA approach, where $m = 10$ and $K = 4N$, and their fast implementation. In all cases, $F = 10N$ and $L = 5$.

4. NUMERICAL EXAMPLES

We proceed to evaluate accuracy of the efficient implementation of the proposed estimator. For this investigation, we used a harmonic signal with $L = 5$ in white Gaussian noise at an SNR of 20 dB, with the SNR being defined as $\text{SNR} = 10\log_{10}(\sigma_w^{-2}\sum_{l=1}^{L}|\alpha_l|^2)$. The number of grid points used for the IAA-based covariance matrix estimate was $K = 1000$, the number of candidate fundamental frequencies was $|\Omega_0| = 5000$, and the fundamental frequency was samples from $\mathcal{U}(0.2, 0.3)$. Using this setup, we measured the average absolute error over all frequency points and Monte-Carlo simulations for different Ns, and the results are provided in Table 1. We note that the error between the brute-force and fast implementations are close to numerical precision for this range of Ns. Then, we evaluate the performance of the proposed method, investigating the influence of K, N, the expected fundamental frequency, and the spacing between fundamental frequencies (the last in a two source scenario). Initially, we consider a noisy harmonic signal as in the previous investigation.Fig. 2a shows the measured mean squared error (MSE) of the proposed estimator defined in (16) (hIAA) as a function of K, with the fundamental frequency being samples from $\mathcal{U}(0.3, 0.4)$.

The results show the performance of the estimators for two different sample lengths, i.e., $N = 40$ and $N = 80$. As is clear from the figure, one needs more frequency points when N is increased to achieve the maximum possible performance. For $N = 40$, $K \approx 400$ seems to be sufficient, whereas at least $K \approx 1200$ frequency points are needed for $N = 80$. Fig. 2b shows the MSE as function of N, for $K = 1000$ frequency grid points, showing the performance of the IAA-based estimator as compared with the harmonic WLS (hWLS) method [1, 5], the harmonic LCMV (hLCMV) method [5], the harmonic approximate NLS (hANLS) method [5], and the harmonic MUSIC (hMUSIC) method [5]. One may note from the figure that the hIAA shows better performance as compared to the other methods for short data lengths, say $N < 30$. Examining the influence of the fundamental frequency, Fig. 2c shows the MSE as a function of the expected fundamental frequencies $\text{E}[\omega_0]$, where in each simulation, the fundamental frequency was sampled from $\text{E}[\omega_0] + \mathcal{U}(-0.001, 0.001)$, using $N = 35$ and $K = 1000$. As is clear from the results, the hIAA estimator clearly outperforms the other methods for low expected fundamental frequencies ($0.25 < \text{E}[\omega_0] < 0.3$).Finally, we compared the discussed methods in a scenario with two harmonic sources, examining two sources with $L = 3$ unit amplitude harmonics. The ratio between each of the sources and a white Gaussian noise source was 40 dB. In each simulation, the fundamental frequency ω_0^1 of first source was sampled from $\mathcal{U}(0.299, 0.301)$ and the fundamental frequency of the second source was $\omega_0^2 = \omega_0^1 + \Delta\omega_0$, where $\Delta\omega_0$ is the spacing, using $N = 60$, and $K = 1000$. As seen in Fig. 2d, the performance of the hIAA method is comparable to that of the hMUSIC method and, generally, better than the performance of the hLCMV and hANLS methods. All the above presented results have been obtained using 500 Monte-Carlo simulations.

N	40	60	80	100	120	140
MAE	407	3.42	0.27	0.17	0.20	0.21

Table 1: Mean absolute error (MAE) ($\cdot 10^{-12}$) measured for different Ns.

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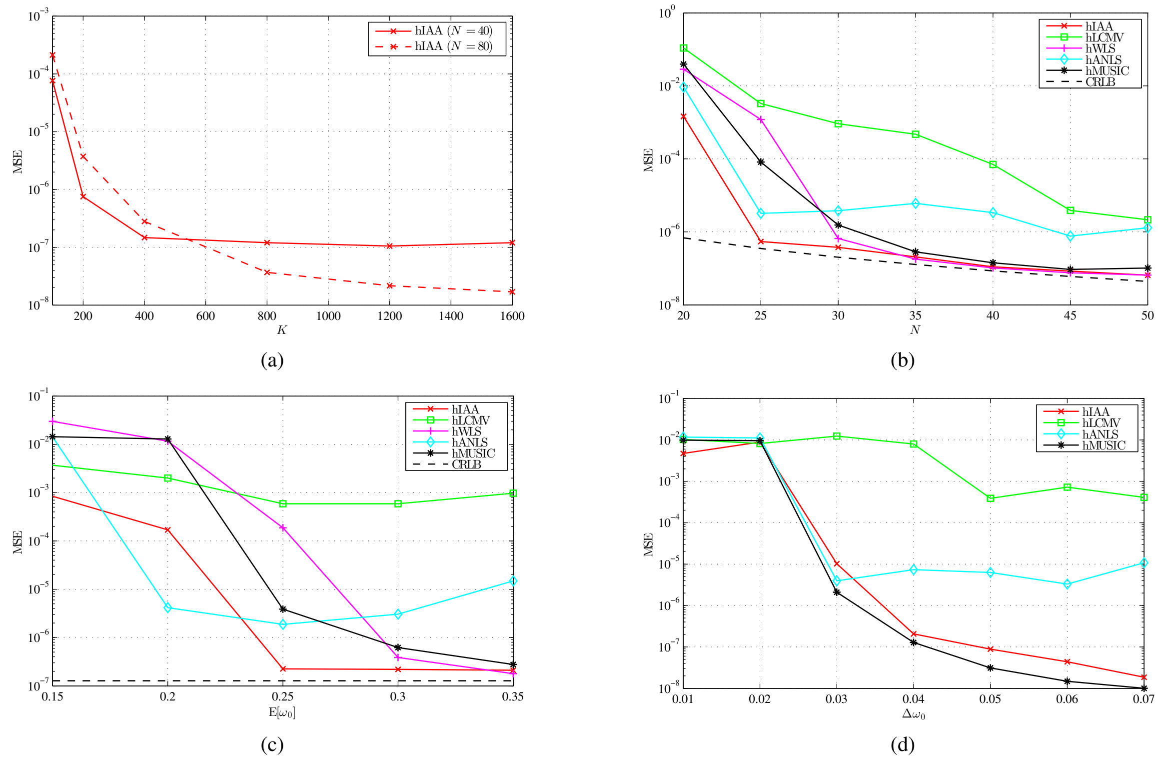

Fig. 2: Mean squared errors of different fundamental frequency estimators as a function of (a) the number of frequency grid points used for the IAA-based covariance matrix estimate, (b) the number of available samples, (c) the expected fundamental frequency, and (d) the spacing between fundamental frequencies in a two source scenario. Moreover, the Cramér-Rao lower bound (CRLB) is depicted in (b) and (c).

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