EXTENDED FOURIER SERIES FOR TIME-VARYING FILTERING AND RECONSTRUCTION FROM LEVEL-CROSSING SAMPLES

Modris Greitans and Rolands Shavelis

Institute of Electronics and Computer Science
14 Dzerbenes Str., Riga LV-1006, Latvia
phone: + (371) 67554500, fax: + (371) 67555337, email: modris_greitans@edi.lv, shavelis@edi.lv
web: www.edi.lv

ABSTRACT

In this paper an extension of the classical Fourier series for analysis of non-stationary signals with time-varying spectral content is presented. This, in comparison to the classical series, allows reducing the number of Fourier coefficients required to represent the signal. The proposed extended Fourier series can be used for time-varying filtering, which in turn can be used for recovery of the signals from level-crossing samples. This has been demonstrated by numerical simulations on different test signals.

***Index Terms*—** Extended Fourier series, time-varying filtering, level-crossing sampling, signal reconstruction

1. INTRODUCTION

The classical Fourier series (FS) allows periodic signals to be expressed in terms of sine and cosine functions with frequencies that are determined by the period of the signal and do not change in time.

When the signals with time-varying spectral content are considered, then instead of FS more advantageous can be a proposed extended Fourier series (EFS), which allows periodic signals to be expressed in terms of sine and cosine functions with frequencies that are integer multiples of the fundamental frequency that varies in time. If this frequency is chosen according to the time-varying bandwidth of the signal, then the number of components representing the signal reduces in comparison to the classical FS.

The time-varying bandwidth, as shown in [1], can also be taken into account to optimally sample the signals with sampling rate determined by the local bandwidth of the signal [2]. However, in practice, such sampling is difficult to implement directly since it requires the knowledge of the local bandwidth. Instead, an alternative signal-dependent sampling technique, called a level-crossing (LC) sampling [3], can be used. In this case the samples are taken every time the signal crosses any of the previously set levels, and the sampling rate naturally adapts to the local bandwidth of the signal.

In order to reconstruct the signal from the obtained LC samples, an iterative algorithm with time-varying filtering can be used. A passband of the filter should vary according to the local bandwidth of the signal and can be estimated from the given LC samples.

For time-varying filtering, an extended Fourier transform (EFT) and its inverse defined in [4] can be used, however, for numerical calculations it is more convenient to use the proposed EFS, since in this case only a limited number of EFS components at certain frequencies are needed to be calculated in order to find the time-varying filtered version of the signal. It can also be shown that EFT follows from EFS if the period of the signal tends to infinity.

2. EXTENDED FOURIER SERIES

Periodic signals $s(t)$ with period T can be expressed in complex Fourier series (FS) as

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}, \tag{1}$$

where coefficients c_n are calculated as

$$c_n = \frac{1}{T}\int_0^T s(t)e^{-jn\frac{2\pi}{T}t}dt \tag{2}$$

These coefficients are obtained by minimizing the mean square error between $s(t)$ and the right side signal of equation (1).

Now, given a periodic positive function $g(t) > 0$ with period T, the same periodic signals $s(t)$ can be expressed in extended Fourier series (EFS) as

$$s(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\frac{2\pi}{\Phi(T)}\Phi(t)}, \tag{3}$$

where

$$d_n = \frac{1}{\Phi(T)}\int_0^T \frac{s(t)}{g(t)} e^{-jn\frac{2\pi}{\Phi(T)}\Phi(t)}dt \tag{4}$$

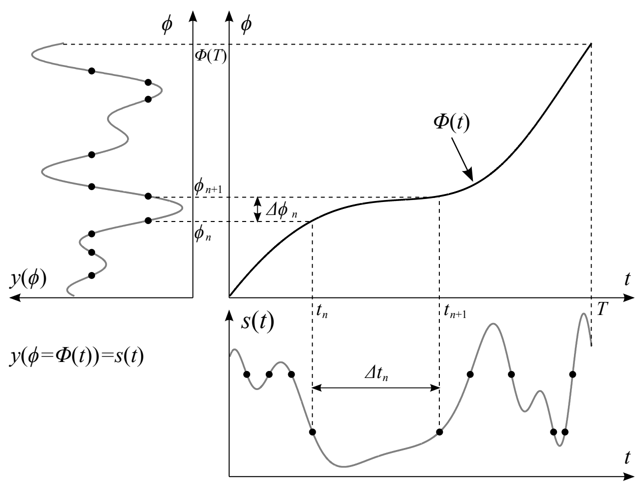


Fig. 1. Transformation of $s(t)$ to $y(\phi)$ by $\varPhi(t)$ such that $y(\varPhi(t)) = s(t)$.

