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DISCRETE EXTENDED KALMAN FILTER ON LIE GROUPS

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ABSTRACT

In this paper, we generalize the Discrete Extended Kalman Filter (D-EKF) to the case where the state and the observations evolve on Lie group manifolds. We propose a new filter called Discrete Extended Kalman Filter on Lie Groups (D-LG-EKF). It assumes that the posterior distribution of the state is a concentrated Gaussian distribution on Lie groups. Our formalism yields closed-form equations for both non-linear discrete propagation and update of the distribution parameters based on the likelihood. We also show that the D-LG-EKF reduces to the traditional D-EKF if the state evolves on an Euclidean space. Our approach leads to a systematic methodology for the design of filters, which is illustrated by the application to a camera pose estimation problem. Results show that the D-LG-EKF outperforms both a constrained D-EKF and a D-EKF applied on the Lie algebra of the Lie group.

***Index Terms*—** Extended Kalman Filter, Lie Groups, Discrete time filtering, Filtering on manifolds

1. INTRODUCTION

This paper deals with the estimation of a state evolving on a manifold [1]. It was motivated by the need of a systematic methodology for the design of filters in the context of localization from wearable camera for personal activity lifelog [1] where both variables to estimate and measurements evolve on Lie groups. Several works tried to extend discrete Euclidean filtering algorithms to manifolds. For example, particle filters for states evolving on a Riemannian, Stiefel or Grassmann manifolds have been proposed. However, in our application, the computational complexity of a particle filter (induced by the particle evolution on the manifold and the highly peaked likelihood) was prohibitive. That is the reason why, in this work, we focus on Kalman filtering and extend the Discrete Extended Kalman Filter (D-EKF) [2] defined for a state and measurements evolving on Euclidean spaces to the case of a state and measurements evolving on matrix Lie groups. Typical examples of such groups include rotation matrices $SO(3)$, unitary quaternions $SU(2)$, rigid-body motion $SE(3)$, homographies $SL(3)$ and invertible matrices $GL(3)$.

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A large amount of works modeling the state on a Lie group have dealt with the specific groups $SO(3)$, $SU(2)$ or $SE(3)$. Among them [3] and [4] modified the unscented Kalman filter to estimate a unitary quaternion. In [5] an algorithm able to estimate the trajectory of a state evolving on $SE(3)$ is described. In [6], an Invariant Momentum-tracking Kalman Filter is derived to estimate a unitary quaternion and an angular momentum vector. Aside from these specific algorithms, a generic constrained filter [7] could be applied to this problem, by embedding the state in a vector space and enforcing an equality constraint to ensure the state remains on the embedded Lie group manifold. This approach does not take the geometry of the Lie group into account in an intrinsic manner and may lead to degenerated configurations as we will discuss later.

In contrast to these approaches, this paper introduces a framework that is both generic and adapted to Lie group geometry. It can be tailored to specific applications by designing the Lie group on which the state is defined. Our formalism is also able to take into account measurements evolving on a Lie group. Assuming the posterior distribution of the state is a concentrated Gaussian on Lie groups, we propose a tractable formulation of discrete error propagation and update that we call Discrete Extended Kalman Filter on Lie groups (D-LG-EKF). Moreover, we show that the D-LG-EKF reduces to the D-EKF if the state evolves on an Euclidean space.

The rest of the paper is organized as follows : Section 2 introduces some Lie group and Lie algebra notions as well as the concentrated Gaussian distribution on Lie group formalism. The D-LG-EKF theory is presented in Section 3 and the differences with the D-EKF as well as the applicability of our formalism are discussed. In section 4, the D-LG-EKF performances are illustrated on a camera pose estimation problem. Finally the conclusion and future research directions are provided in Section 5.

