A NEW WEIGHTING FUNCTION IMPROVING THE CONVERGENCE PERFORMANCE OF IRWLS-BASED ALL-PASS IIR FILTERS

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ABSTRACT

The paper presents a new weighting function that can be used in the method of iteratively reweighted least squares (IRWLS) for designing equiripple all-pass IIR filters. The purpose of introducing this weighting function is to improve the convergence performance in the solution of the IRWLS. The height of each weighting function is designed to be equal to the local maximum of each ripple, and the width of each weighting function is designed so that the area of each weighting function becomes equal to the area of each ripple. We show experimentally that the convergence performance in the solution of the IRWLS can be improved by using the proposed weighting function.

Index Terms— All-Pass IIR filters, equiripple filters, iteratively reweighted least squares (IRWLS)

1. INTRODUCTION

In the paper, we consider the design problems that affect equiripple all-pass IIR filters [5] - [7], [9], [10]. First, we describe a method of solving the phase error minimization problem for the phase of a filter and the desired phase. We utilize a method that involves linearizing a nonlinear optimization problem and then solving the filter design problem in the same manner as the conventional method [1], [5], [11]. Next, we describe the norm which is a measure of the phase error. The L_2 norm and L_∞ (minimax norm) are often used as the values for the norm in a phase-error minimization problems [1] - [4]. When using the L_∞ norm, this is known as the equiripple design method, and many similar design methods have been proposed [6], [13], [14]. In particular, the iteratively reweighted least squares (IRWLS) method, based on a scheme that involves multiplying a least square error by a weighting function [8], is a typical equiripple design method [12] - [14]. However, the convergence performances of the relevant solutions have not been referred to in these papers. Moreover, there are even cases where the convergence of a solution becomes unstable in some design examples.

In order to overcome these instability problems in terms of solution convergence, we introduce a new weighting function that can be used in the IRWLS method. The height of each weighting function is designed to be equal to the local maximum of each ripple, and the width of each weighting function is designed so that the area of each weighting function becomes equal to the area of each ripple. The object of introducing this concept of the weight function design is to properly reflect information about phase errors in the updating of the weighting function. In facts, by implementing weight function design in this way, we can expect to improve the convergence performance in the solution of the IRWLS. In Section 4, we show experimentally how introducing the weight function into some design examples that did not originally result in convergence when using the conventional method improves the convergence performance in the solution of the IRWLS.

The paper is organized as follows. The method of weighted least squares design for all-pass IIR filters is summarized in Section 2. We introduce a new weighting function that is used in the IRWLS method in Section 3. In Section 4, we show that convergence improves when using the introduced weight function by showing some design examples. Finally, the concluding remarks are given in Section 5.

2. WEIGHTED LEAST SQUARES DESIGN

In this Section, we summarize the basic formulation of weighted least squares (WLS) design for all-pass IIR filters according to [8]. Let us consider an N-th-order all-pass IIR filter with the transfer function

$$H(z) = \frac{a_N + \cdots + a_1 z^{-(N-1)} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = z^{-N} \frac{1 + \sum_{n=1}^{N} a_n z^n}{1 + \sum_{n=1}^{N} a_n z^{-n}}, \tag{1}$$

where a_n are real coefficients. Then, the phase response is given by

$$\theta(\omega) = -N\omega + 2 \arctan\left(\frac{a^T \mathbf{s}(\omega)}{1 + a^T \mathbf{c}(\omega)} \right), \tag{2}$$

where

$$\begin{aligned} a &= [a_1 a_2 \cdots a_N]^T, \\ \mathbf{s}(\omega) &= [\sin(\omega)\sin(2\omega)\cdots\sin(N\omega)]^T, \\ \mathbf{c}(\omega) &= [\cos(\omega)\cos(2\omega)\cdots\cos(N\omega)]^T \end{aligned}$$

Suppose that the desired phase response is denoted by $\theta_d(\omega)$, and consider that the following equation holds [8]:

$$\sin(\alpha(\omega))(1 + a^T\mathbf{c}(\omega)) - \cos(\alpha(\omega))(a^T\mathbf{s}(\omega)) \approx 0, \qquad (3)$$

where

$$\alpha(\omega) = \frac{\theta_d(\omega) + N\omega}{2}.$$

If we now square both sides of equation (3) , then we can obtain the following weighted least squares (WLS) form:

$$E = \sum_{k=1}^{M} w(\omega_k)\Big\{a^T(\mathbf{c}(\omega_k)\sin(\alpha(\omega_k)) - \mathbf{s}(\omega_k)\cos(\alpha(\omega_k)) + \sin(\alpha(\omega_k))\Big\}^2,$$

where $w(\omega_k)$ is a frequency-domain weighting function, and M is the number of points at which the desired phase response is sampled.

