ML ESTIMATE AND CRLB OF COVARIANCE MATRIX FOR COMPLEX ELLIPTICALLY SYMMETRIC DISTRIBUTION

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ABSTRACT

This paper derives the "constrained" maximum likelihood (ML) estimators and the Cramér-Rao Lower Bounds (CRLB) for the scatter matrix of Complex Elliptically Symmetric distributions and compares them in the particular cases of complex Gaussian, Generalized Gaussian (GG) and *t*-distributed observation vectors. Numerical results confirm the goodness of the ML estimators and the advantage of a constraint on the matrix trace for small data size.

Index Terms— Matrix estimation, CRLB, complex elliptically distribution.

1. INTRODUCTION

Complex Elliptically Symmetric (CES) distributions constitute a wide family of distributions whose complex Gaussian, Cauchy, Generalized Gaussian, compound-Gaussian, such as *K*-distribution and complex-*t*, are particular cases. The CES distributions are largely applied to model the disturbance in many areas, such as radar, sonar and communications [1-3]. In those applications where adaptive signal processing is performed, for channel estimation or target detection, the estimation of the disturbance covariance matrix is often required [4-7] and the issue of matrix estimation performance must be addressed. This paper focuses on the Maximum Likelihood (ML) estimate of the covariance matrix when the data vector is CES distributed and introduces its Cramér-Rao Lower Bounds (CRLBs). For a detailed and clear review of the CES model we refer the readers to the recent paper by Ollila *et al.* [8] and to the many references therein. Here we summarize only the main properties necessary for the following derivations.

A complex *N*-dimensional vector $\mathbf{z}$ is said CES distributed, in shorthand notation $\mathbf{z} \in CE_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, if its probability density function (PDF) is of the form

$$p(\mathbf{z}) = C_{N,g} \left|\boldsymbol{\Sigma}\right|^{-1} g\left(\left(\mathbf{z}-\boldsymbol{\mu}\right)^H \boldsymbol{\Sigma}^{-1}\left(\mathbf{z}-\boldsymbol{\mu}\right)\right) \tag{1}$$

where $g()$ is the density generator, $C_{N,g}$ is a normalizing constant, $\boldsymbol{\mu}$ is the mean vector and $\boldsymbol{\Sigma}$ is the normalized covariance matrix such that $tr(\boldsymbol{\Sigma}) = N$; for a vector extracted from a stationary process this means $\boldsymbol{\Sigma}_{i,i} = 1$ for $i = 1,\dots,N$. Matrix $\boldsymbol{\Sigma}$ is also called the scatter matrix. It is important to observe that for some CES distributions the un-normalized covariance matrix $\mathbf{M} = E\{\mathbf{z}\mathbf{z}^H\}$ does not exist, but the scatter matrix $\boldsymbol{\Sigma}$ is still well defined. Based upon the Stochastic Representation Theorem reported in [8] any $\mathbf{z} \in CE_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, with $rank(\boldsymbol{\Sigma}) = k \le N$, admits the stochastic representation $\mathbf{z} =_d \boldsymbol{\mu} + R\mathbf{A}\mathbf{u}$, where the non-negative random variable R is called the modular variate, $\mathbf{u}$ is a *k*-dimensional vector uniformly distributed on the unit hypersphere with *k*-1 topological dimensions such that $\mathbf{u}^H\mathbf{u} = 1$, R and $\mathbf{u}$ are independent and $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^H$ is a factorization of $\boldsymbol{\Sigma}$, where $\mathbf{A}$ is a complex *N*x*k* matrix and $rank(\mathbf{A}) = k$. In the following derivations we suppose that $\boldsymbol{\Sigma}$ is full-rank, then $rank(\mathbf{A}) = rank(\boldsymbol{\Sigma}) = N$.

