A NOVEL METHOD FOR MULTICHANNEL SPECTRAL FACTORIZATION

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ABSTRACT

A novel algorithm for the spectral factorization (SF) of a para-Hermitian polynomial matrix (PPM) is presented. This utilizes a series of paraunitary transformations to reconstruct the spectral factors of the PPM from the spectral factors of its eigenvalue polynomial matrix (EPM). The EPM is a diagonal polynomial matrix (DPM) obtained by applying a set of simple shift and rotation operators to the PPM using the second order best rotation (SBR2) algorithm. The spectral factors of the EPM are calculated by factorizing each of its scalar elements independently. In this paper, it is shown how to generate a series of paraunitary matrices which then map the factors of the EPM to the required spectral factors of the original PPM. The method basically introduces a sequence of stable operators which provide a direct connection between the one-dimensional spectral factorization problem and the multidimensional case.

Index Terms— spectral factorization (SF), para-Hermitian polynomial matrix (PPM), second order sequential best rotation (SBR2) algorithm, eigenvalue polynomial matrix (EPM).

1. INTRODUCTION

Causality plays a significant role in analyzing many engineering systems. Constructing a causal system from its autocorrelation function, or equivalently its spectral density function, is referred to as spectral factorization [1]. In the scalar case, let the autocorrelation function of a stationary process $x(t)$ be denoted by $f(\tau) = \mathrm{E}\{x(k+\tau)x^*(k)\}$ and its corresponding Fourier transform by $f(e^{jw})$ where j here denotes $\sqrt{-1}$. If $f(e^{jw})$ belongs to the class of functions in the Hardy space then the z-transform of the autocorrelation function admits an inner-outer factorization of the form $f(z) = f^+(z)f^-(z)$ where $f^+(z)$ is an analytic function outside the unit circle, and $f^-(z) = f^{+^*}(z^{-1})$ [2]. The outer function, $f^+(z)$, can be expressed in the form

$$f^+(z) = \exp(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{j\omega}+z}{e^{j\omega}-z}\ln f(e^{j\omega})d\omega) \qquad (1)$$

However, it is well known that there is no analog of the above formula for vector processes (i.e. the multichannel case) [2], [3]. In the case of a vector process, the autocorrelation function, as shown later, may be represented by means of a para-Hermitian polynomial matrix. To date, amongst all multichannel SF algorithms, the Newton-Raphson based factorisation of the PPM proposed by Wilson [2] seems to be the most stable and reliable. Wilson's algorithm attempts to minimize the Euclidean distance between the product of its estimated spectral factors and the given PPM by means of a stepwise Newton-Raphson iteration [2], [4]. An implementation of this Newton-Raphson method is provided in the polynomial matrix toolbox from PolyX[5].

In this paper, a new algorithm for multichannel SF is introduced. In effect, it constructs a bridge between the multidimensional spectral factors and a set of underlying one dimensional factors by means of totally stable paraunitary transformations. The novel approach adopted here is to transform the initial PPM into the form of a diagonal polynomial matrix (DPM) using a sequence of elementary paraunitary matrix operations. This is achieved fairly simply using the SBR2 algorithm originally proposed by McWhirter et al [6]. An established one dimensional SF algorithm such as Wilson's [2] is then used to compute the inner and outer factors corresponding to each element of the DPM which is, of course, a simple (para-Hermitian) scalar polynomial. The resulting outer (inner) factors are then projected back to generate the required inner (outer) spectral factors of the initial PPM. In principle, this back projection can be implemented very easily, by applying the sequence of elementary paraunitary matrix operations computed by SBR2, in reverse order and paraconjugate form [6]. However, reconstructing the spectral factors of the PPM from the spectral factors of the DPM, is not quite so simple if the analytic properties are to be preserved. A modified sequence of paraunitary matrices is introduced in this paper. This sequence of paraunitary matrices is selected in a way which constructs the spectral factors of the PPM whilst retaining the integrity of the SBR2 algorithm.

