SHORT-TERM POLYNOMIAL PHASE ESTIMATION : APPLICATION TO RADAR SIGNAL IN AN ELECTRONIC WARFARE CONTEXT

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ABSTRACT

With the growing electromagnetic traffic, Electronic Warfare (EW) systems need to be accurate or use new characterizing parameters in order to discriminate at best radar pulses. An accurate study of the Instantaneous Frequency Law (IFL) and Rate (IFR) offers some new opportunities. Multi-linear methods such as the High-order Ambiguity Function (HAF) or any General Representation of Phase Derivatives (GRPD) are often used to estimate those parameters but not in an EW context where short pulses and small chirp rates are recurrent.

In this article, we first describe conditions for which those modulations can be estimated. Then, HAF and GRPD are studied in an EW context and their performances are compared to Cramer-Rao Bounds (CRB) to find the most appropriate one with industrial specifications.

Index Terms— Signal Characterization, Multi-linear Functions, CRB, Electronic Warfare

1. INTRODUCTION

Knowing accurately the parameters of a received signal can be critical in some domains such as Electronic Warfare (EW). Due to the increasing number of radar systems, it is necessary to find, develop and embed new methods to characterize at best radar signals and then the radar system. One of the possible response to this coming problem is to study the intrapulse phase which can often be considered as locally polynomial, especially for radar signals.

Polynomial-Phase Signals (PPS) have been studied for about 20 years to develop new methods able to calculate precisely the signal parameters: the polynomial coefficients or the Instantaneous Frequency Law (IFL) and Rate (IFR). The estimation of the polynomial coefficients can be done with the High-order Ambiguity Function (HAF) [1] or Polynomial-HAF (PHAF). IFL estimation can be performed either from Time-Frequency representation – such as Short-Time Fourier Transform (STFT) or Wigner-Ville distribution (WV) [2]. New methods intending to focus on non-linear IFL have been proposed (Polynomial WV (PWV) [3] or Complex Lag [4]). IFR estimation was proposed in [5] and the concept of any order phase representation in [6]. A cyclostationary approach has also been studied in [7] in a context of compound-Gaussian clutter.

All mentioned methods proved their worth for estimating several constant parameters with long signals but do not take an interest in the intrapulse phase, where intentional modulations (IMOP - Intentional Modulation On Pulse) or not (UMOP - Unintentional Modulation On Pulse) can be found. A study on the whole pulse is therefore inappropriate but studying less parameters and their evolution on a short term is possible.

In this paper, we describe a method to estimate the signal modulation. We also investigate the minimal window length needed to detect modulations. We compare two multi-linear functions: HAF and GRPD (which includes WV and second order phase derivative representation) and their performances in order to find the most suitable algorithm in an EW context. We only consider mono-component signals, *i.e.* we suppose a low probability of superposition of signals within a frequency bandwidth.

This paper is organized as follows. In section 2, constraints coming with multi-linear functions are dealt with. Section 3 describes the studied functions whose performances are presented in section 4. In section 5, we make recommendations in order to find the most suitable algorithms regarding the radar signal and with respect to industrial specifications. Finally, sections 6 and 7 give some concluding remarks and comments.

2. MODELLING AND CONSTRAINTS

2.1. Polynomial-Phase Signal Modelling

A radar signal received by an EW system can be generally modelled by [8]:

$$\begin{aligned} x(t) &= s(t) + w(t) \\ &= A\exp\left(j\phi(t)\right) + w(t) \\ &= A\exp\left(j\sum_{k=0}^{P} a_k t^k\right) + w(t), \qquad 0 \le t \le T, \end{aligned} \tag{1}$$

where:
- A is the amplitude of the signal,
- ϕ is the phase of the signal (P-degree polynomial),
- $\{a_k\}_{k=0\ldots P}$ are the polynomial coefficients,
- w is a white complex circular Gaussian noise with variance σ^2.