2. PRELIMINARIES

2.1. Lie Groups and Lie Algebras

In this section we give the definitions and basic properties of matrix Lie groups and Lie algebra. For a detailed description of these notions the reader is referred to [8]. We focus on matrix Lie groups since they cover most Lie groups of interest in signal and image processing. A Lie group G is a group which has also the structure of a smooth manifold such that group composition and inversion are smooth operations. If G is a matrix Lie group, then $g \in G \subset \mathbb{R}^{n\times n}$ and its operations are matrix multiplication and inversion with the identity matrix as identity element $Id_{n\times n}$. Note that an Euclidean space is a trivial matrix Lie group. The matrix exponential exp_G and matrix logarithm log_G mappings establish a local diffeomorphism between an open neighborhood of $\mathbf{0}_{n\times n}$ in the tangent space at the identity T_eG, called the *Lie Algebra* $\mathfrak{g}$, and an open neighborhood of $Id_{n\times n}$ in G. The Lie Algebra $\mathfrak{g}$ associated to a p-dimensional matrix Lie group is a p-dimensional vector space defined by a basis consisting of real matrices E_i for $i = 1\dots p$. Hence there is a linear isomorphism between $\mathfrak{g}$ and $\mathbb{R}^p$ that we denote as follows : $[\cdot]_G^\vee : \mathfrak{g} \to \mathbb{R}^p$ and $[\cdot]_G^\wedge : \mathbb{R}^p \to \mathfrak{g}$. For example let $\mathfrak{a} \in \mathfrak{g} \subset \mathbb{R}^{n\times n}$, then we have $[\mathfrak{a}]_G^\vee = a \in \mathbb{R}^p$. Thus we can define a basis $[E_i]_G^\vee = e_i$ where $\{e_i\}$ is the natural basis of $\mathbb{R}^p$ and $\mathfrak{a} = \sum_{i=1}^p a_i E_i$ with $a = \left(a_1 \dots a_p\right)^T$. We also define $M \subset G$ and $S \subset \mathbb{R}^p$ as the sets on which exp_G and log_G are bijective functions. The two previous notions are summarized in Fig. 1.

$$M \subset G \subset \mathbb{R}^{n\times n} \quad \underset{exp_G}{\overset{log_G}{\rightleftarrows}} \quad \mathfrak{g} \subset \mathbb{R}^{n\times n} \quad \underset{[\cdot]_G^\wedge}{\overset{[\cdot]_G^\vee}{\rightleftarrows}} \quad S \subset \mathbb{R}^p$$

Fig. 1. Mappings between the Lie group, its Lie algebra and $\mathbb{R}^p$

Lie groups are usually non-commutative. The two following operators capture this property ($X \in G$, $a, b \in \mathbb{R}^p$) :

- The Adjoint representation of G on $\mathbb{R}^p$ is defined as the operator $Ad_G : Ad_G(X)a = \left[X [a]_G^\wedge X^{-1}\right]_G^\vee$
- The adjoint representation of $\mathbb{R}^p$ on $\mathbb{R}^p$ is defined as the operator $ad_G : ad_G(a)b = \left[[a]_G^\wedge [b]_G^\wedge - [b]_G^\wedge [a]_G^\wedge\right]_G^\vee$

Finally let's introduce the Baker-Campbell-Haussdorff formula which expresses the group product directly in $\mathbb{R}^p$:

$$\begin{aligned} &\left[log_G\left(exp_G\left([a]_G^\wedge\right) exp_G\left([b]_G^\wedge\right)\right)\right]_G^\vee \\ &\quad = a + b + O\left(|a,b|^2\right) \end{aligned} \tag{1}$$

The following related formula will be useful for our derivations :

$$\begin{aligned} &\left[log_G\left(exp_G\left([-a]_G^\wedge\right) exp_G\left([a+b]_G^\wedge\right)\right)\right]_G^\vee \\ &\quad = a + \Phi_G(a) b + O\left(|b|^2\right) \end{aligned} \tag{2}$$

where $\Phi_G(a) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} ad_G(a)^m$.