Throughout this paper, matrices are denoted by upper case bold characters and vectors by lower case bold. $[.]_{jk}$ denotes the (j,k) element of the matrix in square brackets. The superscripts $*$, T, and H denote the complex conjugate, matrix transpose and Hermitian conjugate re-

spectively. The underscore is used to denote a polynomial (including the more general case of Laurent polynomials which can include negative powers of the indeterminate variable). A para-Hermitian polynomial matrix is denoted by $\underline{\mathbf{R}}(z) = \sum_{\tau=-N}^{N} \mathbf{R}_\tau z^{-N}$, where $\mathbf{R}_\tau \in \mathbb{C}^{p\times p}$ and for off-diagonal entries we have $[\mathbf{R}_\tau]_{kl} = [\mathbf{R}_{-\tau}]^*_{lk}$. For a given polynomial matrix $\underline{\mathbf{H}}(z)$, the paraconjugate is denoted by $\widetilde{\underline{\mathbf{H}}}(z)$ and defined as $\widetilde{\underline{\mathbf{H}}}(z) = \underline{\mathbf{H}}^H(1/z)$. A paraunitary polynomial matrix $\underline{\mathbf{H}}(z)$ is one which satisfies $\widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{H}}(z) = \underline{\mathbf{H}}(z)\widetilde{\underline{\mathbf{H}}}(z) = \mathbf{I}$. The unit delay is represented throughout by z^{-1}.

The remainder of this paper is organized as follows. In section two the concept of spectral factorization is briefly stated. The SBR2 algorithm is outlined in section three. Section four provides an overview of the novel spectral factorization algorithm, and some comments and conclusion are presented in section five. For clarity, some of the more detailed analysis relating to section four is relegated to the appendix.

2. SPECTRAL FACTORIZATION

Let a polynomial power spectral density matrix be denoted by

$$\underline{\mathbf{R}}(z) = \sum_{\tau=-N}^{N} \mathbf{R}_\tau z^\tau, \tag{2}$$

where the sequence $\{\mathbf{R}_\tau\}_{\tau=-N}^{N}$ denotes the set of constant correlation matrix coefficients $\underline{\mathbf{R}}(\tau) = \mathrm{E}\{\underline{x}(k+\tau)\underline{x}^H(k)\}$. If $\underline{\mathbf{R}}(z)$ is integrable on the unit circle, $\mathbb{T}$, and if the Paley-Wiener condition is satisfied i.e. $\ln\det\underline{\mathbf{R}}(z) \in \mathrm{L}_1(\mathbb{T})$ [1], then $\underline{\mathbf{R}}(z)$ admits the following factorization:

$$\underline{\mathbf{R}}(z) = \underline{\mathbf{S}}^+(z)\underline{\mathbf{S}}^-(z). \tag{3}$$

where

$$\underline{\mathbf{S}}^+(z) = \sum_{k=-N}^{0} \mathbf{S}_k z^k \tag{4}$$

$$\underline{\mathbf{S}}^-(z) = \sum_{k=0}^{N} \mathbf{S}_k^H z^k = \widetilde{\underline{\mathbf{S}}^+}(z) \tag{5}$$

and $\det(\underline{\mathbf{S}}^+(z)) \neq 0$ for all $|z| \geq 1$. $\underline{\mathbf{S}}^+(z)$ is referred to as an outer analytic polynomial matrix from Hardy space [1], [3]. The spectral factors are unique up to a constant right unitary matrix multiplier [1].

3. SECOND ORDER SEQUENTIAL BEST ROTATION ALGORITHM (SBR2)

The second order sequential best rotation algorithm (SBR2) is designed to diagonalize a para-Hermitian polynomial matrix by means of a paraunitary similarity transformation [6]. For a given $p \times p$ PPM , the objective of SBR2 is to compute paraunitary matrices $\underline{\mathbf{H}}(z)$ such that

$$\underline{\mathbf{H}}(z)\underline{\mathbf{R}}(z)\widetilde{\underline{\mathbf{H}}}(z) \approxeq \underline{\mathbf{D}}(z), \tag{6}$$