1.1 Particular cases: Complex *t*-distribution

A complex *N*-dimensional vector $\mathbf{z}$ is *t*-distributed with zero mean value and normalized covariance matrix $\boldsymbol{\Sigma}$ if its PDF is given by:

$$p(\mathbf{z}) = \frac{1}{\pi^N \left|\boldsymbol{\Sigma}\right|} \frac{\Gamma(N+\lambda)}{\Gamma(\lambda)} \left(\frac{\lambda}{\eta}\right)^{\lambda} \left(\frac{\lambda}{\eta} + \mathbf{z}^H \boldsymbol{\Sigma}^{-1}\mathbf{z}\right)^{-(N+\lambda)} \tag{2}$$

where λ is the shape parameter and η is the scale parameter characterizing the distribution. The complex *t*-distribution is a particular case of CES distribution when $R^2 = Q$ is a r.v. independent of $\mathbf{u}$ and it is *F*-distributed such that

$$p(q) = \frac{1}{B(N,\lambda)} q^{N-1} \left(\frac{\lambda}{\eta}\right)^{\lambda} \left(\frac{\lambda}{\eta} + q\right)^{-(N+\lambda)} \tag{3}$$

and $B(N,\lambda) = (N-1)!\,\Gamma(\lambda)\Gamma^{-1}(N+\lambda)$. It is evident that in this case the density generator is $g(t) = \left(\lambda/\eta + t\right)^{-(N+\lambda)}$, where $t = \mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{z}$. This distribution belongs as well to the family of compound-Gaussian, a subclass of CES distribution, and can be obtained with a complex Gaussian distributed speckle and an inverse Gamma (IG) distributed texture. It is quite often used to model the sea clutter data [9].

1.2 Multivariate Generalized Gaussian Distribution

A complex N-dimensional vector $\mathbf{z}$ has a multivariate Generalized Gaussian distribution (MGGD) with zero mean and normalized covariance matrix $\mathbf{\Sigma}$ if its PDF is given by

$$p(\mathbf{z};\mathbf{\Sigma}) = C_N \left|\mathbf{\Sigma}\right|^{-1} \exp\left[-\left(\mathbf{z}^H \mathbf{\Sigma}^{-1}\mathbf{z}\right)^{\beta} \big/ b\right] \qquad (4)$$

where b is the scale parameter, β the shape parameter and C_N a normalization constant. The r.v. R, in distribution, is equivalent to $R =_d G^{1/2\beta}$ where G is a gamma r.v. with scale parameter b and shape parameter $\nu = N/\beta$. In the scalar case (N=1) the PDF of R is:

$$p_R(r) = 2\beta b^{-\frac{1}{\beta}} \Gamma^{-1}\left(\beta^{-1}\right) r \cdot \exp\left(-\frac{r^{2\beta}}{b}\right) u(r) \qquad (5)$$

and $E\left\{R^2\right\} = \Gamma\left(2/\beta\right) b^{1/\beta} \Gamma^{-1}\left(1/\beta\right)$. In this case the density generator is $g(t) = \exp(-t^{\beta}/b)$.

2. ML ESTIMATORS OF THE SCATTER MATRIX

To estimate the covariance matrix (or the scatter matrix) of the observed vector under test $\mathbf{z}$ (primary vector), generally in both radar and communications applications, it is supposed to dispose of $K>N$ secondary data vectors, independent and identically distributed (i.i.d.), that share with the primary vector the same statistical characteristics, then the covariance matrix as well [3-6]. Using these secondary vectors $\left\{\mathbf{z}_i\right\}_{i=1}^{K}$ to build the maximum likelihood function, the ML estimate of $\mathbf{\Sigma}$ for CES distribution is given by the solution of the ML equation [3]:

$$\mathbf{\Sigma} = \frac{1}{K}\sum_{k=1}^{K} \varphi\left(\mathbf{z}_k^H \mathbf{\Sigma}^{-1}\mathbf{z}_k\right)\mathbf{z}_k\mathbf{z}_k^H \qquad (6)$$

where $\varphi(t) \triangleq -g'(t)\, g^{-1}(t)$ is the weight function depending on the shape of the density generator g, and $g'(t) = dg/dt$. Generally eq. (7) is implicit, except in the Gaussian case where $\varphi(t) = 1/\sigma^2$, where σ^2 is the variance of the Gaussian distribution, and the estimator, up to a scalar factor $1/\sigma^2$, is the popular Sample Covariance Matrix (SCM) estimator, and it must be solved iteratively. Proofs of uniqueness and convergence of solution (6) for some CES distributions are in [8].