2.2. Concentrated Gaussian Distribution on Lie Groups

In this section we introduce the concept of concentrated Gaussian on Lie groups [9, 10] as a generalization of the normal distribution in Euclidean space which is used in the D-EKF formalism. In order to define such a distribution, the considered Lie group has to be a connected unimodular matrix Lie group. Henceforth, in the rest of the paper, when referring to Lie groups, we will consider this assumption to hold. Note that this is the case of most Lie groups of interest such as $SO(3)$, $SE(3)$, $SL(3)$, $\mathbb{R}^n$... From [9] the following distribution can be defined :

$$\rho(X) = \alpha e^{-\frac{1}{2}\left(\left[log_G(X)\right]_G^{\vee} P^{-1} \left[log_G(X)\right]_G^{\vee}\right)} \tag{3}$$

Where α is a normalizing constant, $X \in G$, G is a p-dimensional Lie group and P is a definite positive matrix. Probability of elements outside of M is set to zero. Let's define ϵ as follows : $\epsilon = \left[log_G(X)\right]_G^\vee$ where $\epsilon \in S$. When $\rho(X)$ is tightly focused around the group identity (i.e the maximum of the eigenvalues of P is small), the distribution of ϵ can be approximated by a classical Euclidean Gaussian distribution defined on $\mathbb{R}^p$ of mean $\mathbf{0}_{p\times 1}$ and covariance matrix P : $\mathcal{N}_{\mathbb{R}^p}\left(\mathbf{0}_{p\times 1}, P\right)$. In this case, the distribution of X is called a concentrated Gaussian distribution on G around the identity. It can be moved around $\mu \in G$ using the left action of the Lie group, producing a concentrated Gaussian on G centered around μ (denoted $X \sim \mathcal{N}_G(\mu, P)$) :

$$X = \mu\, exp_G\left([\epsilon]_G^\wedge\right) \tag{4}$$

μ will be called the mean of X, ϵ can be seen as a Lie algebraic error of mean $\mathbf{0}_{p\times 1}$ and covariance P. Figure 2 provides a graphical interpretation of the transfer of the probability distribution from ϵ to X. Such a distribution allows us to describe the covariance of the state in $\mathbb{R}^p$ and hence using Euclidean tools while being invariant w.r.t the left action of the group on itself.

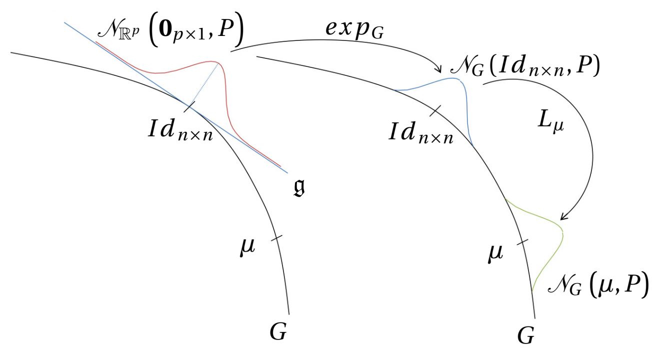


Fig. 2. Concentrated Gaussian on Lie groups

3. DISCRETE EXTENDED KALMAN FILTER ON LIE GROUPS THEORY

The Discrete Extended Kalman Filter (D-LG-EKF) theory is built upon the formalism of the concentrated Gaussian distribution on Lie groups.

3.1. System Model

Let the system state be modeled as satisfying the following equation :

$$
\begin{aligned}
X_k &= f(X_{k-1}, u_{k-1}, n_{k-1}) \\
&= X_{k-1}\, exp_G\left([\Omega(X_{k-1}, u_{k-1}) + n_{k-1}]_G^{\wedge}\right)
\end{aligned}
\tag{5}
$$

where $X_k \in G$ is the state we wish to estimate at time k and G is a p-dimensional Lie group. $u_{k-1} \in \mathbb{R}^w$ corresponds to a control input and $n_{k-1} \sim \mathcal{N}_{\mathbb{R}^p}\left(\mathbf{0}_{p\times 1}, R_{k-1}\right)$ is a white Gaussian noise. $\Omega : G \times \mathbb{R}^w \to \mathbb{R}^p$ is a non-linear $\mathscr{C}^2$ function.

We also consider discrete measurements on a q-dimensional Lie group G' :

$$
z_k = h(X_k)\, exp_{G'}\left([w_k]_{G'}^{\wedge}\right) \tag{6}
$$

where $z_k \in G'$, $h : G \to G'$ is a $\mathscr{C}^1$ function and $w_k \sim \mathcal{N}_{\mathbb{R}^q}\left(\mathbf{0}_{q\times 1}, Q_k\right)$ is a white Gaussian noise.