According to the considered assumptions (*i.e.* variant polynomial coefficients), equation (1) becomes:

$$x(t) = A\exp\left(j\sum_{k=0}^{P} a_k(t)t^k\right) + w(t). \tag{2}$$

Polynomial coefficients are directly linked to IFL and IFR by:

$$IFL(t) \triangleq \frac{1}{2\pi}\frac{d\phi(t)}{dt},$$
$$IFR(t) \triangleq \frac{1}{2\pi}\frac{d^2\phi(t)}{dt^2}.$$

The proposed algorithm is inspired by a frequency tracking algorithm proposed in [9] which aims at following the frequency evolution using STFT and Kalman filtering. Its principle is to take one part of the signal called sliding window around time step t_i, consider it as a chirp with constant polynomial coefficients and apply any multi-linear function on it to estimate the polynomial coefficients $\{a_k(t_i)\}_{k=0\ldots P}$.

2.2. Cramer-Rao Bound (CRB)

In [8], CRB for polynomial phase estimation are introduced for a 2-degree PPS with constant parameters:

$$CRB\{\widehat{a}_1\} = \frac{6\,f_s^2}{N^3\,SNR} = \frac{6}{N\,T^2\,SNR}, \tag{3}$$

$$CRB\{\widehat{a}_2\} = \frac{90\,f_s^4}{N^5\,SNR} = \frac{90}{N\,T^4\,SNR}, \tag{4}$$

where f_s the sampling frequency, N is the length of the signal and T its duration ($N = T/f_s$) and SNR is the baseband signal-to-noise ratio defined by $SNR = A^2/\sigma^2$.

Let $s_w[t_i]$ be the restriction of signal s on a window of length N_w, duration T_w centred on time step t_i. Around time step t_i, one can consider $a_k(t) \approx a_k(t_i)$. Thus, $s_w[t_i]$ is also polynomial with constant coefficients $\{b_k\}_{k=0\ldots P}$ expressed by:

$$b_1 = a_1(t_i) + 2a_2(t_i)t_i = 2\pi\, IFL(t_i),$$
$$b_2 = a_2(t_i) = \pi\, IFR(t_i).$$

From equations 3 and 4 which can be applied to coefficients b_1 and b_2, CRB for $\widehat{a}_i(t_i)$ would be [10]:

$$CRB\{\widehat{a}_1(t_i)\} = \frac{6}{N_w\,T_w^2\,SNR}\left(1 + 60\left(\frac{t_i}{T_w}\right)^2\right), \tag{5}$$

$$CRB\{\widehat{a}_2(t_i)\} = \frac{90}{N_w\,T_w^4 SNR}. \tag{6}$$

Therefore, the knowledge of $a_1(t_i)$ and its time modulations depend on the time step the window is centred in (equation (5)). In others words, $\widehat{a_1}(t_i)$ at the end of a long pulse will highly vary due to extrapolation and the needed knowledge of $a_2(t_i)$ to estimate $a_1(t_i)$. The alternative to this high variance is to follow the modulation of the IFL, whose CRB do not depend on the time step t_i. Those CRB are:

$$CRB\left\{\widehat{IFL}\,(t_i)\right\} = \frac{1}{4\pi^2}\frac{6}{N_w\,T_w^2\,SNR}, \tag{7}$$

$$CRB\left\{\widehat{IFR}\,(t_i)\right\} = \frac{1}{\pi^2}\frac{90}{N_w\,T_w^4\,SNR}. \tag{8}$$

In what follows, CRB$\left\{\widehat{IFL}(t_i)\right\}$ and CRB$\left\{\widehat{IFR}(t_i)\right\}$ will be expressed in dB$_{MHz}$ and in dB$_{MHz/\mu s}$.