2.1 ML for complex *t*-distribution

From eq. (6) we can derive that the ML estimate of the scatter matrix in this case is given by the solution of the following recursive equation:

$$\hat{\mathbf{\Sigma}}_{n+1} = \frac{N+\lambda}{K}\sum_{k=1}^{K} \frac{\mathbf{z}_k\mathbf{z}_k^H}{\lambda/\eta + \mathbf{z}_k^H \hat{\mathbf{\Sigma}}_n^{-1}\mathbf{z}_k} \qquad (7)$$

Since by definition for the scatter matrix we have $tr\left(\mathbf{\Sigma}\right) = N$, we can introduce the constraint on the trace in the iteration to calculate the ML estimation, as proposed in [10], so yielding:

$$\hat{\mathbf{\Sigma}}_{n+1} = \sum_{k=1}^{K} \frac{\mathbf{z}_k\mathbf{z}_k^H}{\lambda/\eta + \mathbf{z}_k^H \hat{\mathbf{\Sigma}}_n^{-1}\mathbf{z}_k} \text{ with } tr\left(\hat{\mathbf{\Sigma}}_{n+1}\right) = N \qquad (8)$$

2.2 ML for MGGD

The constrained ML estimate of the scatter matrix in this case is given by the unique solution of [11]:

$$\hat{\mathbf{\Sigma}}_{n+1} = \sum_{k=1}^{K} \frac{\mathbf{z}_k\mathbf{z}_k^H}{\left(\mathbf{z}_k^H \mathbf{\Sigma}_n^{-1}\mathbf{z}_k\right)^{1-\beta}} \quad \text{with } tr\left(\hat{\mathbf{\Sigma}}_{n+1}\right) = N \qquad (9)$$

3. CRLB ON COVARIANCE MATRIX ESTIMATION

Supposing that $\mathbf{\Sigma}$ depends on a vector of M real unknown parameters $\mathbf{\theta} = [\theta_1,\ldots\theta_M]^T$ that we want to estimate, we can calculate the CRLB($\mathbf{\theta}$) by inverting the Fisher Information Matrix (FIM) whose elements are given by

$$\left[\mathbf{J}\right]_{i,j} = E\left\{\frac{\partial \ln p(\mathbf{z})}{\partial\theta_i}\frac{\partial \ln p(\mathbf{z})}{\partial\theta_j}\right\} \qquad (10)$$

Moreover, we suppose here that the only unknown parameters in the data vector distribution relate to the scatter matrix, whereas the shape and scale parameters of the distribution are known.

3.1 CRLB for *t*-distributed CES

Using matrix and derivative properties [12] we can verify that in this case,

$$\frac{\partial \ln p(\mathbf{z})}{\partial\theta_i} = -tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right) + \frac{N+\lambda}{\lambda/\eta + \mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}}\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{z} \qquad (11)$$

where we set $\partial\mathbf{\Sigma}/\partial\theta_i = \mathbf{A}_i$, for $i = 1,\ldots,M$. Then, after some algebra, we obtain

$$\begin{aligned}\left[\mathbf{J}\right]_{i,j} &= tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right)tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right)\\ &-\left(N+\lambda\right)tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right)E\left\{\frac{\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_j\mathbf{\Sigma}^{-1}\mathbf{z}}{\lambda/\eta + \mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}}\right\}\\ &-\left(N+\lambda\right)tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right)E\left\{\frac{\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{z}}{\lambda/\eta + \mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}}\right\}\\ &+\left(N+\lambda\right)^2 E\left\{\frac{\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{z}\,\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_j\mathbf{\Sigma}^{-1}\mathbf{z}}{\left(\lambda/\eta + \mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}\right)^2}\right\}\end{aligned} \qquad (12)$$

In the Appendix we verified that eq. (12) yields to

$$[\mathbf{J}]_{i,j} = -\frac{1}{N+1+\lambda} tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right) tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) + \frac{N+\lambda}{N+1+\lambda} tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) \tag{13}$$

As verified in [9] for $\lambda \to +\infty$ the PDF tends to the multivariate Gaussian PDF. In this case, as expected [12]:

$$[\mathbf{J}]_{i,j} = tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) \tag{14}$$