Then, the WLS estimate of coefficients a_n is given by minimizing the phase error E. Therefore, by setting $\frac{\partial E}{\partial a_i} = 0$, for $i = 1, 2, \cdots, N$, $\mathbf{Qa} = \mathbf{d}$ holds. Here, the elements of $\mathbf{Q}$ and $\mathbf{d}$ are given as follows:

$$\begin{aligned} \mathbf{Q} &= \sum_{k=1}^{M} w(\omega_k)\mathbf{sc}(\omega_k)\mathbf{sc}^T(\omega_k), \\ \mathbf{d} &= -\sum_{k=1}^{M} w(\omega_k)\sin(\alpha(\omega_k)\mathbf{sc}(\omega_k), \\ \mathbf{sc}(\omega_k) &= (\mathbf{c}(\omega_k)\sin(\alpha(\omega_k)) - \mathbf{s}(\omega_k)\cos(\alpha(\omega_k))\,. \end{aligned}$$

As mentioned in Section 1, in order to apply the WLS method to an equiripple design problem, a frequency-domain weighting function $w(\omega_k)$ as used in the IRWLS method [8] plays an important role. Now, let us give a rough sketch of the desired properties of the frequency-domain weighting function $w(\omega_k)$ in IRWLS. Suppose that the absolute phase error is denoted by $e_n(\omega_k)$, that the sum of phase error is denoted by $E_n^{'}$, and that a frequency-domain weighting function is denoted by $w_n(\omega_k)$, all at the n-th iteration. Then, it is desirable to design a frequency-domain weight function such that the sum of the phase error $E_n^{'}$ and the weighting function $w_n(\omega_k)$ may be changed similarly in each iteration (see Fig.1).

3. IRWLS METHOD FOR EQUIRIPPLE ALL-PASS IIR FILTER DESIGN

In this Section, we introduce a new weighting function that can be used in the method of iteratively reweighted least squares (IRWLS) for equiripple all-pass IIR filter design.

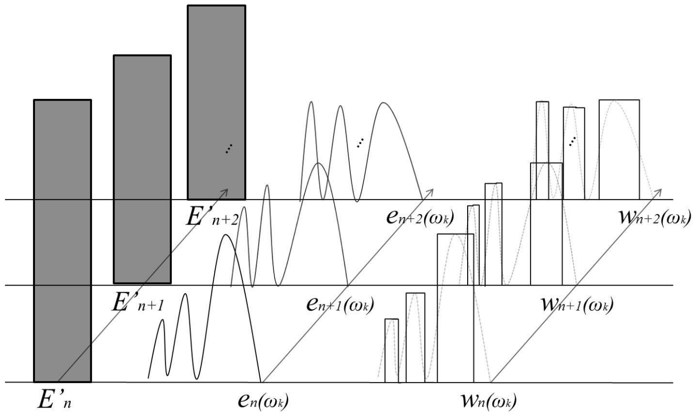

Fig. 1. A rough sketch of the desired properties of the freqeuncy-domain weighting function in IRWL:$E_n^{'}$, $e_n(\omega_k)$, and $w_n(\omega_k)$ should to be changed similarly in each iteration.

A frequency-domain weighting function based on the phase error in the conventional method has been proposed [12] - [14]. However, in the conventional method, the frequency-domain weighting function $w_n(\omega_k)$ is not well related to the absolute phase error $e_n(\omega_k)$.

In this paper, to cope with this problem, we will relate a minimax (L_∞) phase error more closely than the conventional method to the weighting function $w_n(\omega_k)$. That is, in each iteration, frequency-domain weighting functions are designed so that the local area of the functions becomes equal to the local area of the phase error. In the following subsection, we will describe in detail the design method for the weighting function.