and

$$\varPhi(t) = \int_0^t \frac{1}{g(\tau)} d\tau \tag{5}$$

In this case the coefficients d_n are obtained by minimizing the mean $1/g(t)$-weighted square error between both sides of (3) and considering the $1/g(t)$-weighted orthogonality of the base functions

$$\int_0^T \frac{1}{g(t)} e^{jn\frac{2\pi}{\varPhi(T)}\varPhi(t)} e^{-jk\frac{2\pi}{\varPhi(T)}\varPhi(t)} dt = \varPhi(T)\delta_{n,k} \tag{6}$$

EFS can also be obtained from Fig. 1 by expressing the signal $y(\phi)$, where $y(\phi = \varPhi(t)) = s(t)$, in the classical FS as

$$y(\phi) = \sum_{n=-\infty}^{\infty} d_n e^{jn\frac{2\pi}{\varPhi(T)}\phi} \tag{7}$$

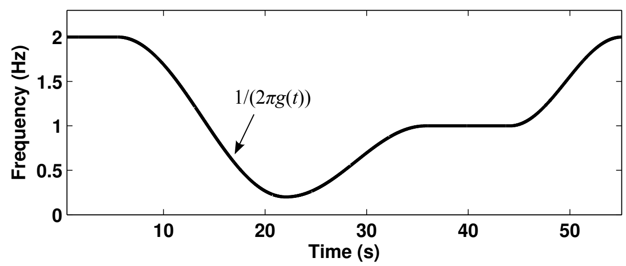
with coefficients d_n being

$$d_n = \frac{1}{\varPhi(T)} \int_0^{\varPhi(T)} y(\phi) e^{-jn\frac{2\pi}{\varPhi(T)}\phi} d\phi \tag{8}$$

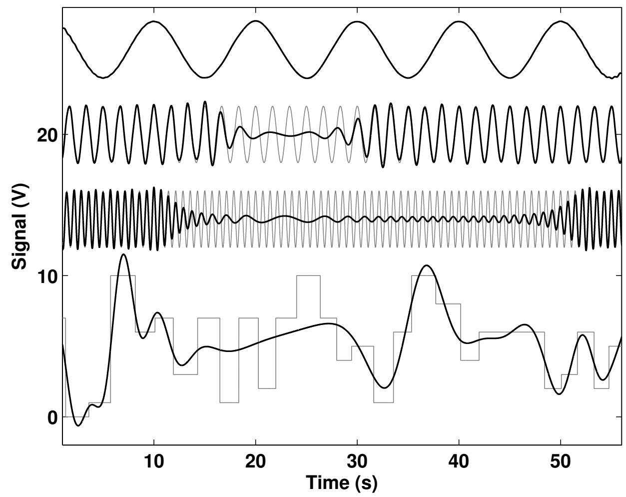
Now, by putting $\phi = \varPhi(t)$ into (7) and (8), the expressions (3) and (4) are obtained.

If only $|n| \leq N$ terms of EFS are used to approximate the signal, then the mean $1/g(t)$-weighted square error between $s(t)$ and its approximation $\hat{s}_N(t)$ is

$$\begin{aligned} P_{errN} &= \frac{1}{T} \int_0^T \frac{1}{g(t)} \big(s(t) - \hat{s}_N(t)\big)^2 dt = \\ &= \frac{1}{T} \int_0^T \frac{1}{g(t)} s^2(t) dt - \frac{\varPhi(T)}{T} \sum_{n=-N}^{N} |d_n|^2 \end{aligned} \tag{9}$$



(a) Time-varying bandwidth of the low-pass filter



(b) Input (gray) and output (black lines) signals of the filter

Fig. 2. Examples of time-varying filtering.