2.3. Windows Length Constraints

Usually, the only considered constraint for PPS is [5]:

$$|a_k| \le \frac{\pi}{k\left(\frac{N-1}{2}\right)^{(k-1)} f_s^k}. \tag{9}$$

But constraints can also appear for small parameters. For example, in a radar context, chirp rates usually ranges from 1 to 10MHz/μs (some exceptions can be found with several hundred MHz/μs). From equation (6), it ensues that the minimal window length (with no SNR threshold consideration and in the case of an unbiased estimator) is:

$$N \ge \sqrt[5]{\frac{90\,f_s^4}{\pi^2\;CRB\left\{\widehat{IFR}(t_i)\right\}\;SNR}}. \tag{10}$$

In figure 1, the minimal window size is depicted according to SNR and the chirp rate. For instance, for SNR=10dB, at least 250 samples are needed to detect a chirp rate of 5MHz/μs with a mean squared error (MSE) of -10dB$_{MHz/\mu s}$ (*i.e.* a mean error of 30%, which is a typical domain value), not counting the SNR threshold.

To detect any modulation of the chirp rate (expressed with coefficient a_3 - around 10MHz/μs^2), according to the formula available in [8], the minimal window size of signal would be around 400,000 samples. Because it is not calculable, a_3 coefficient will be considered as negligible.

3. MULTI-LINEAR FUNCTIONS METHODS

3.1. High-order Ambiguity Function (HAF)

The High-order Instantaneous Moment (HIM) of order K for a signal s is defined by [1]:

$$\mathcal{M}_K^{HAF}[s, t_i, \tau](t) = \prod_{q=0}^{K-1}\left[s^{(*q)}(t - t_i - q\tau)\right]^{\binom{K-1}{q}}, \tag{11}$$

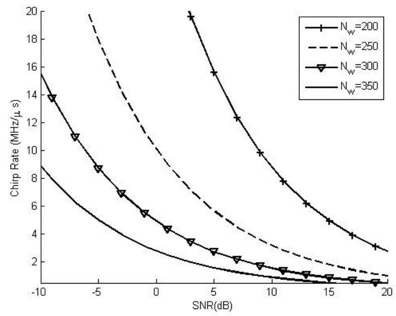


Fig. 1: Minimal window size needed to estimate different chirp rates for a MSE$\left\{\widehat{IFR}(t_i)\right\}$=-10dB$_{MHz/\mu s}$ with a sampling frequency $f_s = 1$GHz.

where:

- $t \in \left[t_i - \frac{T_w}{2}, t_i + \frac{T_w}{2}\right]$,
- $s^{(*q)}(t) = \begin{cases} s(t) & \text{if } q \text{ is even} \\ s^*(t) & \text{if } q \text{ is odd} \end{cases}$.

The phase of moment $\mathcal{M}_k^{HAF}[s, t_i, \tau]$ is expressed by

$$\Phi_{\mathcal{M}_K^{HAF}}(t) = k!\tau^{k-1}b_k(t - t_i) + \left[(k-1)!\tau^{k-1}b_{k-1} - 0.5k!(k-1)\tau^k b_k\right].$$

If we consider the Fourier transform $\mathcal{P}_K[s, t_i, \tau]$ of $\mathcal{M}_K^{HAF}[s, t_i, \tau]$, called the High-order Ambiguity Function, defined by

$$\mathcal{P}_K[s, t_i, \tau](f) = \int_{-\infty}^{\infty} \mathcal{M}_K^{HAF}[s, t_i, \tau](t)e^{-j2\pi f t}dt,$$

it results that

$$\hat{b}_k = \frac{1}{2\pi k!\tau^{k-1}} \operatorname*{argmax}_f |\mathcal{P}_K[s, t_i, \tau](f)|. \qquad (12)$$

According to [8], the minimal asymptotic variance for $K = 2$ or 3 is given for $\tau = \frac{N_w}{K}$. Thus, coefficient b_2 is estimated by calculating moment $\mathcal{M}_2^{HAF}[s, t_i, \tau](t) = s(t - t_i)s^*(t - t_i - \frac{N_w}{2})$ and its Fourier transform. Coefficient b_1 is given after demodulation of signal s by $\exp(-j\widehat{b}_2(t - t_i)^2)$ and the calculation of the Fourier transform of these demodulated signal. The main drawback of this method is that the estimate of b_1 depends on $\widehat{b}_2$.