3.2 CRLB for MGGD

In this case we have

$$\begin{aligned}[\mathbf{J}]_{i,j} &= tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right) tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) \\ &-\frac{\beta}{b} tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right) E\left\{\left(\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}\right)^{\beta-1}\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_j\mathbf{\Sigma}^{-1}\mathbf{z}\right\} \\ &-\frac{\beta}{b} tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) E\left\{\left(\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}\right)^{\beta-1}\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{z}\right\} \\ &+\frac{\beta^2}{b^2} E\left\{\left(\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{z}\right)^{2\beta-2}\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{z}\,\mathbf{z}^H\mathbf{\Sigma}^{-1}\mathbf{A}_j\mathbf{\Sigma}^{-1}\mathbf{z}\right\}\end{aligned} \tag{15}$$

In the Appendix we prove that

$$\begin{aligned}[\mathbf{J}]_{i,j} &= -\left(1-\frac{\beta+N}{(N+1)}\right) tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\right) tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_j\right) \\ &+\frac{\beta+N}{N+1} tr\left(\mathbf{\Sigma}^{-1}\mathbf{A}_i\mathbf{\Sigma}^{-1}\mathbf{A}_j\right)\end{aligned} \tag{16}$$

In the complex Gaussian case $\beta = 1$. Hence, as expected, again the FIM is given by (14).

4. NUMERICAL ANALYSIS

To investigate numerically the behaviour of the CRLBs and to compare it with the accuracy of the "constrained" ML estimator we considered an AR(1) stationary process with normalized covariance function $C(m) = \rho^{|m|}$, where ρ is the one-lag correlation coefficient. We investigated different cases with different levels of a-priori knowledge on the scatter matrix, that is real. For lack of space, we report here only the results for the following cases: (C1) *Case* 1: Completely unknown matrix. In this case the unknown parameters are all the elements of the scatter matrix, thus M=N^2; (C2) *Case* 2: Completely unknown matrix with the constraint on the trace that $tr(\mathbf{\Sigma}) = N$, that amounts to say that the last element on the main diagonal is known to be $[\mathbf{\Sigma}]_{N,N} = N - \sum_{i=1}^{N-1}[\mathbf{\Sigma}]_{i,i}$. In this case M=N^2-1. To analyze the impact of the constraint on the trace, we consider first the easiest CES model, the complex Gaussian one and we compare the square root of the CRLBs (RCRLBs) in the case C1 (C1-G) and C2 (C2-G) of $\mathbf{\Sigma}_{11}$ and $\mathbf{\Sigma}_{12}$ with the Root Mean Square Error (RMSE) of the same matrix elements of the Sample Covariance Matrix (SCM), the well known ML estimate on Gaussian covariance matrix [12], and the SCM where we impose the constrain $tr(\hat{\mathbf{\Sigma}}) = N$. The constrained SCM is named SCMNT in the figures.

In Fig.1 both CRLB and RMSE refer to the element $\mathbf{\Sigma}_{12} = \rho$, in Fig. 2 we refer to $\mathbf{\Sigma}_{11} = 1$. On the *x*-axis there is the number of samples, that is the dimension of the vector **z,** that spans in the interval [4-30]. If not otherwise stated, $\rho = 0.9$ and K=2N. The gain in constraining the trace is evident for both estimator and bound, particularly for low values of N. The initial increasing in the behaviour of both RMSE and RCRLB must not be surprising, since, with increasing N the number of unknowns is increasing as well.

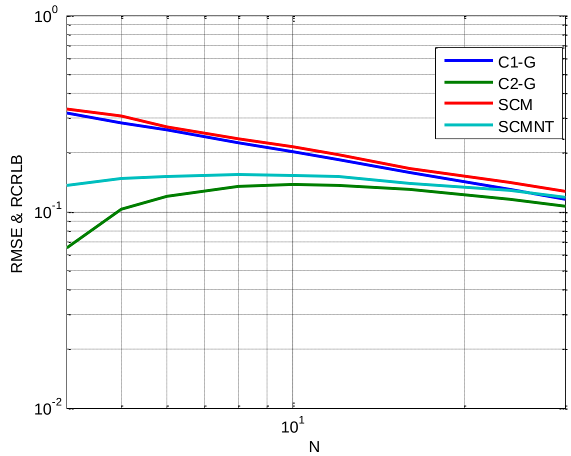

Fig.1 - CRLBs and RMSEs for $\mathbf{\Sigma}_{12} = \rho$, complex Gaussian process.