From (9) it follows that Bessel's inequality and Parseval's identity in this case can be written as

$$\frac{1}{\varPhi(T)} \int_0^T \frac{1}{g(t)} s^2(t) dt \geq \sum_{n=-N}^{N} |d_n|^2 \tag{10}$$

and

$$\frac{1}{\varPhi(T)} \int_0^T \frac{1}{g(t)} s^2(t) dt = \sum_{n=-\infty}^{\infty} |d_n|^2 \tag{11}$$

The special case when $g(t) = 1$ leads to classical Fourier series.

3. EXTENDED FOURIER TRANSFORM

Non-periodic signals of finite length can also be decomposed into EFS components – in this case the period T and so the value $\varPhi(T)$ due to monotonic increase of $\varPhi(t)$ are assumed to tend to infinity. In result the coefficients d_n and the distance $n\frac{2\pi}{\varPhi(T)}$ between components become infinitely small. To avoid such small values, both sides of (4) are multiplied by $\varPhi(T)$ and the discrete values $n\frac{2\pi}{\varPhi(T)}$ are replaced by continuous variable ω_g. In result the obtained expression

$$\lim_{T\to\infty} d_n \Phi(T) = \int_0^T \frac{s(t)}{g(t)} e^{-j\omega_g \Phi(t)} dt = S(\omega_g) \qquad (12)$$

conforms to the definition of an extended Fourier transform (EFT) given in [4].

EFT can also be obtained from the classical FT by applying it to the signal $y(\phi)$ shown in Fig. 1 and putting $\phi = \Phi(t)$ into

$$Y(\omega_g) = \int_{-\infty}^{\infty} y(\phi) e^{-j\omega_g \phi} d\phi \qquad (13)$$

4. TIME-VARYING FILTERING

After decomposing the signal $s(t)$ into EFS, the coefficients d_n corresponding to frequencies $n\frac{2\pi}{\Phi(T)}$ are given. In order to perform the time-varying filtering of $s(t)$ from Ω_{min} to Ω_{max}, only those coefficients d_n with indices n following from $\Omega_{min} \leq |n\frac{2\pi}{\Phi(T)}| \leq \Omega_{max}$ are used in (3) to calculate the approximation of $s(t)$

$$y(t) = \sum_{|n|\in[N_{min}, N_{max}]} d_n e^{jn\frac{2\pi}{\Phi(T)}\Phi(t)}, \qquad (14)$$

where

$$N_{min} = \lceil \frac{\Omega_{min}\Phi(T)}{2\pi} \rceil, \qquad (15)$$

and

$$N_{max} = \lfloor \frac{\Omega_{max}\Phi(T)}{2\pi} \rfloor \qquad (16)$$

The obtained signal $y(t)$ can be assumed to be the output of the time-varying filter with passband changing according to the function $1/g(t)$.

Examples of low-pass filtering with time-varying bandwidth (Fig. 2a) are shown in Fig. 2b – the gray lines are the input, while the black lines – output signals of the filter. The first three signals (starting from up) are cosines with frequencies 0.1, 0.6 and 1.4 Hz, and the fourth is the piecewise constant signal. As it follows from Fig. 2, the frequency properties of the output signals correspond to time-varying bandwidth of the filter.

5. RECONSTRUCTION FROM LEVEL-CROSSING SAMPLES

Time-varying filtering can be used for reconstructing the signals from samples obtained in signal-dependent way with sampling density depending on the local bandwidth of the signal. One such method is a level-crossing (LC) sampling with signal samples taken every time the signal crosses any of the previously set levels [3]. Due to obtained samples $s(t_k)$ are placed non-uniformly, an appropriate algorithm for recovery of the signal $s(t)$ is needed. One such algorithm given in [5] allows recovery of bandlimited to $[-\Omega, \Omega]$ signals, if the maximum distance between the sampling points

$$\Delta t_{max} = \sup_{k\in Z}(t_{k+1} - t_k) < \frac{\pi}{\Omega} \qquad (17)$$

The problem is that in LC sampling case the condition (17) can often be only satisfied if a large number of closely located levels are used, which leads to large sampling densities and reduced energy efficiency of data acquisition.