3.2. Generalized Representation of Phase Derivative (GRPD)

To calculate coefficients b_1 and b_2, consider the moments:

$$\mathcal{M}_1^{GRPD}[s, t_i](\tau) = s(t_i + \frac{\tau}{2})s^*(t_i - \frac{\tau}{2}), \quad \tau \in [-T_w, T_w], \qquad (13)$$

$$\mathcal{M}_2^{GRPD}[s, t_i](\tau) = s(t_i - \sqrt{\tau})s(t_i + \sqrt{\tau}), \quad \tau \in \left[-T_w^2/4, T_w^2/4\right], \qquad (14)$$

whose respective phases $\Phi_{\mathcal{M}_1^{GRPD}}$ and $\Phi_{\mathcal{M}_2^{GRPD}}$ are:

$$\Phi_{\mathcal{M}_1^{GRPD}}(\tau) = (b_1 + 2b_2t_i)\tau,$$
$$\Phi_{\mathcal{M}_2^{GRPD}}(\tau) = 2b_2\tau + \left[2b_0 + 2b_1t_i + 2b_2t_i^2 + 4b_2t_i\sqrt{\tau}\right].$$

As proposed in [11], if we consider a time origin of $-\frac{N_w-1}{2}$, coefficient b_1 is estimated by the relation:

$$\hat{b}_1 = 2\pi \operatorname*{argmax}_f \left|\int_{-\infty}^{\infty} \mathcal{M}_1^{GRPD}[s, t_i](\tau)e^{-j2\pi f\tau}d\tau\right|. \quad (15)$$

After substitution, a Fourier transform can be applied to $\mathcal{M}_2[s, t_i]$ to estimate b_2:

$$\hat{b}_2 = \pi \operatorname*{argmax}_f \left|\int_{-\infty}^{\infty} \mathcal{M}_2^{GRPD}[s, t_i](\tau)e^{-j2\pi f\tau}d\tau\right|. \quad (16)$$

Unlike HAF, estimating b_1 with GRPD does not depend on $\hat{b}_2$.

4. RESULTS

To evaluate differences between algorithms, we compare HAF and GRPD methods to two others used for less critical issues:

Direct Derivation: this method consists in a phase calculation, a phase unwrapping and calculations of the first and second derivatives to obtain the IFL and the IFR and then the polynomial coefficients:

$$\widehat{IFL}(t_i) = \frac{1}{2\pi}\frac{\text{angle}\left(s(t_i + \frac{T_w}{4})\right) - \text{angle}\left(s(t_i - \frac{T_w}{4})\right)}{\frac{T_w}{2}},$$

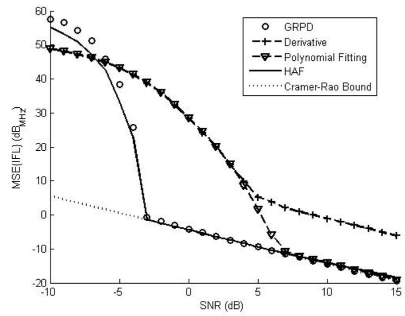
$$\widehat{IFR}(t_i) = \frac{\widehat{IFL}\left(t_i + \frac{T_w}{4}\right) - \widehat{IFL}\left(t_i - \frac{T_w}{4}\right)}{\frac{T_w}{2}}.$$

Polynomial Fitting: the phase is calculated, unwrapped and fitted to a polynomial using the method of least squares to obtain the coefficients.