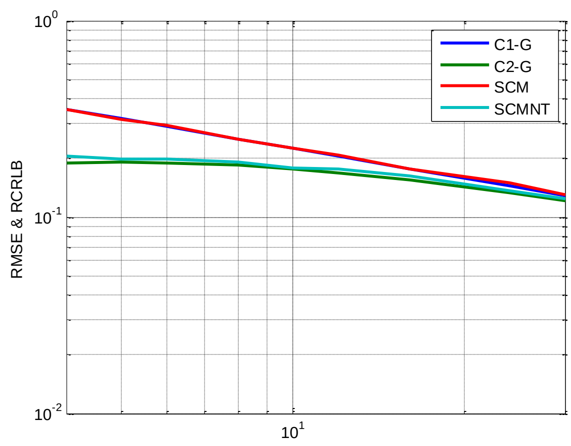

Fig.2 - CRLBs and RMSEs for $\mathbf{\Sigma}_{11} = 1$, complex Gaussian process.

In Figs 3 and 4 we plot the RMSEs of the constrained ML estimator of $\hat{\mathbf{\Sigma}}_{11}$ and $\hat{\mathbf{\Sigma}}_{12}$ and the RCRLBs in the case of *t*-distributed data. The number of iteration to solve eq. (8) has been set to N_{it}=3, and the matrix estimator has been initialized using the SCMNT estimate. The performance is quite similar to the Gaussian case. C1-T and C2-T are the RCRLBs for C1 and C2 with *t*-distributed data and the

RMSEs are quite close to the RCRLBs C2-T for both elements.

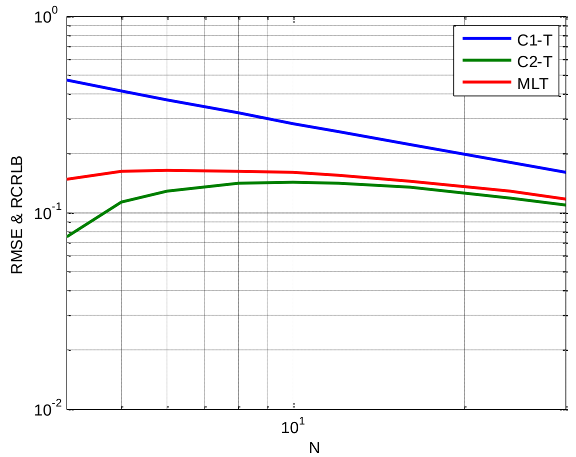

Fig.3 - CRLBs and RMSEs for $\boldsymbol{\Sigma}_{12} = \rho$, complex *t*-distribution.

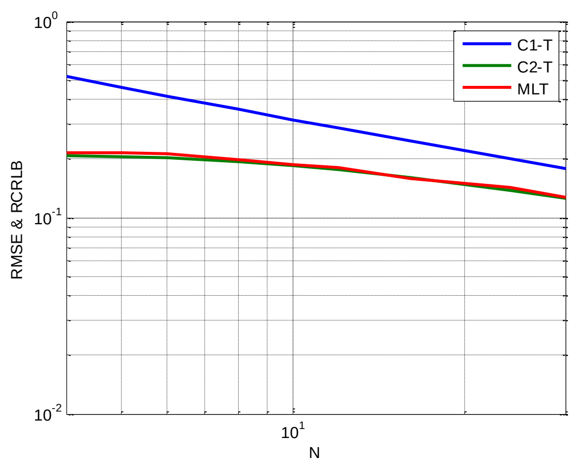

Fig.4 - CRLBs and RMSEs for $\boldsymbol{\Sigma}_{11} = 1$, complex *t*-distribution.

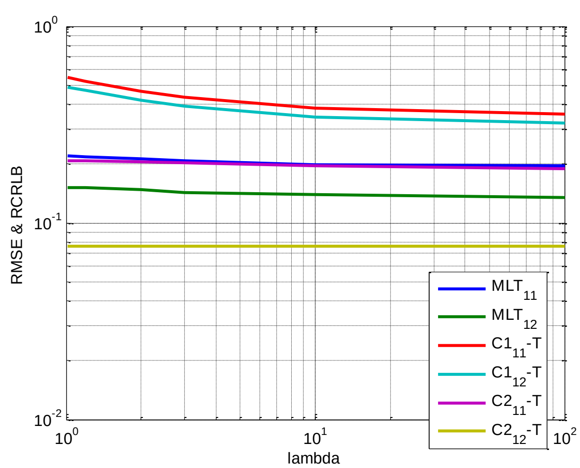

Fig.5 - CRLBs and RMSEs vs λ, complex *t*- distribution.