where $\underline{\mathbf{D}}(z) = \mathrm{diag}\{d_{11}(z), d_{22}(z), ...d_{pp}(z)\}$ is a diagonal polynomial matrix. The diagonal elements of $\underline{\mathbf{D}}(z)$ may be viewed as polynomial eigenvalues of the matrix $\underline{\mathbf{R}}(z)$ [6]. In effect SBR2 generates a set of one-dimensional power spectral densities from which the PPM can be reconstructed. $\underline{\mathbf{H}}(z)$ is a paraunitary polynomial matrix with $\det\underline{\mathbf{H}}(z) = z^{-k}$. $\underline{\mathbf{H}}(z)$ is generated iteratively during the SBR2 procedure which continues until the PPM is sufficiently diagonal [6]. $\underline{\mathbf{H}}(z)$ takes the general form

$$\underline{\mathbf{H}}(z) = \prod_{i=n}^{1} \underline{\mathbf{H}}_i(z) \tag{7}$$

where $\underline{\mathbf{H}}_i(z)$, $1 \leq i \leq n$ is an elementary paraunitary matrix, and n represents the number of iteration required. If in the β^{th} iteration, $\underline{\mathbf{R}}_\beta(z) = \underline{\mathbf{H}}_\beta(z)\underline{\mathbf{R}}_{\beta-1}(z)\widetilde{\underline{\mathbf{H}}}_\beta(z)$ is still not sufficiently diagonal, and the dominant off-diagonal coefficient is the (j,k) element of the matrix coefficient of z^M, then $\underline{\mathbf{H}}_\beta(z)$ is given by

$$\underline{\mathbf{H}}_\beta(z) = \begin{pmatrix} \mathbf{I}_1 & & & & \mathbf{0} \\ & cos(\theta) & & sin(\theta)e^{i\phi}z^{-M} & \\ & & \mathbf{I}_2 & & \\ & -sin(\theta)e^{-i\phi} & & cos(\theta)z^{-M} & \\ \mathbf{0} & & & & \mathbf{I}_3 \end{pmatrix} \tag{8}$$

where $\underline{\mathbf{H}}_\beta(z)$ is equal to the $p \times p$ identity matrix except for elements in the (j,k) plane which are replaced by a 2×2 paraunitary matrix. The angles θ and ϕ are selected to ensure that the dominant coefficient is driven to zero. The entries $\mathbf{I}_r$, $r \in \{1,2,3\}$ constitute identity matrices of dimension $(\min\{j,k\}-1)$, $(|j-k|-1)$ and $(p-\max\{j,k\}+1)$ respectively. It should be clear that $\underline{\mathbf{H}}(z)$ is paraunitary by construction and so SBR2 is a stable, reversible algorithm. Given $\underline{\mathbf{D}}(z)$ and $\underline{\mathbf{H}}(z)$, it is possible to reconstruct the matrix $\underline{\mathbf{R}}(z)$ to a high degree of accuracy [6]. The reverse procedure, starting with $\underline{\mathbf{R}}_n(z) = \underline{\mathbf{D}}(z)$, can be stated very simply as

$$\underline{\mathbf{R}}_{\beta-1}(z) = \widetilde{\underline{\mathbf{H}}}_\beta(z)\underline{\mathbf{R}}_\beta(z)\underline{\mathbf{H}}_\beta(z) \tag{9}$$

$\beta = n, ... 1$. The reconstruction is finally given by $\underline{\mathbf{R}}_0(z)$.

4. NOVEL SPECTRAL FACTORIZATION ALGORITHM

The main idea behind our algorithm is to provide a framework for extending scalar spectral factorization to the multidimensional case. To this end, after the PPM has been diagonalized by SBR2 as in (6), each entry within $\underline{\mathbf{D}}(z)$ is factorizable in

accordance with the scalar SF formula (1), leading to inner and outer polynomial matrices for $\underline{\mathbf{D}}(z)$ in the form

$$\begin{aligned}\underline{\mathbf{D}}(z) &= \text{diag}\{d_{11}(z), d_{22}(z), ..., d_{pp}(z)\} = \\ &\text{diag}\{d_{11}^{+}(z), d_{22}^{+}(z), ..., d_{pp}^{+}(z)\}\text{diag}\{d_{11}^{-}(z), d_{22}^{-}(z), ..., d_{pp}^{-}(z)\} \\ &= \underline{\mathbf{D}}^{+}(z)\underline{\mathbf{D}}^{-}(z)\end{aligned} \tag{10}$$