3.2. Proposed solution

We assume the state posterior distribution to be a concentrated Gaussian distribution on Lie groups : $p(X_k|z_1, \ldots, z_l) \approx \mathcal{N}_G\left(\mu_{k|l}, P_{k|l}\right)$. We focus on $l = k-1$ (propagation) and $l = k$ (update). Therefore, the aim of the D-LG-EKF is to propagate and update the distribution parameters $\mu_{k-1|k-1}$ and $P_{k-1|k-1}$. In our formalism, μ is the state estimate. The maximum of the eigenvalues of the white Gaussian noises covariance matrices considered in Section 3.1 are assumed to be small in order to apply the concentrated Gaussian distribution formalism.

3.2.1. *Propagation*

We assume that the state posterior distribution at time $k-1$ is represented by $\mathcal{N}_G\left(\mu_{k-1|k-1}, P_{k-1|k-1}\right)$. Therefore, the aim of this section is to show how to propagate $\mu_{k-1|k-1}$ and $P_{k-1|k-1}$ between two consecutive sensor measurements.

Mean Propagation The state estimate is propagated using the state model without noise :

$$
\mu_{k|k-1} = \mu_{k-1|k-1}\, exp_G\left(\left[\hat{\Omega}_{k-1}\right]_G^{\wedge}\right) \tag{7}
$$

where $\hat{\Omega}_{k-1} = \Omega\left(\mu_{k-1}, u_{k-1}\right)$.

Covariance Propagation In order to propagate the covariance, we study the Lie algebraic error propagation. The state error on G can be expressed as follows :

$$
\begin{aligned}
exp_G\left([\epsilon_{k|k-1}]_G^{\wedge}\right) &= \mu_{k|k-1}^{-1} X_k \\
= exp_G\left(\left[-\hat{\Omega}_{k-1}\right]_G^{\wedge}\right) & exp_G\left(\left[\epsilon_{k-1|k-1}\right]_G^{\wedge}\right) \\
exp_G\left([\Omega(X_{k-1}, u_{k-1}) + n_{k-1}]_G^{\wedge}\right) &
\end{aligned}
\tag{8}
$$

Linearizing Ω in $\mu_{k-1|k-1}$ and using equations (1) and (2), one can obtain the following Lie algebraic error propagation :

$$
\epsilon_{k|k-1} = \mathscr{F}_{k-1}\epsilon_{k-1|k-1} + \Phi_G\left(\hat{\Omega}_{k-1}\right) n_{k-1} + O\left(\left|\epsilon_{k-1|k-1}, n_{k-1}\right|^2\right) \tag{9}
$$

where

$$
\mathscr{F}_{k-1} = Ad_G\left(exp_G\left(-\hat{\Omega}_{k-1}\right)\right) + \Phi_G\left(\hat{\Omega}_{k-1}\right)\mathscr{C}_{k-1} \tag{10}
$$

and

$$
\mathscr{C}_{k-1} = \frac{\partial}{\partial \epsilon}\Omega\left(\mu_{k-1|k-1}\, exp_G\left([\epsilon]_G^{\wedge}\right), u_{k-1}\right)|_{\epsilon=\mathbf{0}} \tag{11}
$$

As in the D-EKF case, terms in $O\left(\left|\epsilon_{k-1|k-1}\right|^2\right)$ are neglected. Moreover, we do not consider terms in $O\left(\left|\epsilon_{k-1|k-1}, n_{k-1}\right|^2\right)$ since, because of the concentrated Gaussian assumption, n_{k-1} is assumed to be small.

Under these conditions : $\mathbb{E}\left[\epsilon_{k|k-1}\right] = m_{k|k-1} = \mathbf{0}_{p\times 1}$. Finally, we obtain the following covariance propagation formula :

$$
\begin{aligned}
P_{k|k-1} &= \mathbb{E}\left[\epsilon_{k|k-1}\epsilon_{k|k-1}^T\right] \\
&= \mathscr{F}_{k-1}P_{k-1|k-1}\mathscr{F}_{k-1}^T + \Phi_G\left(\hat{\Omega}_{k-1}\right)R_{k-1}\Phi_G\left(\hat{\Omega}_{k-1}\right)^T
\end{aligned}
\tag{12}
$$

Propagation step summary At the end of the propagation step, the estimated state is parametrized as follows :

$$
X_k|z_1, \ldots, z_{k-1} \sim \mathcal{N}_G\left(\mu_{k|k-1}, P_{k|k-1}\right) \tag{13}
$$

where $\epsilon_{k|k-1} \sim \mathcal{N}_{\mathbb{R}^p}\left(m_{k|k-1} = \mathbf{0}_{p\times 1}, P_{k|k-1}\right)$.