For gaining a deeper insight into the constrained ML matrix estimator for the *t*-distributed process, in Fig. 5 we plot the RCRLBs and the RMSEs as a function of the shape parameter λ, for N=8 and K=16. In this figure the subscript "11" refers to the element $\boldsymbol{\Sigma}_{11}$, and "12" to $\boldsymbol{\Sigma}_{12}$, C1 and C2 refer to the RCRLBs in the two analyzed cases, "T" denotes the *t*-distribution, and ML for the RMSEs of the ML estimator.

The variations of RMSEs and CRLBs are generally very slow as a function of λ, particularly for the constrained case, meaning that the ML estimator is almost insensitive to the variations of this parameter. The analyses performed on the MGGD process have shown pretty similar results and are omitted here for lack of space.

5. CONCLUSIONS

In this paper we have analyzed the MLEs and the CRLBs for the scatter matrix of CES distributions, focusing in particular on the *t*-distribution and the MGG-distribution. We provided closed form expressions of the FIM in both cases and we compared some numerical results for the RCRLBs and RMSEs in the case of Gaussian (MGG with β=1) and *t*-distributed complex processes. The results show that: (*i*) There is always a gain in constraining the trace, particularly for low values of N; (*ii*) The performances of the "constrained " SCM in the Gaussian case are pretty close to those of the constrained MLE for the *t*-distribution. (*iii*) The MLE for the *t*-distribution is almost insensitive to the variations of the distribution shape parameter, which is a measure of data non-Gaussianity.

6. REFERENCES

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APPENDIX - CRLBs calculation

Before proceeding in the calculation of CRLBs for *t*-distributed CES of eq. (15), we must notice that a white, unit power, Gaussian distributed vector $\mathbf{w}$ is a particular case of the CES distributed vector, with $\boldsymbol{\Sigma}=\mathbf{I}$ and amplitude $R=R_w$ central χ^2 -distributed random variable of order $2N$, scale parameter $b=1/2$ such that $E\left\{R_w^2\right\}=2Nb=N$ and $E\left\{R_w^4\right\}=N\left(N+1\right)$. Based upon this property, we can write in general that $\mathbf{u}=\mathbf{w}/R_w$, then $\mathbf{z}=R\boldsymbol{\Lambda}\mathbf{u}=R\boldsymbol{\Lambda}\mathbf{w}R_w^{-1}$, where $\mathbf{z}$ is independent of R_w. Observing that $\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{z}=R^2\mathbf{u}^H\boldsymbol{\Lambda}^H\boldsymbol{\Lambda}^{-H}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}\mathbf{u}=R^2\mathbf{u}^H\mathbf{u}=R^2$, since $\mathbf{u}^H\mathbf{u}=1$, and calling $\mathbf{x}=\boldsymbol{\Lambda}\mathbf{u}$, we obtain

$$E\left\{\frac{\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{z}}{\lambda/\eta+R^2}\right\}=E\left\{\frac{R^2}{\lambda/\eta+R^2}\right\}E\left\{\mathbf{x}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{x}\right\} \quad \text{(A.1)}$$

where we used the independence of R of $\mathbf{x}$. If we introduce the vector $\mathbf{t}=\boldsymbol{\Lambda}\mathbf{w}=R_w\boldsymbol{\Lambda}\mathbf{u}\in CN\left(\mathbf{0},\boldsymbol{\Sigma}\right)$, we can use the relation $E\left\{\mathbf{t}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{t}\right\}=tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)$ [Kay] and, observing that $E\left\{\mathbf{t}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{t}\right\}=E\left\{R_w^2\right\}E\left\{\mathbf{x}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{x}\right\}$, we obtain

$$E\left\{\mathbf{x}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{x}\right\}=\frac{E\left\{\mathbf{t}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{t}\right\}}{E\left\{R_w^2\right\}}=\frac{1}{N}tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right) \quad \text{(A.2)}$$