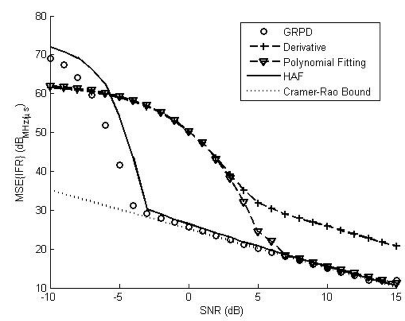
4.1. Mean Squared Error Evaluation

To evaluate the performances of the algorithms, we consider a sampling frequency of 1GHz. Polynomial coefficients have been calculated so as to describe a chirp with central frequency of 100MHz and a chirp rate of 5MHz/μs. The simulation is run over 10,000 times on 256-sample windows. FFTs are calculated with a 2^{14}-sample zero padding. Results of this simulation are depicted in figure 2.

Figure 2a shows the mean squared error for the IFL estimation for each algorithm. For HAF and GRPD algorithm, both MSE have a similar behaviour according to the SNR. After filtering, signals usually have a SNR ranging from 0dB to 10dB, a choice between those two



(a) MSE$\left\{\widehat{IFL}(t_i)\right\}$ in dB$_{MHz}$.



(b) MSE$\left\{\widehat{IFR}(t_i)\right\}$ in dB$_{MHz/\mu s}$.

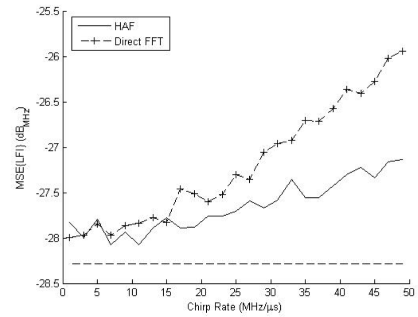
Fig. 2: Estimation of MSE$\left\{\widehat{IFR}(t_i)\right\}$ and MSE$\left\{\widehat{IFL}(t_i)\right\}$ for $IFL(t_i)$ = 100MHz and $IFR(t_i)$ = 5MHz/μs with f_s = 1GHz and N_w=256 samples.

algorithms cannot be done just by focusing on the IFL estimation.

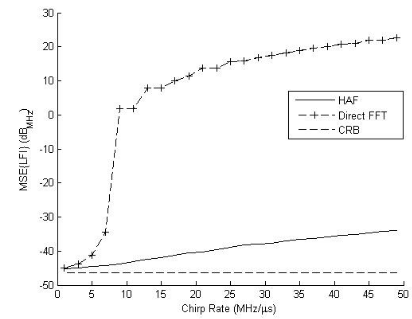
After studying figure 2b which depicts the MSE for the IFR, we see that the SNR threshold is 1dB smaller for the GRPD algorithm than for the HAF. We can also remark that the GRPD is closer to the BCR. This SNR threshold difference is due to the dependence of the IFR to estimate the IFL in the HAF algorithm. But it is not significant enough to prefer one algorithm to an other. We prefer to focus on the bigger gap between the derivative or polynomial fitting that are actually used and the multilinear methods, the latter having a smaller SNR threshold and whose MSE tend towards the CRB.

Estimation of both parameters by the direct derivation algorithm is unbiased but less efficient than the other one due to its high vulnerability to noise and the use of arctangent function which is not linear. Estimation via polynomial fitting also tends towards the CRB but for higher SNR because of the use of the same arctangent function.

Note that a plateau can appear for SNR>20dB for HAF algorithm for both figures, due to the FFT precision. This can be corrected by increasing the zero padding.



(a) MSE$\{\widehat{IFL}\}$ for N_w=256 samples



(b) MSE$\{\widehat{IFL}\}$ for N_w=1024 samples

Fig. 3: Estimation of MSE$\left\{\widehat{IFL}\right\}$ in dB$_{MHz}$ for IFL= 100MHz and IFR=5MHz/μs *without* considering IFR for the calculation of IFL estimate. f_s=1GHz.

