HANKEL STRUCTURED MATRIX RANK MINIMIZATION APPROACH TO SIGNAL DECLIPPING

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ABSTRACT

This paper proposes a new algorithm for the restoration of the clipped signal based on the structured matrix rank minimization. We assume that the signal is modeled by deterministic autoregressive model with unknown model order and propose the matrix rank minimization approach to recover the clipped signal. The main result of this paper is to formulate the signal declipping problem as the Hankel structured matrix rank minimization problem with inequality constraint and to provide an algorithm to solve this problem by modifying the null space based alternating optimization (NSAO) algorithm. Numerical examples show that the proposed algorithm recovers the clipping signal efficiently.

Index Terms— signal restoration, signal declipping, matrix rank minimization, compressed sensing

1. INTRODUCTION

This paper proposes a signal declipping algorithm based on the matrix rank minimization approach. Signal clipping is a signal distortion process as illustrated in Fig. 1 and is occurred when the signal goes beyond the dynamic range of the system (also known as the clipping level). Several algorithms have been proposed for the signal declipping [1, 2, 3, 4]. In [1] the double sides period substitution method is proposed for packet voice communications. In [2] the signal declipping algorithm is proposed based on the assumption that the signal is modeled by the autoregressive (AR) model (and more statical models). In [3, 4] the sparse representation based signal declipping algorithm is proposed using the orthogonal matching pursuit (OMP). The performance of the algorithm [3] has better than that of other conventional algorithms such as [2] when a desirable overcomplete dictionary is given. However, we focus on the AR model based signal declipping and proposed the matrix rank minimization approach because the performance of the OMP based algorithm highly depends on the given dictionaries.

Similarly to [2], this paper assumes that the signal is modeled by a numerical model and takes a rank minimization approach proposed in [5], where the signal is modeled by the autoregressive-moving average with exogenous terms (ARMAX) model and is recovered by estimating the model order to achieve the video inpainting. In this approach, the signal recovery problem is formulated as the Hankel structured matrix rank minimization problem, and the signal is recovered by minimizing the matrix rank. The advantage of the rank minimization approach is that the signal is restored even if both the coefficients and the model order are unknown. Therefore this paper proposes the matrix rank minimization approach to signal declipping problems.

Although the rank minimization problem is NP hard in general, several useful and practical algorithms are proposed to obtain its approximate solution [6, 7, 8, 9, 10]. This paper utilizes and modifies the null space based alternating optimization (NSAO) algorithm proposed in [7], where a low-rank solution is provided by optimizing the null space matrix. The advantage of the NSAO algorithm is suitable for parallel computing. In [7], this algorithm is implemented on parallel GPGPU, and numerical experiments indicates its high computing efficiency.

The contribution of this paper is to modify the NSAO algorithm to guarantee that the solution matrix has the Hankel structure and satisfies the inequality constraints, and to provide a matrix rank minimization based signal declipping algorithm.

2. MAIN RESULTS

2.1. Problem formulation

We define the undistorted signal $\boldsymbol{s} = [s_1\ s_2 \dots s_L]^T \in \boldsymbol{R}^L$ and the observed signal with clipping $\boldsymbol{y} = [y_1\ y_2 \dots y_L]^T \in \boldsymbol{R}^L$, where y_i is described by the distortion function g_C as

$$y_i = g_C(s_i) = \begin{cases} s_i & \text{if } -C \leq s_i \leq C \\ C & \text{if } C < s_i \\ -C & \text{if } s_i < -C \end{cases}, \qquad (1)$$

where C is a constant corresponding to the clipping level. Let us assume that the signal is modeled by the following model,

$$s_i = \sum_{j=1}^{r} a_j s_{i-j}, \qquad (2)$$

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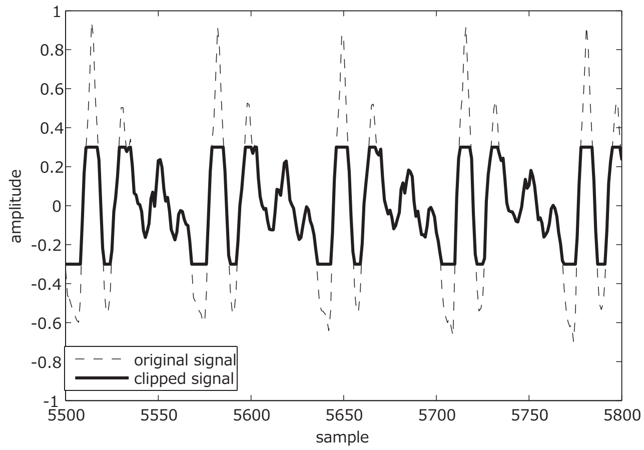