3.1. A design algorithm for a new weighting function

In this subsection, we describe a design algorithm for a new weighting function that aims to improve the convergence performance in the solution of the IRWLS. Let the weighting function $w_{n+1}(\omega)$ at the n-th iteration be given by

$$w_{n+1}(\omega) = w_n(\omega)\beta_n(\omega), \qquad (4)$$

where the updating function, $\beta_n(\omega)$, is a rectangular function that is related to the local phase error. Note that the updating function proposed by [8] is a function of the envelope of the phase error. Here, the distribution of the phase error $e_\theta(\omega)$ is defined as the absolute phase error between the phase response $\theta(\omega)$ and the desired phase response $\theta_d(\omega)$

$$e_\theta(\omega) = |\theta(\omega) - \theta_d(\omega)|. \qquad (5)$$

At the n-th iteration, let us suppose that the phase error distribution has P ripples (see Fig.2; P is 5 in Fig.2.). When the total area of a p-th ripple is denoted by $\hat{E}_p$ and the local

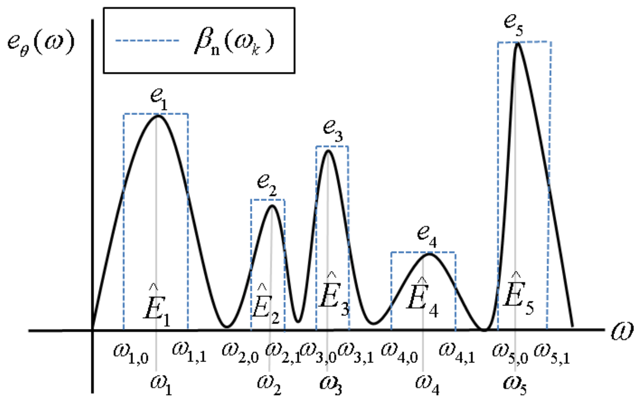

Fig. 2. A rectangular updating function $\beta_n(\omega_k)$ for a frequency-domain weighting function

maximum of a p-th ripple is denoted by e_p, the width L of each rectangle is designed so that the area of each rectangle becomes equal to the area of each ripple. That is, $\hat{E}_p = e_p \cdot L$. Therefore, we can calculate the width L of each rectangle by

$$L = \frac{\hat{E}_p}{e_p}. \tag{6}$$

Then, in each iteration, a rectangular updating function $\beta_n(\omega)$ can be constructed as

$$\beta_n = \begin{cases} e_p & (\omega_{p,0} \le \omega \le \omega_{p,1}) \\ e_p \cdot e_\varepsilon & (\omega_{p,1} < \omega < \omega_{p+1,0}) \end{cases} \; for \; 1 \le p \le P, \tag{7}$$

where ω_{step} is the sampling width, and

$$|\omega_{p,0} - \omega_p| = |\omega_{p,1} - \omega_p| = \omega_{step} \cdot (L/2), \tag{8}$$

and ω_p is the frequency of the local maximum of a p-th ripple, and e_ε is a small positive number. Let us now define

$$\mathbf{q} = [e_1 \quad e_2 \quad \cdots \quad e_p] \tag{9}$$

$$q_{max} = \max(\mathbf{q}), \quad q_{min} = \min(\mathbf{q}), \tag{10}$$

$$C = \frac{q_{max} - q_{min}}{q_{max}} \le \varepsilon, \tag{11}$$

where ε is a small positive number (say, 0.001). Here, C is a measure of the convergence determination in the IRWLS. C can be viewed as the measure of equirippleness. If the measure of convergence determination C satisfies equation (11), the algorithm terminates. If this is not the case, we update the weighting function as in equation (4). We show below the equiripple all-pass IIR filter design algorithm.