Similarly, we observe that

$$E\left\{\frac{\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{z}\,\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{z}}{\left(\lambda/\eta+\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{z}\right)^2}\right\}=E\left\{\frac{R^4}{\left(\lambda/\eta+R^2\right)^2}\right\}\frac{\left[tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\right)tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)+tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)\right]}{N(N+1)} \quad \text{(A.3)}$$

where we used the property of complex Gaussian vectors $E\left\{\mathbf{t}^H\mathbf{C}\mathbf{t}\,\mathbf{t}^H\mathbf{D}\mathbf{t}\right\}=tr\left(\mathbf{C}\boldsymbol{\Sigma}\right)tr\left(\mathbf{D}\boldsymbol{\Sigma}\right)+tr\left(\mathbf{C}\boldsymbol{\Sigma}\mathbf{D}\boldsymbol{\Sigma}\right)$, where $\mathbf{C}$ and $\mathbf{D}$ are Hermitian matrices and again $\boldsymbol{\Sigma}=E\left\{\mathbf{t}\mathbf{t}^H\right\}$.

Replacing eqs (17-18) in eq. (15) we obtain the FIM

$$[\mathbf{J}]_{i,j}=\begin{bmatrix}1-2\dfrac{N+\lambda}{N}E\left\{\dfrac{R^2}{\left(\lambda/\eta+R^2\right)}\right\}\\+\dfrac{\left(N+\lambda\right)^2}{N\left(N+1\right)}E\left\{\dfrac{R^4}{\left(\lambda/\eta+R^2\right)^2}\right\}\end{bmatrix}tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\right)tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)+\frac{\left(N+\lambda\right)^2}{N\left(N+1\right)}E\left\{\frac{R^4}{\left(\lambda/\eta+R^2\right)^2}\right\}tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right) \quad \text{(A.4)}$$

Now we notice that

$$E\left\{\frac{R^2}{\left(\lambda/\eta+R^2\right)}\right\}=E\left\{\frac{q}{\lambda/\eta+q}\right\}=\frac{B\left(N+1,\lambda\right)}{B\left(N,\lambda\right)}=\frac{N}{N+\lambda} \quad \text{(A.5)}$$

Similarly

$$E\left\{\left(\frac{R^2}{\lambda/\eta+R^2}\right)^2\right\}=\frac{B\left(N+2,\lambda\right)}{B\left(N,\lambda\right)}=\frac{N\left(N+1\right)}{\left(N+\lambda\right)\left(N+1+\lambda\right)} \quad \text{(A.6)}$$

Replacing (A.5-6) in (A.4) yields the closed form FIM of eq. (13).

With similar observations as in the t-distribution case we obtain, in the MGGD case, that

$$E\left\{\left(\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{z}\right)^{\beta-1}\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{z}\right\}=E\left\{R^{2\beta}\right\}tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)/N \quad \text{(A.7)}$$

and

$$E\left\{\left(\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{z}\right)^{2\beta-2}\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{z}\,\mathbf{z}^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\boldsymbol{\Sigma}^{-1}\mathbf{z}\right\}=\frac{E\left\{R^{4\beta}\right\}}{N(N+1)}\left[tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\right)tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)+tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)\right] \quad \text{(A.8)}$$

Replacing eqs (A.7-8) in eq. (A.4) we obtain the FIM

$$[\mathbf{J}]_{i,j}=\begin{bmatrix}1-\dfrac{2\beta}{Nb}E\left\{R^{2\beta}\right\}\\+\dfrac{\beta^2}{N\left(N+1\right)b^2}E\left\{R^{4\beta}\right\}\end{bmatrix}tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\right)tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right)+\left[\frac{\beta^2}{N\left(N+1\right)b^2}E\left\{R^{4\beta}\right\}\right]tr\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}_i\boldsymbol{\Sigma}^{-1}\mathbf{A}_j\right) \quad \text{(A.9)}$$

Observing that $E\left\{R^{2\beta}\right\}=E\left\{G\right\}=Nb/\beta$, $E\left\{R^{4\beta}\right\}=E\left\{G^2\right\}=Nb^2\beta^{-1}+N^2b^2\beta^{-2}$, and replacing these relations in eq. (A.4) we obtain the FIM in eq. (16).