The necessary condition (17) for recovery of $s(t)$ from $s(t_k)$ can be restated if the signal $s(t)$ is transformed to $y(\phi)$ by the function $\Phi(t)$ as shown in Fig. 1. Now, if $y(\phi)$ is bandlimited to $[-\Omega_g, \Omega_g]$, then it can be reconstructed from its non-uniformly placed samples $y(\phi_k)$, if the maximum distance between the sampling points

$$\Delta \phi_{max} = \sup_{k\in Z}(\phi_{k+1} - \phi_k) < \frac{\pi}{\Omega_g} \qquad (18)$$

By putting $\phi = \Phi(t)$ into (18), the necessary condition for recovery of EFT bandlimited to $[-\Omega_g, \Omega_g]$ signal $s(t)$ from $s(t_k)$ becomes:

$$\sup_{k\in Z}\big(\Phi(t_{k+1}) - \Phi(t_k)\big) < \frac{\pi}{\Omega_g}, \qquad (19)$$

and the reconstruction is performed similarly as in [5] by the iterative algorithm

$$\begin{aligned} \hat{s}_0(t) &= A[\breve{s}_{s(t_k)}(t)], \\ \hat{s}_{i+1}(t) &= \hat{s}_i(t) + A[\breve{s}_{(s-s_i)(t_k)}(t)], \end{aligned} \qquad (20)$$

where $i \geq 0$ denotes the number of iteration, $\breve{s}_{s(t_k)}(t)$ is obtained by piecewise constant or linear interpolation of $s(t_k)$, and the operator A, given the function $\Phi(t)$, denotes the time-varying filtering of the signal to limit its bandwidth to $[-\Omega_g, \Omega_g]$ as described in the previous section. The reconstruction result, when i tends to infinity, is $\lim_{i\to\infty} \hat{s}_i(t) = s(t)$.

In case of piecewise constant interpolation the signal $\breve{s}_{s(t_k)}(t)$, $k = 1, 2, \ldots, K$, is written as

$$\breve{s}_{s(t_k)}(t) = \sum_{k=1}^{N} s(t_k)\psi_k(t), \qquad (21)$$

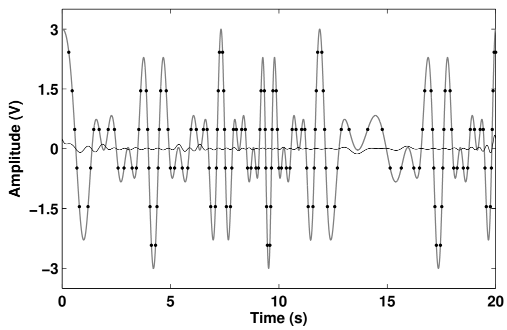
where

$$\psi_k(t) = \begin{cases} 1, & \text{if} \quad \tau_k \leq t < \tau_{k+1} \\ 0, & \text{other} \end{cases} \qquad (22)$$

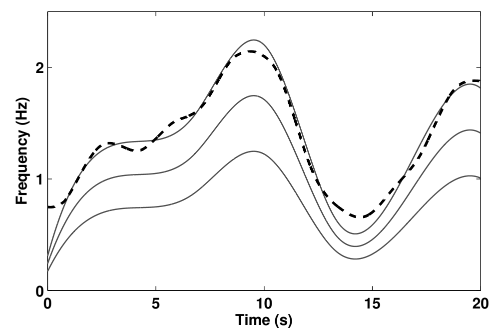
and $\tau_1 = t_1, \tau_{K+1} = t_K$ and $\tau_k = \frac{t_{k-1}+t_k}{2}$ if $k \in [2, K]$. The EFS coefficients of $\breve{s}_{s(t_k)}(t)$ in this case can be calculated as

$$d_n = \frac{j}{2\pi n} \sum_{k=1}^{K} s(t_k)\Big(e^{-j\frac{2\pi n}{\Phi(T)}\Phi(\tau_{k+1})} - e^{-j\frac{2\pi n}{\Phi(T)}\Phi(\tau_k)}\Big) \quad (23)$$

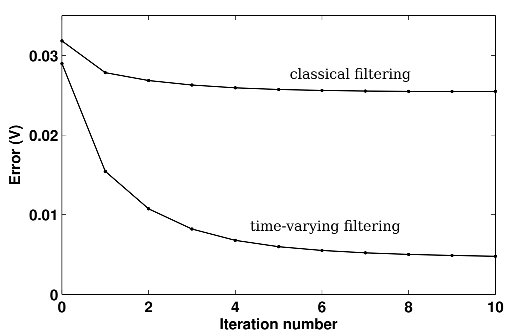
By comparing the condition (19) with (17), it can be concluded that the maximum distance Δt_{max} between the sampling points in the latter case can exceed the Nyquist step and



