STRATIFIED SENSOR NETWORK SELF-CALIBRATION FROM TDOA MEASUREMENTS

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ABSTRACT

This paper presents a study of sensor network calibration from time-difference-of-arrival (TDOA) measurements. Such calibration arises in several applications such as calibration of (acoustic or ultrasound) microphone arrays, bluetooth arrays, and radio antenna networks. We propose a non-iterative algorithm that applies a three-step stratification process, (i) using a set of rank constraints to determine the unknown time offsets, (ii) applying factorization techniques to determine transmitters and receivers up to unknown affine transformation and (iii) determining the affine stratification using the remaining constraints. This results in novel algorithms for direct recovery of both transmitter and receiver positions using TDOA measurements, down to 6 receivers and 8 transmitters. Experiments are shown both for simulated and real data with promising results.

Index Terms— Network Self-Calibration, TOA, TDOA, Minimal Problem

1. INTRODUCTION

Determining the sound source positions using a number of microphones at known locations and measuring the time-difference of arrival of sounds have been an important application in sound ranging and localization. Such techniques are used with microphone arrays to enable beamforming and speaker tracking. However, in most of such applications, calibrating microphone positions and time of transmission for sound sources are difficult. Self-calibration of sensor networks using TDOA measurements is a nonlinear optimization problem, for which proper initialization is essential. It has been shown e.g. in [1,2] that poor initialization potentially gives local minima that are far off the ground truth in both synthetic and real experiments. Several previous works rely on prior knowledge or extra assumptions of locations of the sensors to initialize the problem. In [3], the distances between pairs of microphones are manually measured and multi-dimensional scaling is used to compute microphone positions. Other options include using GPS [4] to get approximate locations, or using transmitter-receiver pairs (radio or

audio) that are close to each other [5, 6].

Another line of works focuses on solving the initialization without any additional assumptions. Initialization of TOA networks has been studied in [7, 8]. Initialization of TDOA networks is studied in [9], where solutions were given to non-minimal cases (e.g. 10 receivers, 5 transmitters in 3D). For 2D TDOA calibration, a recursive search algorithm is proposed in [10]. In [11], the minimal cases where all receivers lie on a line are solved for both TOA and TDOA. Though the same rank constraint as ours is explored in [12, 13], both methods are iterative and dependent on initialization.

In this paper we study the initialization network calibration problem from only TDOA measurements for general dimensions. We utilize constraints on the rank of measurement matrix and propose a non-iterative scheme for calculating the time offsets. After calibrating the TDOA measurements with the offsets, we use a two-step scheme for the subsequent TOA problem. These schemes allow for a wider class of solvable cases which are closer to minimal cases for initializing TDOA network calibration problem. This gives non-iterative algorithms for direct recovery of both transmitter and receiver positions using as few as 6 receivers using TDOA measurements. Previous state of the art method [9], required at least 10 receivers. Solving these cases is of theoretical importance. The solvers can also be used in RANSAC [14] schemes to remove outliers in noisy data. The methods are validated both on synthetic and real data. The node localization is cross-validated against independent recordings as well as against computer vision based approaches.

2. PROBLEM FORMULATION

Let $\mathbf{r}_i$, $i = 1, \ldots, m$ and $\mathbf{s}_j$, $j = 1, \ldots, n$ be the spatial coordinates of m transmitters and n receivers, respectively. For measured time of arrival t_{ij} from transmitter $\mathbf{r}_i$ to receiver $\mathbf{s}_j$, we have $v(t_{ij} - t_j) = ||\mathbf{r}_i - \mathbf{s}_j||_2$, where t_j is the unknown offset for each transmitter, and v is the speed of measured signals (assumed to be constant). We will in the sequel work with the distance measurements ($f_{ij} = vt_{ij}$, $o_j = vt_j$). The TDOA calibration problem can then be defined as follows.