where $d_{ii}^{+}(z)$ and $d_{ii}^{-}(z)$ respectively, denote the outer and inner spectral factors of $d_{ii}(z)$, $1 \leq i \leq p$. We have

$$\begin{aligned}\underline{\mathbf{R}}(z) &\approxeq \widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{D}}(z)\underline{\mathbf{H}}(z) \\ &= \widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{D}}^{+}(z)\underline{\mathbf{D}}^{-}(z)\underline{\mathbf{H}}(z) \\ &= \widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{D}}^{+}(z)\underline{\mathbf{H}}(z)\widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{D}}^{-}(z)\underline{\mathbf{H}}(z)\end{aligned}$$

$$= \{\prod_{i=1}^{n}\widetilde{\underline{\mathbf{H}}}_i(z)\underline{\mathbf{D}}^{+}(z)\prod_{i=n}^{1}\underline{\mathbf{H}}_i(z)\}\{\prod_{i=1}^{n}\widetilde{\underline{\mathbf{H}}}_i(z)\underline{\mathbf{D}}^{-}(z)\prod_{i=n}^{1}\underline{\mathbf{H}}_i(z)\}$$

$$\doteq \underline{\mathbf{A}}^{+}(z)\underline{\mathbf{A}}^{-}(z). \tag{11}$$

The equality above shows how the spectral factors of $\underline{\mathbf{D}}(z)$ can be projected back by inverting the effect of SBR2. Furthermore, the polynomial matrices $\underline{\mathbf{A}}^{+}(z)$ and $\underline{\mathbf{A}}^{-}(z)$ are both clearly paraunitary. However, $\underline{\mathbf{A}}^{+}(z)$ and $\underline{\mathbf{A}}^{-}(z)$ may contain both positive and negative powers of z in which case they do not comply with the definition of spectral factors as set out in equations (4) and (5). For this reason, $\underline{\mathbf{A}}^{+}(z)$ and $\underline{\mathbf{A}}^{-}(z)$ will be referred to as inner-like and outer-like polynomial matrices, respectively. To overcome this limitation, we apply an additional paraunitary polynomial matrix from the right to $\underline{\mathbf{A}}_i^{+}(z)$ and from the left to $\underline{\mathbf{A}}_i^{-}(z)$ in each step of the following iterative reconstruction starting with $\underline{\mathbf{A}}_n^{+}(z) = \underline{\mathbf{D}}^{+}(z)$ and $\underline{\mathbf{A}}_n^{-}(z) = \underline{\mathbf{D}}^{-}(z)$:

$$\underline{\mathbf{A}}_{\beta-1}^{+}(z) = \widetilde{\underline{\mathbf{H}}}_\beta(z)\underline{\mathbf{A}}_\beta^{+}(z)\underline{\mathbf{H}}_\beta(z)\underline{\mathbf{C}}_\beta(z), \tag{12}$$

$$\underline{\mathbf{A}}_{\beta-1}^{-}(z) = \widetilde{\underline{\mathbf{C}}}_\beta(z)\widetilde{\underline{\mathbf{H}}}_\beta(z)\underline{\mathbf{A}}_\beta^{-}(z)\underline{\mathbf{H}}_\beta(z) \tag{13}$$

for $\beta = n, ... 1$. The outer matrix is given by $\underline{\mathbf{S}}^{+}(z) = \underline{\mathbf{A}}_0^{+}(z)$ while the inner matrix is given by $\underline{\mathbf{S}}^{-}(z) = \underline{\mathbf{A}}_0^{-}(z)$. We call each of the matrices $\underline{\mathbf{C}}_\beta(z)$ a counteractive polynomial matrix (CPM). The purpose of introducing a CPM is to eliminate the unwanted positive or negative powers which would otherwise occur in $\underline{\mathbf{S}}^{+}(z)$ and $\underline{\mathbf{S}}^{-}(z)$ without affecting the integrity of the SBR2 reconstruction process.