Algorithm

Step:1 set $n = 1$ and $w_n(\omega_k) = 1$ for $k = 1, 2, \cdots, M$
Step:2 Compute **Q**,**d** and solve **a**
Step:3 Evaluate the error function $e_\theta(\omega_k) = 1$ for k =1,2,$\cdots$, M
Step:4 Compute **q**, C
Step:5 If $C \le \varepsilon$, then exit
Step:6 Update a rectangular updating function as $w_{n+1}(\omega) = w_n(\omega)\beta_n(\omega)$
Step:7 set $n = n + 1$ and go to **Step:2**

4. DESIGN EXAMPLE

As mentioned in Section 2 and Section 3, we now show some examples of the filter design using the proposed weighting function for IRWLS (Below, we call it "the proposed method".). We now experimentally show cases where the convergence of a solution becomes unstable in some design examples. To evaluate the design accuracy, we use the following four types of evaluation criteria. The first is the maximum absolute phase error e_{max} as defined by the following equation

$$e_{max} = \max\{|e_\theta(\omega)|, \omega \in [0, \pi]\}. \tag{12}$$

The second is the normalized root-mean-square (NRMS) error $e_{\theta 2}$ as defined by the following equation

$$e_{\theta 2} = \left[\frac{\int_0^\pi e_\theta^2(\omega)\, \mathrm{d}\omega}{\int_0^\pi \theta_d^2(\omega)\, \mathrm{d}\omega} \right]^{1/2} \cdot 100\%. \tag{13}$$

The third is the design time (sec), and the fourth is the number of iterations required. Here, the design time is the result of the execution environment for the following. (CPU:Intel Core2 Quad CPU Q9550(2.83GHz), Memory:4GB, OS:Windows Vista[32bit])

4.1. Design example

We approximate the desired phase response $\theta_d(\omega)$ given by

$$\theta_d(\omega) = \begin{cases} -12\omega & (0 \le \omega \le 0.3\pi) \\ -\dfrac{6.4}{0.7}\omega + \left(\dfrac{6.4}{0.7} - 10\right)\pi & (0.3\pi < \omega \le \pi). \end{cases} \tag{14}$$

Here, the order is $N = 10$, and both ε and e_ε are set to 0.001.

4.1.1. Comparison of the basic performance

Fig.3 shows the result for the phase response using our method. Fig.4 shows the result of the phase error. Fig.5 shows the shape of the weighting function at the time of convergence. Furthermore, Table.1 shows the maximum absolute phase error e_{max}, NRMS error $e_{\theta 2}$, the design time, and the number of iterations for convergence. Since the conventional method [13] did not converge, it is not indicated in the table.

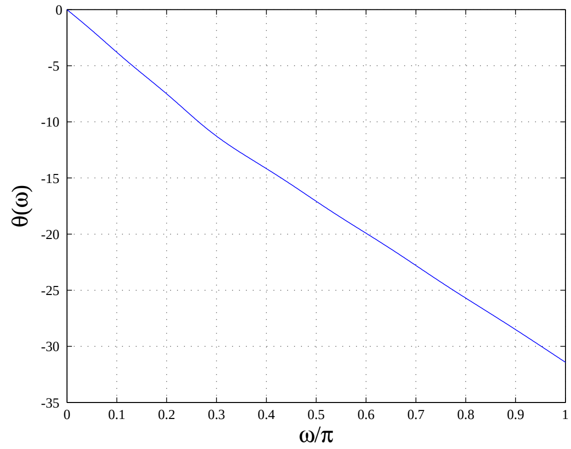

Fig. 3. Phase response using our method [Design example

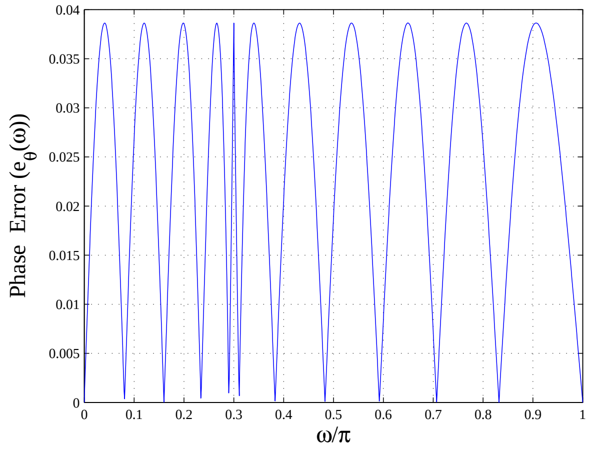

Fig. 4. Frequency response error using our method [Design example] : $e_\theta(\omega)$

From these figures and from the table, we can check the basic performance of our method. Compared with conventional methods, our method is able to achieve convergence in fewer iterations, which is the aim of convergence improvement. In the following subsection, we investigate in detail the improvement in the number of iterations required for convergence when using our method.