Fig. 1. An example of the original speech signal and the clipped signal. Clipping level is 0.3.

where r denotes the model order. In order to simplify the discussion, we consider here a noiseless case. In [2] the AR model is utilized for the signal declipping and dequantization, and this paper proposes a declipping algorithm based on the model similarly. As in [5], we take an approach to recover the signal by estimating the model order instead of estimating a_j, that is, we recover the unclipped samples by estimating the signal such that has a proper model order.

Let us define the Hankel matrix S by

$$S = \begin{bmatrix} s_1 & s_2 & \cdots & s_N \\ s_2 & s_3 & \cdots & s_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_N & s_{N+1} & \cdots & s_{2N-1} \end{bmatrix} \in \boldsymbol{R}^{N \times N}. \tag{3}$$

Since it holds that $\textbf{rank} S = n$ if and only if the model order equals n, this paper proposes the following matrix rank minimization problem to estimate the model order and to recover the declipped samples for given y_i, $i = 1, \ldots, 2N-1$,

$$\begin{array}{lll} \text{Minimize} & \textbf{rank}\hat{S} & \\ \text{subject to} & \hat{S} \in \mathcal{H}, & \\ & C \leq \hat{S}_{i,j} & \text{for } i+j-1 \in \Pi^+, \\ & \hat{S}_{i,j} \leq -C & \text{for } i+j-1 \in \Pi^-, \\ & \hat{S}_{i,j} = y_i & \text{for } i+j-1 \notin \Pi^+ \cup \Pi^-\}, \end{array} \tag{4}$$

where $\mathcal{H}$ denotes the set of matrices with the Hankel structure defined in (3), $\hat{S}_{i,j}$ denotes the (i,j)-element of matrix $\hat{S}$, and Π^+ and Π^- denote the index sets of positive and negative clipped samples in y_i. This problem is difficult to obtain the exact solution because the matrix rank minimization problem is NP hard. In order to solve (4) approximately, this paper proposes a Hankel structured matrix rank minimization algorithm using the null space based alternating optimization algorithm.

2.2. The Null Space Based Alternating Optimization Algorithm

This subsection presents the null space based alternating optimization (NSAO) algorithm proposed in [7]. Let us consider the following matrix rank minimization problem,

$$\text{Minimize } \textbf{rank} Z \quad \text{subject to } Z \in \Omega \subset \boldsymbol{R}^{m_z \times n_z}, \tag{5}$$

where Z is an optimization matrix, $m_z \leq n_z$, and Ω is a given convex set. Because the problem of minimizing the matrix rank is equal to the problem of maximizing its nullity, (5) is equal to the following matrix rank maximization problem,

$$\text{Maximize } \textbf{rank} W \quad \text{subject to } ZW = \mathbf{0}_{m_z, n_z},\ Z \in \Omega, \tag{6}$$

where $W \in \boldsymbol{R}^{n_z \times n_z}$ and Z are variable matrices, and $\mathbf{0}_{m_z,n_z}$ denotes the $m_z \times n_z$ zero matrix. Based on the fact that we can maximize the rank of W by minimizing $\|W\|_F^2$ under only the constraints $W_{i,i} = 1$ for all i, where $\|\cdot\|_F$ denotes the Frobenius-norm. In [7] proposes the following problem to obtain an approximate solution of (5),

$$\begin{array}{ll} \text{Minimize} & \|W\|_F^2 \\ \text{subject to} & ZW = \mathbf{0}_{m_z,n_z},\ W_{ii} = 1\, \forall i,\ Z \in \Omega. \end{array} \tag{7}$$

and provides NSAO-GPM (Algorithm 2 of [7]).