3.2.2. *Update*

This step consists in incorporating the information coming from the measurement z_k into the Lie algebraic error. It is followed by a reparametrization of the state to satisfy to the concentrated Gaussian distribution assumption.

Lie algebraic error update Let's define the following innovation term :

$$
\begin{aligned}
\tilde{z}_k &= \left[log_{G'}\left(h\left(\mu_{k|k-1}\right)^{-1} z_k\right)\right]_{G'}^{\vee} \\
&= \left[log_{G'}\left(exp_{G'}\left(\mathscr{H}_k\epsilon_{k|k-1} + O\left(\left|\epsilon_{k|k-1}\right|^2\right)\right) exp_{G'}\left([w_k]_G^{\wedge}\right)\right)\right]_{G'}^{\vee}
\end{aligned}
\tag{14}
$$

where

$$
\mathscr{H}_k = \frac{\partial}{\partial \epsilon}\left[log_{G'}\left(h\left(\mu_{k|k-1}\right)^{-1} h\left(\mu_{k|k-1}\, exp_G\left([\epsilon]_G^{\wedge}\right)\right)\right)\right]_G^{\vee}|_{\epsilon=\mathbf{0}} \tag{15}
$$

Using equation (1), we obtain :

$$
\tilde{z}_k = \mathscr{H}_k\epsilon_{k|k-1} + w_k + O\left(\left|\epsilon_{k|k-1}, w_k\right|^2\right) \tag{16}
$$

As in the D-EKF case, terms in $O\left(\left|\epsilon_{k|k-1}\right|^2\right)$ are neglected. Moreover, we do not consider terms in $O\left(\left|\epsilon_{k|k-1}, w_k\right|^2\right)$ since, because of the concentrated Gaussian assumption, w_k is assumed to be small.

Equation (16) is linear in $\epsilon_{k|k-1}$ which evolves on $\mathbb{R}^p$. Therefore, we can apply the classical update equations of the

Kalman filter [11] to update $\epsilon_{k|k-1}$ into the posterior distribution as $\epsilon^-_{k|k} \sim \mathcal{N}_{\mathbb{R}^p}\left(m^-_{k|k}, P^-_{k|k}\right)$ where $m^-_{k|k}$ and $P^-_{k|k}$ can be calculated as follows :

$$\begin{cases} K_k = P_{k|k-1}\mathcal{H}_k^T\left(\mathcal{H}_k P_{k|k-1}\mathcal{H}_k^T + Q_k\right)^{-1} \\ m^-_{k|k} = \mathbf{0}_{p\times 1} + K_k\left(\tilde{z}_k - \mathcal{H}_k\mathbf{0}_{p\times 1}\right) \\ P^-_{k|k} = (Id - K_k\mathcal{H}_k)P_{k|k-1} \end{cases} \tag{17}$$

State Reparametrization At the end of the update step, we expect to have $X_k = \mu_{k|k} exp_G\left(\left[\epsilon_{k|k}\right]^\wedge_G\right)$ with $\mathbb{E}\left[\epsilon_{k|k}\right] = \mathbf{0}_{p\times 1}$ (conditionally to $z_1, \ldots, z_k$), to satisfy the concentrated Gaussian distribution definition (4). However we have $\mathbb{E}\left[\epsilon^-_{k|k}\right] = m^-_{k|k} \neq \mathbf{0}_{p\times 1}$. Hence, we perform the following reparametrization :

$$\mu_{k|k} = \mu_{k|k-1} exp_G\left(\left[m^-_{k|k}\right]^\wedge_G\right) \tag{18}$$