(a) Original (gray) and the error (black line) signals

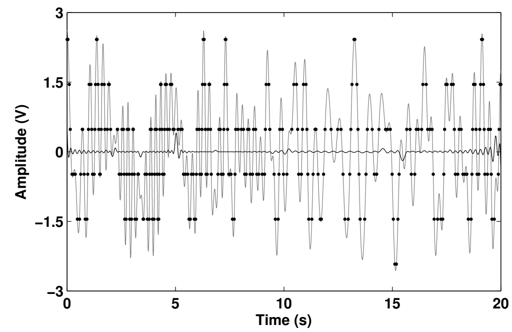


(b) Frequency traces of the signal

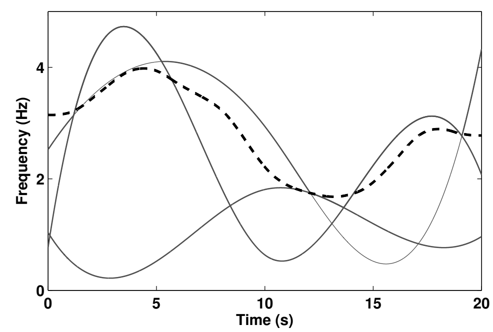


(c) Reconstruction error

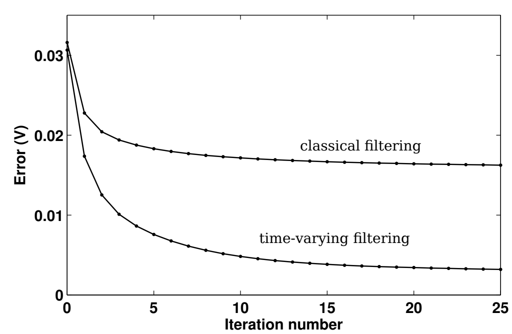
Fig. 3. Reconstruction of the first signal.



(a) Original (gray) and the error (black line) signals



(b) Frequency traces of the signal



(c) Reconstruction error

Fig. 4. Reconstruction of the second signal.

is determined by the function $\Phi(t)$. The question that remains to be answered in order to apply the algorithm (20) for recovery of $s(t)$ from LC samples is: how to estimate $\Phi(t)$ (or time-varying bandwidth of the signal) from the given LC samples? This is the topic for the next section.

6. ESTIMATION OF TIME-VARYING BANDWIDTH OF THE SIGNAL FROM LC SAMPLES

The local bandwidth of the filter should conform to the local bandwidth $f_{max}(t)$ of the signal. In [6] it was proposed that $f_{max}(t)$ equals the low-pass filtered version of the instantaneous frequency of the first Intrinsic Mode Function (IMF) of the signal found by empirical mode decomposition (EMD) [7]. Similar result can be obtained by analyzing the first IMF of the signal found by local mean decomposition (LMD) [8],[9]. In this case the time instants of local maxima and minima of the first IMF closely match the time instants t_m of local maxima and minima of the signal, therefore the conclusion is that $f_{max}(t)$ can be estimated from the increasing time sequence t_m, $m = 1, 2, \ldots$, as follows. At first, the discrete values $\Phi(t_m) = m\pi$ are found and the continuous function $\Phi(t)$ is obtained by the monotone cubic interpolation of these values. Then, the derivative $\frac{1}{2\pi}\frac{d\Phi(t)}{dt}$ is found and the time-varying bandwidth of the signal is assumed to be equal to the low-pass filtered version of this derivative.

In level-crossing sampling case the function $\Phi(t)$ can be estimated from LC samples by considering that at least one or more local maxima or minima of the signal are located between two samples $s(t_k)$ and $s(t_{k+1})$ if $s(t_k) = s(t_{k+1})$. Given all the indices $k'_1 < k'_2 < \ldots < k'_L \in [1, K]$ with $s(t_{k'_l}) = s(t_{k'_l+1})$, the discrete values $\hat{\Phi}(\frac{t_{k'_l}+t_{k'_l+1}}{2}) = l\pi$ are found. Then, as previously, the continuous function $\hat{\Phi}(t)$ is obtained by the monotone cubic interpolation of these values, and the time-varying bandwidth of the signal is assumed to equal the low-pass filtered version of the derivative of $\hat{\Phi}(t)$.