2.3. Signal Declipping Algorithm

Let us define $\mathcal{I}$ and $\mathcal{W}$ by

$$\begin{aligned} \mathcal{I} = \{ & \hat{S} \in \boldsymbol{R}^{N \times N} : \\ & C \leq \hat{S}_{i,j} \text{ for } i+j-1 \in \Pi^+, \\ & \hat{S}_{i,j} \leq -C \text{ for } i+j-1 \in \Pi^-, \\ & \hat{S}_{i,j} = y_i \text{ for } i+j-1 \notin \Pi^+ \cup \Pi^-\}, \end{aligned} \tag{8}$$

and

$$\mathcal{W} = \{W \in \boldsymbol{R}^{N \times N} : W_{i,i} = 1\}. \tag{9}$$

The set $\mathcal{I}$ corresponds to the inequality constraints in (4). Letting $Z = \hat{S}$ and $\Omega = \mathcal{H} \cap \mathcal{I}$ in the NSAO algorithm, the re;axed problem of (4) is obtained as follows,

$$\begin{array}{ll} \text{Minimize} & \|W\|_F^2 \\ \text{subject to} & \hat{S}W = \mathbf{0}_{m_z,n_z},\ W \in \mathcal{W},\ \hat{S} \in \mathcal{I} \cap \mathcal{H}. \end{array} \tag{10}$$

If the sequence s_i is completely modeled by the AR model, (10) has a solution such that $\hat{S}W = 0$. However, s_i usually contains a model error, and therefore there seldom exists W such that $\hat{S}W = 0$. Hence we deal with the following relaxed problem instead of (4),

$$\begin{array}{ll} \text{Minimize} & \gamma\|W\|_F^2 + \|\hat{S}W\|_F^2 \\ \text{subject to} & W \in \mathcal{W}, \\ & \hat{S} \in \mathcal{I} \cap \mathcal{H} \end{array} . \tag{11}$$

In the NSAO-GPM of [7], it is hard to compute the projection P_Ω exactly, and therefore this paper proposes the modified NSAO-GPM algorithm as shown in Algorithm 1. In this algorithm, $\hat{S}$ is projected on $\mathcal{I}$ after its update on $\mathcal{H}$ and is always included in $\mathcal{H} \cap \mathcal{I}$.

Although this algorithm is not the GPM exactly, it guarantees that $\hat{S}$ remains in $\mathcal{I} \cap \mathcal{H}$ in each iteration, and it tales a low computational cost.

Next we focus on the computations of projected gradient matrices F_S and projection $P_\mathcal{I}$. Since the initial value of $\hat{S}$ is a Hankel matrix and $P_\mathcal{H}$ is the projection on $\mathcal{H}$, $\hat{S}$ and F_S remain to have the Hankel structure. Therefore F_S can be described as

$$F_S = \begin{bmatrix} f_1 & f_2 & \dots & f_N \\ f_2 & f_3 & \dots & f_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_{N+1} & \dots & f_{2N-1} \end{bmatrix}. \tag{12}$$

Because F_s is obtained as the least squares solution of the simultaneous equation $F_s = -D_s$, f_l is the least squares solution of the following equation,

$$f_l [1\,1 \dots 1]^T = -\boldsymbol{d}_l. \tag{13}$$

where $\boldsymbol{d}_l \in \boldsymbol{R}^l$ is the vector defined by

$$\boldsymbol{d}_l = [d_{1,l}\ d_{2,l-1}\ d_{3,l-2}\ \dots\ d_{l,1}]^T,$$

and $d_{i,j}$ denotes the (i,j)-element of the gradient matrix D_S, D_S in this subsection is equal to D_Z in [7]. Then we can obtain f_l simply as $f_l = -\frac{1}{l}\sum_{i+j-1=l} d_{i,j}$.

Next we move onto the calculation of $P_\mathcal{I}(S)$. Since it is difficult to compute $P_\mathcal{I}(S)$ exactly, this paper proposes a simple approximation of the projection $\hat{S} = P_\mathcal{I}(\bar{S})$ as follows,

$$\hat{S}_{i,j} = \begin{cases} C & \text{if } q \in \Pi^+ \text{ and } \bar{S}_{i,j} < C \\ -C & \text{if } q \in \Pi^- \text{ and } -C < \bar{S}_{i,j} \\ \bar{S}_{i,j} & \text{if otherwise} \end{cases}, \tag{14}$$

where $\bar{S}_{i,j}$ denotes the (i,j)-element of $\bar{S}$, $q = i+j-1$. This algorithm fixes to satisfy the inequality constraint.

Though Algorithm 1 utilizes a rough approximation to compute P_Ω, it provides a good solution, which can be seen in the next section.