The CPM is defined in each step of the reconstruction, based on the sign of the determinant of $\underline{\mathbf{H}}_\beta(z)$ and the location of the element to be reconstructed (referred to as the target element). Let $\det \underline{\mathbf{H}}_\beta(z) = z^{-M}$; then, if $M \leq 0$ and the target element is located in the (i, j) plane, $\underline{\mathbf{C}}_\beta(z)$ is defined as

$$\underline{\mathbf{C}}_\beta(z) = \begin{pmatrix} \mathbf{I}_1 & & & & & \mathbf{0} \\ & \frac{1}{\sqrt{2}} & & & \frac{1}{\sqrt{2}} & \\ & & & \mathbf{I}_2 & & \\ & \frac{-z^M}{\sqrt{2}} & & & \frac{z^M}{\sqrt{2}} & \\ \mathbf{0} & & & & & \mathbf{I}_3 \end{pmatrix} \tag{14}$$

where $\mathbf{I}_r$ $r \in \{1, 2, 3\}$ and all matrix entries are in one to one correspondence with those in (8). On the other hand, if $M > 0$, $\underline{\mathbf{C}}_\beta(z)$ takes the form

$$\begin{pmatrix} f_{11}(z) & & & & & & \\ & f_{22}(z) & & & & & \\ . & . & . & . & . & . & \\ & & \frac{z^{-M}}{\sqrt{2}} & & \frac{-z^{-M}}{\sqrt{2}} & . & \\ . & . & . & . & . & . & \\ . & . & . & f_{rr}(z) & . & . & \\ . & . & . & . & . & . & \\ & & \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} & . & \\ . & . & . & . & . & & \\ & & & & & & f_{p,p}(z) \end{pmatrix}$$

where, for elements not in the (i, j) plane, $f_{kk}(z)$ ($k \in \{1, ...p\}$) is equal to one if the (i, k) and (j, k) entries are both equal to zero; otherwise $f_{kk}(z) = z^{-M}$.

In summary, our proposed algorithm computes a set of one dimensional spectral factors and transforms them to the required spectral factors of the initial PPM by means of paraunitary matrices. In the appendix we show that the additional CPM operations defined above do not have any effect on the integrity of the SBR2 algorithm.

5. COMMENTS AND CONCLUSIONS

A novel multi-channel spectral factorization algorithm has been presented in this paper. It provides the means for a series of paraunitary transformations to map a set to one-dimensional spectral factors to the spectral factors of a PPM. In this respect, our algorithm is entirely distinct from any which have been published before. Furthermore, it is based on paraunitary transformations which are extremely stable and well suited to DSP programming. Due to the limitation on space, it has not been possible in this paper, to present any of the numerical computations carried out to date. However, successful results have been obtained for a number of sample multichannel SF problems. We intend to include various representative examples in a future publication.

6. APPENDIX

This appendix is intended to illustrate that the computed spectral factors from the algorithm specified in section 4 satisfy all the conditions given in section 2. This amounts to showing that

A The inner-like and outer-like spectral factors are generated by the recursion in (12) i.e that $\underline{\mathbf{A}}_0^+(z)\underline{\mathbf{A}}_0^-(z) \approxeq R(z)$.

B The computed outer spectral factor $\underline{\mathbf{S}}^+(z)$ has no terms with positive power as specified in equation (4). It is required to show that $\det \underline{\mathbf{S}}^+(z) \neq 0$ for all $|z| < 1$. Noting that this is true for $\underline{\mathbf{A}}_n^+(z)$, it suffices to show that the same property is maintained following the iterative step defined in (12),

6.1. Proof of claim A

From (12) it follows immediately that

$$\underline{\mathbf{A}}_{\beta-1}^+(z)\underline{\mathbf{A}}_{\beta-1}^-(z) = \widetilde{\underline{\mathbf{H}}}_\beta(z)\underline{\mathbf{A}}_\beta^+(z)\underline{\mathbf{A}}_\beta^-(z)\underline{\mathbf{H}}_\beta(z) \tag{15}$$

By iterating this equation for $\beta = n, n-1, ..., 1$, and noting that

$$\underline{\mathbf{A}}_n^+(z)\underline{\mathbf{A}}_n^-(z) = \underline{\mathbf{D}}^+(z)\underline{\mathbf{D}}^-(z) = \underline{\mathbf{D}}(z) \tag{16}$$

it follows as required that

$$\underline{\mathbf{A}}_0^+(z)\underline{\mathbf{A}}_0^-(z) = \widetilde{\underline{\mathbf{H}}}(z)\underline{\mathbf{D}}(z)\underline{\mathbf{H}}(z) \approxeq \underline{\mathbf{R}}(z) \tag{17}$$

where $\underline{\mathbf{H}}(z)$ is given by the product of elementary paraunitary matrices $\underline{\mathbf{H}}_i(z)$ as in (7)