Thus, using equation (2) and neglecting terms in $O\left(\left|\epsilon^-_{k|k}\right|^2\right)$, we obtain :

$$m_{k|k} = \mathbf{0}_{p\times 1} \tag{19}$$

$$P_{k|k} = \Phi_G\left(m^-_{k|k}\right)P^-_{k|k}\Phi_G\left(m^-_{k|k}\right)^T \tag{20}$$

Update Step Summary At the end of the update step, the estimated state is parametrized as follows :

$$X_k | z_1, \ldots, z_k \sim \mathcal{N}_G\left(\mu_{k|k}, P_{k|k}\right) \tag{21}$$

where $\epsilon_{k|k} \sim \mathcal{N}_{\mathbb{R}^p}\left(m_{k|k} = \mathbf{0}_{p\times 1}, P_{k|k}\right)$. The LG-EKF algorithm is summarized below :

Algorithme 1 D-LG-EKF Algorithm

Inputs : $\mu_{k-1|k-1}$, $P_{k-1|k-1}$, u_{k-1}, z_k
Outputs : $\mu_{k|k}$, $P_{k|k}$

Propagation :
$\mu_{k|k-1} = \mu_{k-1|k-1} exp_G\left(\left[\hat{\Omega}_{k-1}\right]^\wedge_G\right)$
$P_{k|k-1} = \mathcal{F}_{k-1}P_{k-1|k-1}\mathcal{F}_{k-1}^T + \Phi_G\left(\hat{\Omega}_{k-1}\right)R_{k-1}\Phi_G\left(\hat{\Omega}_{k-1}\right)^T$

Update :
$K_k = P_{k|k-1}\mathcal{H}_k^T\left(\mathcal{H}_k P_{k|k-1}\mathcal{H}_k^T + Q_k\right)^{-1}$
$m^-_{k|k} = K_k\left(\left[log_{G'}\left(h\left(\mu_{k|k-1}\right)^{-1} z_k\right)\right]^\vee_{G'}\right)$
$\mu_{k|k} = \mu_{k|k-1} exp_G\left(\left[m^-_{k|k}\right]^\wedge_G\right)$
$P_{k|k} = \Phi_G\left(m^-_{k|k}\right)(Id_{l\times l} - K_k\mathcal{H}_k)P_{k|k}\Phi_G\left(m^-_{k|k}\right)^T$

3.3. Discussion

The D-LG-EKF generalizes the D-EKF. An Euclidean space is a trivial Lie group where functions exp, log, $[\cdot]^\wedge$, $[\cdot]^\vee$, Ad and Φ are identity mappings. Furthermore, in this case, group composition and inversion correspond to vector addition and subtraction. Consequently, equations (5) and (6) generalize the traditional additive noise Euclidean equations $X_k = f(X_{k-1}, u_{k-1}) + n_{k-1}$ and $z_k = h(X_k) + w_k$ (see [2]) . Moreover, for Euclidean space, matrices $\mathcal{F}_{k-1}$ and $\mathcal{H}_k$ correspond to Jacobians of f and h calculated at $\mu_{k-1|k-1}$ and $\mu_{k|k-1}$ respectively. Therefore, it is straightforward to see that the D-LG-EKF (see alg.1) reduces to the D-EKF (see [2]) when G and G' are Euclidean spaces.

Why not employing a D-EKF to solve our problem ? Estimating a state $X \in G \subset \mathbb{R}^{n\times n}$ while considering measurements $z \in G' \subset \mathbb{R}^{m\times m}$, where G and G' are Lie groups of dimension p and q respectively, is not coherent with the D-EKF theory which was developed to estimate states evolving on Euclidean spaces. However, it is possible to adapt the constrained D-EKF formalism [7] in an ad hoc manner to fit to this problem, assuming $X \in \mathbb{R}^{n\times n}$, vectorizing it and considering the group geometry as a state constraint. Such an algorithm (noted D-EKF Constr in the experiments) treats the geometry of the Lie group as an extrinsic constraint, thus the filtering is performed in the Euclidean embedding space $\mathbb{R}^l$ of the Lie group, where $l > p$. Consequently, both the state and the measurement covariance matrices are singular which causes issues during the Kalman gain computation.