3. NUMERICAL EXAMPLES

This section presents numerical examples for the proposed algorithm. We utilize the 4 kind of 6 second speech signal (sampling frequency = 16 kHz, $L = 95612$ samples) of University of Tsukuba Multilingual (UT-ML) Speech Corpus, which is available at the web site[1]. We use $\gamma = 1$ and $\varepsilon = 10^{-4}$, which achieve the best performance. We separate 95612 samples into 212 blocks consisting of 451 samples without overlapping and apply the proposed algorithm 212 times.

[1] http://research.nii.ac.jp/src/eng/list/index.html

Algorithm 1 Proposed signal declipping algorithm.

Require: $\boldsymbol{y}$, $\gamma > 0$, $\varepsilon > 0$
  Construct Y from $\boldsymbol{y}$.
  Set $\hat{S} \leftarrow Y$.
  repeat
    $\hat{S}_{old} \leftarrow \hat{S}$.
    $D_W \leftarrow 2\left(\gamma W + \hat{S}^T \hat{S} W\right)$.
    $F_W \leftarrow W - P_\mathcal{W}(W - D_W)$.
    $\alpha_W \leftarrow \mathbf{tr}(F_W^T D_W)/2 f_\gamma(F_W, \hat{S})$.
    $W \leftarrow W - \alpha_W F_W$.
    $D_S \leftarrow 2\hat{S} W W^T$.
    $F_S \leftarrow \hat{S} - P_\mathcal{H}(\hat{S} - D_S)$.
    $\alpha_S \leftarrow \mathbf{tr}(D_S^T F_S)/2\|F_S W\|_F^2$.
    $\hat{S} \leftarrow P_\mathcal{I}(\hat{S} - \alpha_S F_S)$.
  until $\|\hat{S} - \hat{S}_{old}\|_F / \|\hat{S}\|_F \le \varepsilon$
Ensure: $\hat{\boldsymbol{s}} = [S_{1,1}\ S_{1,2}\ \dots\ S_{1,N}\ S_{2,N}\ \dots\ S_{N,N}]^T$

Fig. 2 shows the declipping results in the case of the clipping level C=0.2 and 0.4. We can see that the proposed algorithm recovers the clipped signal well and that the declipped signal has few large spikes. These results are available for download at the web site[2].

Next we compare the proposed algorithm with the dual-constrained OMP based algorithm(given $\theta_{\max} = 1$) proposed in [3] in the case of the clipping levels C=0.2, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8. The results are shown in Fig. 3, where the performance of the algorithms are evaluated by the signal-to-noise ratio (SNR) computed as (15).

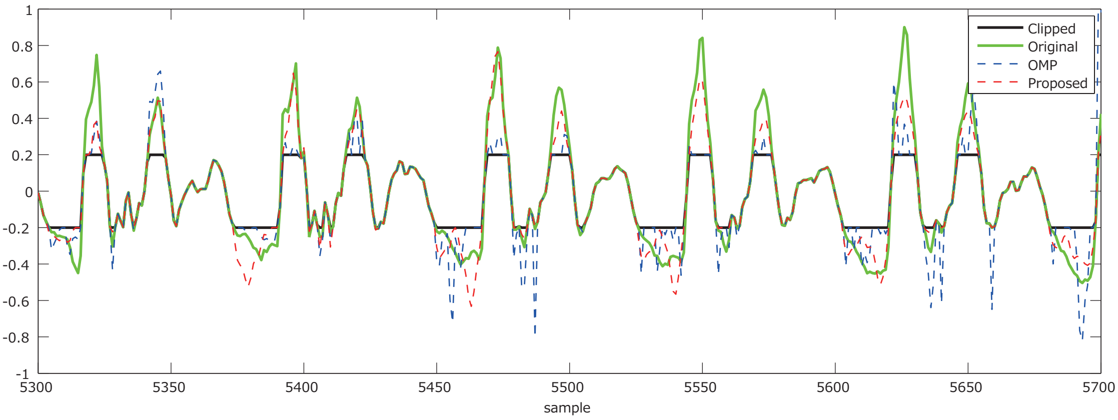
$$\text{SNR} = 20\log_{10}\frac{\|\boldsymbol{s}\|_2}{\|\boldsymbol{s} - \hat{\boldsymbol{s}}\|_2}. \tag{15}$$

We can see that the proposed algorithm enhances the SNR about 5.0 dB compared with the OMP based algorithm in clipping level = 0.2. Fig. 4 shows the computing time. Because the computational cost of the proposed algorithm depends on the number of signals to estimate, the recovery for higher clipping level requires less computing time. As can be seen, the proposed algorithm is much faster than the OMP based algorithm.