Table 1. e_{max}, $e_{\theta 2}$, The design time, the number of iterations for convergence [Design example] : our method, the conventional method ([13], [14])

Design example	e_{max}	$e_{\theta 2}$	The design time	The number of iterations
Our method	0.03863	0.14416	0.31200	13
Sunder[13]	Not convergent			
Yong[14]	0.03866	0.15197	2.26201	97

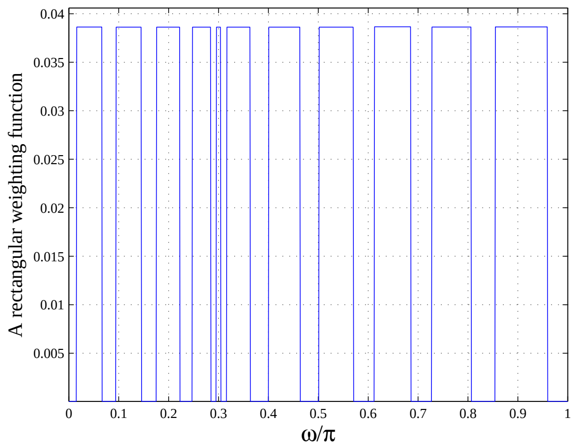

Fig. 5. A new weighting function [Design example] : $\beta_n(\omega_k)$

4.1.2. *Comparison of the convergence performance*

In order to compare the convergence, at the n-th iteration, we shall illustrate the measure of convergence determination C and the measure of change in the weighting function D. The measure of change in the weighting function D is defined by

$$D = |w_n(\omega) * w_n^T(\omega) - w_{n+1}(\omega) * w_{n+1}^T(\omega)|. \qquad (15)$$

Fig.6 shows the measure of convergence determination C for our method and for the conventional method. Fig.7 shows the measure of change in the weighting function D for ou method and for the conventional method. Fig.6 shows that our method is superior to the conventional method with respect to the number of iterations required to achieve convergence. Fig.7 shows that our method has quick standup capability towards convergence. These results show that our method is an effective method towards improving the number of iterations to achieve convergence.

5. CONCLUSION

We have proposed a new weighting function for designing equiripple all-pass IIR filters. The purpose of introducing this weighting function is to improve the convergence performance in the solution of the IRWLS. We have shown experimentally that the convergence performance in the solution of the IRWLS improves by using the proposed weighting func-

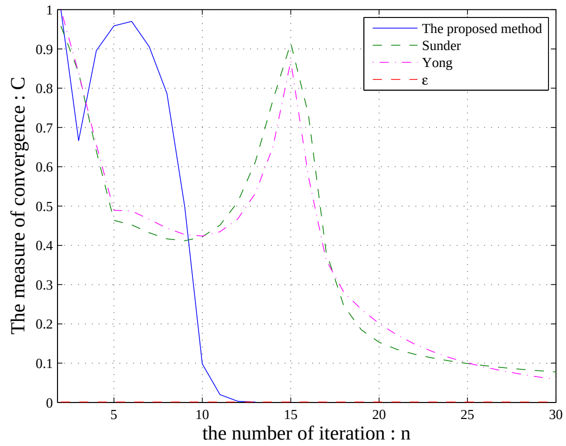

Fig. 6. The measure of convergence determination C [Design example] : ,Our method Sunder[13], Yong[14]

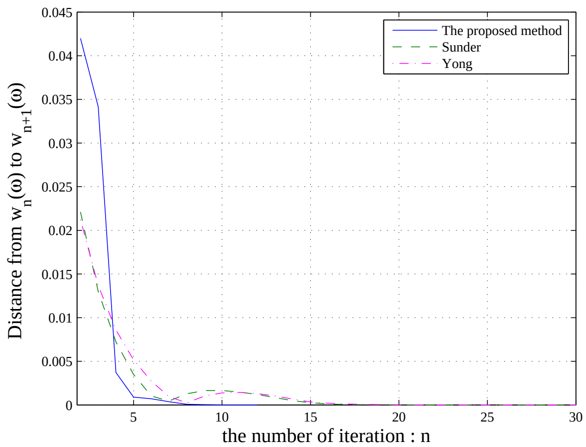

Fig. 7. The measure of change in the weighting function D [Design example] : Our method, Sunder[13], Yong[14]

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