6.2. Proof for claim B

In order to simplify the following proof, some further notation is introduced. A linear combination of analytic functions $X_1(z)$ and $X_2(z)$ is denoted by $f_{X_1 \bigoplus X_2}(z)$ and given as follows

$$f_{X_1 \bigoplus X_2}(z) = aX_1(z) + bX_2(z) \tag{18}$$

where a and b are constant complex numbers. If we wanted to represent a linear combination of $X_1(z)$ and $X_2(z)$ located in the (i, j) entry of a matrix at the δ^{th} iteration, we denote it by $f^{ij,\delta}_{X_1 \bigoplus X_2}(z)$. We begin with a proof for any 2×2 PPM and then generalise it to the $p \times p$ case.

For a 2×2 PPM, $\underline{\mathbf{D}}(z)$ and its spectral factors take the form given in (10) i.e.

$$\underline{\mathbf{D}}(z) = \text{diag}\{d_{11}(z), d_{22}(z)\} = \text{diag}\{d_{11}^+(z), d_{22}^+(z)\}\text{diag}\{d_{11}^-(z), d_{22}^-(z)\} = \underline{\mathbf{D}}^+(z)\underline{\mathbf{D}}^-(z) \tag{19}$$

The reconstruction, based on 'inverting' the SBR2 algorithm, starts with the diagonal matrix $\underline{\mathbf{D}}(z)$, and then, after the first step $\underline{\mathbf{R}}_1(z)$ can be represented as follows

$$\underline{\mathbf{R}}_1(z) = \begin{pmatrix} f^{11,1}_{d_{11} \bigoplus d_{22}}(z) & f^{12,1}_{d_{11} \bigoplus d_{22}}(z)z^{-k_1} \\ f^{21,1}_{d_{11} \bigoplus d_{22}}(z)z^{k_1} & f^{22,1}_{d_{11} \bigoplus d_{22}}(z) \end{pmatrix}, \tag{20}$$

where $\det \underline{\mathbf{H}}_1(z) = z^{-k_1}$. The corresponding outer-like matrix from the equation (11) can be given as

$$\underline{\mathbf{A}}_1^+(z) = \begin{pmatrix} f^{11,1}_{d_{11}^+ \bigoplus d_{22}^+}(z) & f^{12,1}_{d_{11}^+ \bigoplus d_{22}^+}(z)z^{-k_1} \\ f^{21,1}_{d_{11}^+ \bigoplus d_{22}^+}(z)z^{k_1} & f^{22,1}_{d_{11}^+ \bigoplus d_{22}^+}(z) \end{pmatrix}, \tag{21}$$

Without loss of generality, if $k_1 > 0$, then by selecting the CPM as follow

$$\underline{\mathbf{C}}_1^+(z) = \begin{pmatrix} \frac{z^{-k_1}}{\sqrt{2}} & -\frac{z^{-k_1}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{22}$$

the outer spectral PM of the stage one can be given by

$$\underline{\mathbf{S}}_1^+(z) = \begin{pmatrix} g^{11,1}_{f^{11,1} \bigoplus f^{12,1}}(z)z^{-k_1} & g^{11,1}_{f^{11,1} \bigoplus f^{12,1}}(z)z^{-k_1} \\ g^{21,1}_{f^{21,1} \bigoplus f^{22,1}}(z) & g^{22,1}_{f^{21,1} \bigoplus f^{22,1}}(z) \end{pmatrix}, \tag{23}$$