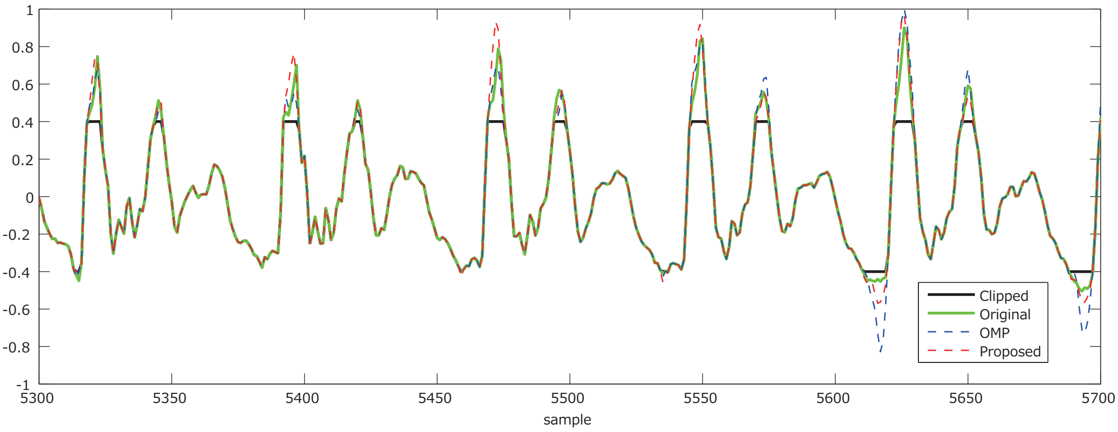
4. CONCLUSION

This paper deals with the signal declipping problem, which is formulated as the matrix rank minimization problem. In order to solve this problem, the linear inequality constrained and the Hankel structured matrix rank minimization algorithm is proposed by modifying the NSAO algorithm. Numerical examples show that the proposed algorithm can recover clipped

[2] http://p.tl/uCkW



(a) C=0.2



(b) C=0.4

Fig. 2. Declipping results of 400 samples of 95612 samples recovered by the proposed algorithm for the clipping level C=0.2 and 0.4.

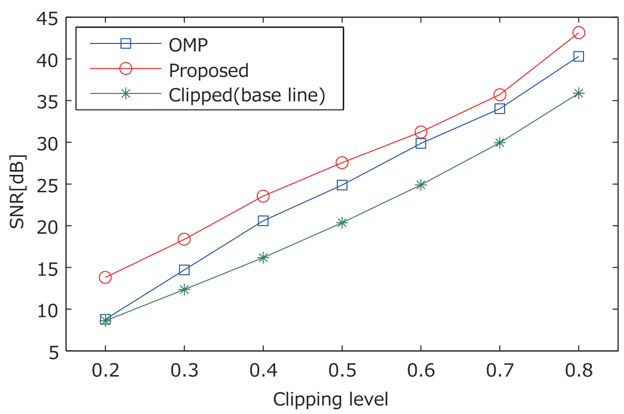


Fig. 3. Average performance of the algorithms for the clipping level C=0.2, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8.

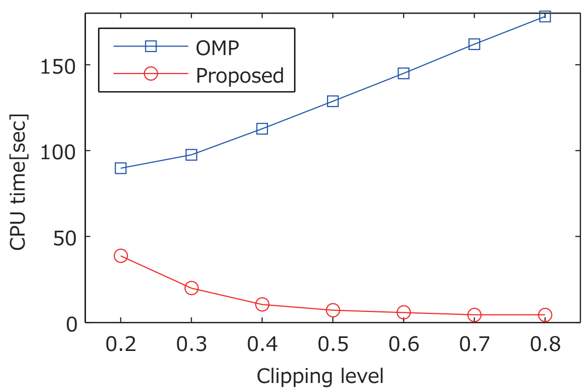


Fig. 4. CPU times of the algorithms.

signal efficiently and has better performance than the OMP based algorithm.

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