7. NUMERICAL RESULTS

Two test signals have been sampled by LC and then reconstructed using the algorithm (20). The first signal shown in Fig. 3a is composed of 3 similar components $s_1(t) = \sum_{n=1}^{3}\cos(k_n\Phi(t))$ with coefficients $k_1 < k_2 < k_3$ and frequencies shown in Fig. 3b by the solid lines, while the second signal (Fig. 4a) is composed of 3 different components $s_2(t) = \sum_{n=1}^{3}\cos(\Phi_n(t))$ with instantaneous frequencies shown in Fig. 4b.

The first step after obtaining the LC samples (black dots in Fig. 3a and Fig. 4a) was to estimate the time-varying bandwidth $\hat{f}_{max}(t)$ of the signal as described in Section 6. The obtained results are shown in Fig. 3b and Fig. 4b by the black dashed lines. As it follows the estimated functions $\hat{f}_{max}(t)$ are close to the upper frequency traces present in the signals.

Given the functions $\hat{f}_{max}(t)$, the signals were reconstructed using the iterative algorithm (20). The error signals $s(t) - \hat{s}_i(t)$ after $i = 20$ iterations are shown in Figures 3a and 4a by the black thin lines, while the mean square values of the error signals after each iteration are shown in Figures 3c and 4c. The upper curves in both figures correspond to the reconstruction when instead of time-varying filtering the classical filtering with constant bandwidth is used, and the lower curves are obtained when the estimated bandwidth functions $\hat{f}_{max}(t)$ are used. As it follows, better reconstruction is obtained in time-varying filtering case since larger distances between sampling points are allowed.

8. CONCLUSIONS

The proposed extended Fourier series allows decomposing the signals into sines and cosines with time-varying frequencies. This allows reducing the number of discrete components that represent the signals with time-varying spectral content, if the fundamental frequency of EFS is chosen according to the time-varying bandwidth of the signal.

The proposed series can be used for time-varying filtering of the signal by finding its approximation composed of only a limited number of components from the given frequency band. The time-varying filtering with the passband corresponding to the bandwidth of the signal can also be used for recovery of the signal from the samples obtained in signal-dependent way according to its local bandwidth. In comparison to the classical filtering, larger distances between consecutive sampling points are allowed in order to recover the signal.

9. ACKNOWLEDGEMENT

This research is supported by Latvian State research program in innovative materials and technologies.

10. REFERENCES

[1] J.J. Clark, M.R. Palmer, and P.D. Lawrence, "A transformation method for the reconstruction of functions from nonuniformly spaced samples," *IEEE Trans. on Acoustics, Speech and Signal Processing*, vol. 33, pp. 1151–1165, October 1985.

[2] N.N. Brueller, N. Peterfreund, and M. Porat, "Non-stationary signals: optimal sampling and instantaneous bandwidth estimation," in *Proc. of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis*. IEEE, 1998, pp. 113 – 115.

[3] E. Allier and G. Sicard, "A new class of asynchronous a/d converters based on time quantization," *Proc. of ASYNC 2003*, pp. 196–205, 2003.

[4] L. Heyoung and Z.Z. Bien, "A variable bandwidth filter for estimation of instantaneous frequency and reconstruction of signals with time-varying spectral content," *IEEE Trans. on Signal Processing*, vol. 59, pp. 2052–2071, May 2011.

[5] H.G. Feichtinger and K. Grchenig, "Theory and practice of irregular sampling," *Wavelets: Mathematics and Applications*, pp. 305–363, 1993.

[6] R. Shavelis and M. Greitans, "Signal sampling according to time-varying bandwidth," in *Proc. EUSIPCO 2012*, 2012, pp. 1164–1168.

[7] N.E. Huang and et al., "The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis," in *Proc. R. Soc. Lond. A*, 1998, vol. 454, pp. 903–995.

[8] J.S. Smith, "The local mean decomposition and its application to eeg perception data," *J. R. Soc. Interface*, vol. 2, pp. 443–454, July 2005.

[9] Y. Wang, Z. He, and Y. Zi, "A comparative study on the local mean decomposition and empirical mode decomposition and their applications to rotating machinery health diagnosis," *J. Vib. Acoust.*, vol. 132, pp. 10, April 2010.