where $g^{ij,1}$ denotes a linear combination between the functions $f^{ij,1}$ in the equation (21).The emerging spectral factor from our algorithm satisfies the power conditions as mentioned in equation (4), since by comparing the negative power in $\underline{\mathbf{R}}_1(z)$ and $\underline{\mathbf{S}}_1^+(z)$, it is possible to see that the maximum negative powers of $\underline{\mathbf{S}}_1^+(z)$ and $\underline{\mathbf{R}}_1(z)$ are equal. In the next iteration, the linear combination followed by an appropriate selection of the CPM is guaranteed to generate valid factors for the second stage. In other words, in the first iteration the spectral factors of the PM are extracted and following that it is possible to preserve the spectral factor properties from one iteration to another. Hence, the final spectral factors satisfy the power conditions as given in (4), (5). Similarly, if $k_1 \leq 0$, a suitable CPM is given by

$$\underline{\mathbf{C}}_1^+(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{z^{k_1}}{\sqrt{2}} & \frac{z^{k_1}}{\sqrt{2}} \end{pmatrix}, \tag{24}$$

In the $p \times p$ case, let the target entry be reconstructed from the j^{th} entry and k^{th} entry, in the l^{th} iteration. Assume also that by the $(l-1)^{th}$ step, the j^{th} entry and the i^{th} entry have already been combined and similarly the k^{th} and j^{th} entries and without loss of the generality let $i < j < k < r$. The previous statement relates to the general situation where two entries have already been combined during the SBR2 reconstruction process and a third entry is going to be combined with one of them at the next iteration. Furthermore, we assume that the third entry has also been combined with another entry at some stage.

Without lose of generality, for $k_\tau > 0$ and $k_\upsilon > 0$, the outer polynomial matrix in the $(l-1)^{th}$ stage, $\underline{\mathbf{S}}^{+}_{l-1}(z)$, can be computed by the outer PM of the 2×2 sub-matrix and has the general form as follows,

$$\underline{\mathbf{S}}^{+}_{1-1}(z) = \begin{pmatrix} f^{ii}_{+} z^{-k_\tau} & f^{ij}_{+} z^{-k_\tau} & & \mathbf{0} \\ f^{ji}_{+} & f^{jj}_{+} & & \\ . & . & . & . \\ . & . & f^{kk}_{+} z^{-k_\upsilon} & f^{kr}_{+} z^{-k_\upsilon} \\ \mathbf{0} & . & f^{rk}_{+} & f^{rr}_{+} \end{pmatrix}, \tag{25}$$

where f^{hf}_{+} represents the linear combination between the outer entries corresponding to the h and f entries from the last time they were combined ; $\{h, f\} \in \{\{i, j\}, \{k, r\}\}$. If $\det \underline{\mathbf{H}}_l(z) = z^{-k_l}$ the outer-like PM can be given by

$$\underline{\mathbf{A}}^{+}_{l}(z) = \begin{pmatrix} u^{ii}_{+} z^{-k_\tau} & u^{ij}_{+} z^{-k_\tau} & g^{ik}_{+} z^{-k_l} & \mathbf{0} \\ u^{ji}_{+} & g^{jj}_{+} & g^{jk}_{+} z^{-k_l} & u^{jr}_{+} \\ . & . & . & . \\ & g^{kj}_{+} z^{k_l} & u^{kk}_{+} & g^{kr}_{+} z^{k_l} \\ \mathbf{0} & u^{rj}_{+} & u^{rk}_{+} z^{-k_l} & u^{rr}_{+} \end{pmatrix}, \tag{26}$$

where u^{hf}_{+} represents a linear combination of the entries h and f in the $(l-1)^{th}$ iteration; $\{h, f\} \in \{\{i, j\}, \{k, r\}\}$, and g^{hf}_{+} represents a para-linear combination of entries h and f in the $(l-1)^{th}$ iteration (i.e. when a and/or b in equation (18) are replaced by cz^{k_l}, where $k_l \in \mathbb{Z}$ and $c \in \mathbb{R}$). For $k_l > 0$, by selecting a CPM of the form

$$\underline{\mathbf{C}}^{+}_{l}(z) = \begin{pmatrix} f_{ii}(z) = z^{-k_l} & & & & \\ & \frac{z^{-k_l}}{\sqrt{2}} & \frac{-z^{-k_l}}{\sqrt{2}} & . \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & . \\ . & . & . & . & \\ & & & & f_{rr}(z) = z^{-k_l} \end{pmatrix}, \tag{27}$$

it is straightforward to show that the CPM retains the actual power of the outer-like matrix except for eliminating the positive powers (i.e. k_l).

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