Another way to employ a D-EKF to solve our problem is to estimate $x = \left[log_G(X)\right]^\vee_G$ instead of X [12] (noted D-EKF LieAlg in the experiments) and to consider measurements $\left[log_{G'}(z)\right]^\vee_{G'}$. To apply such a filter, log_G and $log_{G'}$ must be defined over the whole group. In this case, the D-EKF LieAlg is a suitable alternative to the D-LG-EKF and does not produce singular covariance matrices. However, $log_{G'}$ may be discontinuous for some groups such as $SO(3)$ which would yield the innovation to be incorrectly large even with a small error on the group.

When is the D-LG-EKF applicable ? The D-LG-EKF applies for Lie groups of interest such as : $SO(3)$ and $SU(2)$ (rotation), $SL(3)$ (homographies), $SE(2)$ and $SE(3)$ (rigid body motion), $(\mathbb{R}^{+*}, \times)$ (scale factor),$(\mathbb{R}^n, +)$ (any element of an Euclidean space) or products of these spaces. For commutative Lie groups, our formalism greatly simplifies since every term dealing with non-commutativity disappear such as Φ. For $SO(3)$ and $SE(3)$, analytic expressions of log, exp and Φ are available [13]. For $SL(3)$, these functions have to be numerically approached.

When implementing a D-LG-EKF, the choice of the appropriate Lie group (direct product '$\times$', semi-direct product '$\rtimes$' or twisted product '$\star$' between the spaces) as well as the choice of its associated Lie algebra basis, remain a modeling question for the practitioner.

4. SIMULATION RESULTS

We choose to evaluate the proposed formalism on a camera pose estimation problem. It deals with estimating the camera position $T \in \mathbb{R}^3$ and orientation $R \in SO(3)$ using a white-noise acceleration model. Therefore, both the angular velocity $\omega \in \mathbb{R}^3$ and the radial velocity $\nu \in \mathbb{R}^3$ are also estimated. We assume R and T are directly observed. Finally, we choose to consider the following Lie groups : $G = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ and $G' = SO(3) \times \mathbb{R}^3$ with the classical basis of $SO(3)$ [13].

To compare the results of the D-LG-EKF, we implemented two other filters, a D-EKF Constr and a D-EKF LieAlg (introduced in 3.3). We simulate a room as a 3D point cloud cube and assume the camera to be perfectly calibrated. Then, we generate trajectories and create sequences of measurements in G' using a maximum likelihood algorithm such as [14]. The covariance of each measurement is estimated by propagating the covariance from each 3D observed point. For all the filters, T and R are perfectly initialized with small variances whereas ω and v are set to zero with large variances.

Figure 3 reports the RMSE of each filter w.r.t sampling rate δt. The RMSE is defined as the square root of the average of the following errors : $\left\|\mu_T - T\right\|_2^2$ (position error) and $\left\| log_{SO3}\left([\mu_R{}^T R]^\vee_{SO(3)} \right) \right\|_2^2$ (orientation error).

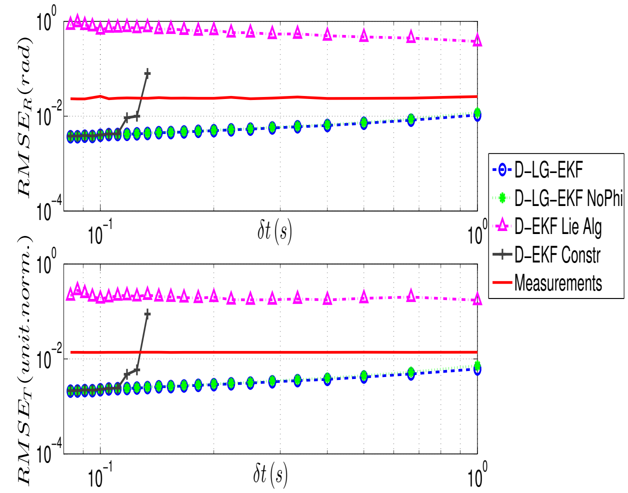
As it was expected by the theoretical differences outlined in Section3.3, the D-LG-EKF outperforms both the D-EKF Constr and the D-EKF LieAlg. Indeed, in the D-EKF Constr, the more δt grows, the further from the optimal state is the state estimate projected, which results in the incorrect estimates of the filter and numerical instabilities. For small δt, these effects are limited. Concerning the D-EKF LieAlg case, when the norm of the vector describing the rotation in the Lie algebra goes over π, the estimation becomes incorrect because of the $SO(3)$ logarithm discontinuity. As opposed to these two filters, the D-LG-EKF does not suffer from those limitations and consequently it does not diverge, and efficiently smoothens the camera trajectory. As δt grows, the state model becomes less informative which is why the D-LG-EKF RMSE comes closer to the measurements RMSE.