The small difference between the two multi-linear method can be considered as negligible. On longer pulses this difference no longer exists. Because the EW context is critical, a choice would be done on the reliability of the values. Because estimation of the IFL by the HAF algorithm depends on the IFR estimation, we would tend to prefer the GRPD algorithm.

4.2. Need of demodulation in the HAF algorithm

Because coefficient b_2 is negligible compared to f_s^2, one can wonder if not considering b_2 in the algorithm would affect the estimation of coefficient b_1. In other words, does the demodulation step during HAF algorithm worth it in terms of computing operations? Simulations are performed considering a chirp rate ranging from 1 to 50MHz/μs. IFL is estimated with 256-sample and 1024-sample windows with algorithms considering, or not, the IFR estimate. In figure 3, MSE for both methods can be compared to the CRB.

According to figure 3a, not considering the b_2 coefficient does not affect estimation of b_1 in the HAF until IFR<20MHz/μs if the window has 256 samples. Beyond this value, difference can be observed and increases to 1dB. On the other hand, with 1024-sample window (fig-

Category	1	2	3	4
IFL(MHz)	some hundred			
IFR (MHz/μs)	1 to 20	0	>50	0
Number of Samples	>1000	some dozen or hundred		>1000
Applied Methods	HAF GRPD	GRPD	GRPD	GRPD
Window length	>1024	>256	>64	$< T_s/f_s$
Gain for $\widehat{IFR}$	>20dB	>16dB	>8dB	–

Table 1: Radar categories, their parameters and corresponding chosen methods with $f_s \approx$ 1GHz.

ure 3b), differences between algorithms can be seen from some MHz/μs.

Determining from figures 3a and 3b a threshold beyond which the demodulation is needed may be too ambitious, nevertheless, we can consider that demodulation is not necessary in the special case of EW and UMOP applications.

5. APPLICATION TO EW CONTEXT

Radar signals can be splitted into 4 categories:

1. Long pulse, lasting several μs *e.g.* FMCW radar, with generally a small chirp rate,
2. Short pulse with no chirp rate, lasting less than 1μs,
3. Short pulse with chirp rate,
4. Long pulse with phase code like QPSK or Barker signals with symbol duration T_s.

In table 1, parameters for each category are listed and we propose an algorithm to study any signal frequency. The last row describes the gain of MSE$\left\{\widehat{IFR}\right\}$ of the applied method versus the direct derivation for given window size and for a signal with parameters given in 4.2.

The choice highly depends on the signal length. A short signal implies the use of GRPD whereas a long one can also be processed by HAF. Nevertheless, from table 1, we can conclude that polynomial methods are highly recommended versus direct derivation which is currently used in EW systems. For phase code signal, we must have $T_w \ll T_s$ to detect any frequency modulation between symbol change. Because symbol duration is often small, the use of GRPD algorithm is thus recommended.

6. DISCUSSION

Results presented in this article have to be moderated. First, we consider chirp rates around 5MHz/μs. This range of chirp rates is often found but it is not characteristic. Moreover, radar signals can also have phase codes. That's why the sliding window principle can not be generalized to all signals. Usually, long radar pulses have a small chirp rate which enables the use of large windows. But short pulses have a greater chirp rate which enables the use of shorter windows with 64 samples or even 32.

Secondly, the minimal size given by (10) and depicted in figure 1 is a lower bound. Parameters such as the SNR threshold or the estimator efficiency must be considered and increasing the size of the window to have at the end the given MSE is recommended.

7. CONCLUSION & FUTURE WORK

We have introduced theoretically the limits of multi-linear functions by showing the minimal length of a signal to describe parameters that can be found in an EW context. We have compared different methods to finally choose the most appropriate one in a radar context and for several categories of radar signals. The next step of this study is the application of the appropriate method to actual radar signals with a view of discrimination and identification.

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