Eventually, we also considered the case where the matrices Φ_G in the D-LG-EKF algorithm are replaced by identity matrices. We call this version : D-LG-EKF NoPhi. It turns out that neglecting the matrices Φ_G only slightly reduces the performances of the algorithm in the considered case. Therefore, depending on the required accuracy of the considered application, one can choose to replace them by identity matrices.

5. CONCLUSION

In this paper, we proposed a new generic algorithm called Discrete Extended Kalman Filter on Lie groups that generalizes the Discrete Extended Kalman Filter to the case where the state and the observations evolve on Lie group manifolds. Assuming the posterior distribution is a concentrated Gaussian distribution, we showed how to propagate and update the distribution parameters. The systematic methodology of our algorithm was illustrated by a camera pose estimation problem where both a constrained D-EKF and a D-EKF applied on the Lie algebra of the Lie group were outperformed. We believe that, in this type of application, our new algorithm allows one to replace the D-EKF.

Fig. 3. RMSE of the filters ($RMSE_R$: orientation and $RMSE_T$: position) calculated on 2000 trajectories



6. REFERENCES

[1] Hazem Wannous, Vladislavs Dovgalecs, Rémi Mégret, and Mohamed Daoudi, "Place recognition via 3d modeling for personal activity lifelog using wearable camera," in *Advances in Multimedia Modeling*, pp. 244–254. Springer, 2012.

[2] Simon S Haykin et al., *Kalman filtering and neural networks*, Wiley Online Library, 2001.

[3] E. Kraft, "A quaternion-based unscented kalman filter for orientation tracking," in *International Conference on Information Fusion*, 2003, vol. 1, pp. 47–54.

[4] J. Crassidis and F. Markley, "Unscented filtering for spacecraft attitude estimation," *Journal of Guidance, Control, and Dynamics*, vol. 26, pp. 536–542, 2003.

[5] Paul Smith, Tom Drummond, and Kimon Roussopoulos, "Computing map trajectories by representing, propagating and combining pdfs over groups," in *ICCV*, 2003, pp. 1275–1282.

[6] S. Persson and I. Sharf, "Invariant momentum-tracking kalman filter for attitude estimation," in *Robotics and Automation (ICRA), 2012 IEEE International Conference on*, 2012, pp. 592–598.

[7] D. Simon, "Kalman filtering with state constraints : a survey of linear and nonlinear algorithms," *IET Control Theory & Applications*, vol. 4, no. 8, pp. 1303–1318, 2010.

[8] G.S. Chirikjian, *Stochastic Models, Information Theory, and Lie Groups*, Springer, 2012.

[9] Y. Wang and G. Chirikjian, "Error propagation on the euclidean group with applications to manipulators kinematics," *IEEE Transactions on Robotics*, vol. 22, 2006.

[10] K. Wolfe, M. Mashner, and G. Chirikjian, "Bayesian fusion on lie groups," *Journal of Algebraic Statistics*, vol. 2, pp. 75–97, 2011.

[11] P. Maybeck, *Stochastic Models, Estimation, and Control*, Academic Press, 1979.

[12] Y. Ma, S. Soatto, J. Kosecka, and S. Sastry, *An Invitation to 3D Vision : From Images to Geometric Models*, Springer, 2004.

[13] J Selig, "Lie groups and lie algebras in robotics," *Computational Noncommutative Algebra and Applications*, pp. 101–125, 2005.

[14] Chien-Ping Lu, Gregory D. Hager, and Eric Mjolsness, "Fast and globally convergent pose estimation from video images," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 22, no. 6, pp. 610–622, 2000.