Problem 2.1 *(TDOA-based Network Self-Calibration) Given relative distance measurements f_{ij} determine receiver positions $\mathbf{r}_i$, $i = 1, \ldots, m$, transmitter positions $\mathbf{s}_j$, $j = 1, \ldots, n$ and offsets $o_j, j = 1, \ldots, n$ such that $f_{ij} = ||\mathbf{r}_i - \mathbf{s}_j||_2 + o_j$.*

$m \backslash n$	4	5	6	7	8	9
5	-	-	-	-	-	I
6	-	-	III	-	III	-
7	-	I	III	-	-	-
8	-	-	-	-	-	-
9	-	III	-	-	-	-
10	I	II	-	-	-	-

$m \backslash n$	3	4	5	6
4	-	-	I	-
5	-	-	-	III
6	I	-	-	-
7	-	III	-	-
8	-	II	-	-
9	-	-	-	-

K	m	n	N_{sol}
3	9	5	1
3	7	6	5
3	6	8	14
2	7	4	1
2	5	6	5

Fig. 1. Cases for TDOA problem for 3D (left) and 2D (middle). (I) minimal cases, (II) solvable cases for [9] and (III) solvable cases for our proposed method. Right: number of solutions to the polynomial systems of rank constraints on unknown offsets for different cases in (III) (3D and 2D).

If the offsets are known or have been estimated, the conversion from TDOA to TOA problems (where the absolute distance $d_{ij} = ||\mathbf{r}_i - \mathbf{s}_j||_2$ are given) are straightforward, i.e. by setting $d_{ij} = f_{ij} - o_j$. Note that for both TDOA and TOA problems, one can only reconstruct locations of receivers and transmitters up to euclidean transformation and mirroring. In the following discussion, we assume that the dimensionality K of the affine space spanned by $\mathbf{r}_i$ and $\mathbf{s}_j$ is the same, e.g. $K = 3$ for 3D problems. The minimal configurations have previously been determined in [7] and also shown in Table 1.

3. METHODS

To solve the TDOA calibration problem, we use a stratified approach to solve first for the offsets $\{o_j\}$ (Section 3.1), and then solve the TOA calibration problem (Section 3.2).

3.1. Estimating the Offsets

In this section we present two new techniques for solving the unknown offsets. The first scheme is an improved version of the linear factorization in [9]. Another one is to make full use of the rank constraints on the measurement matrix.

3.1.1. Linear Method

We know that $d_{ij}^2 = (f_{ij} - o_j)^2 = (\mathbf{r}_i - \mathbf{s}_j)^T(\mathbf{r}_i - \mathbf{s}_j) = \mathbf{r}_i^T\mathbf{s}_i - 2\mathbf{r}_i^T\mathbf{s}_j + \mathbf{s}_j^T\mathbf{s}_j$. By constructing the vectors $\mathbf{R}_i = \begin{bmatrix} 1 & \mathbf{r}_i^T & \mathbf{r}_i^T\mathbf{r}_i \end{bmatrix}^T$ and $\mathbf{S}_j = \begin{bmatrix} \mathbf{s}_j^T\mathbf{s}_j - o_j^2 & -2\mathbf{s}_j^T & 1 \end{bmatrix}^T$, we obtain $f_{ij}^2 - 2f_{ij}o_j = \mathbf{R}_i^T\mathbf{S}_j$. By collecting $\mathbf{R}_i$ and $\mathbf{S}_j$ into matrix $\mathbf{R}$ ($(K+2) \times m$) and $\mathbf{S}$ ($(K+2) \times n$), we have $\mathbf{F} = \mathbf{R}^T\mathbf{S}$, where $\mathbf{F}$ is the $m \times n$ matrix containing $\{f_{ij}^2 - 2f_{ij}o_j\}$. This suggests that matrix $\mathbf{F}$ is at most of rank $K + 2$ as we increase m and n. A slight modification to scheme in [9] can reduce the required number of receivers by 1. The idea is to exploit the structure of $\mathbf{S}$ - ones in the last row, by multiplying $\mathbf{F}$ from the right by a $n \times (n-1)$ matrix $\mathbf{C}_n$ of the form $\begin{bmatrix} -\mathbb{1}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix}^T$ where $\mathbb{1}_{n-1}$ is a $(n-1) \times 1$ vector with 1 as entries and $\mathbf{I}_{n-1}$ is identity matrix of size $(n-1)$. This operation subtracts from each column j ($j \geq 2$) of $\mathbf{S}$ the first column and gives a matrix with all zeros at the last row. Equivalently, this gives $\bar{\mathbf{F}} = \mathbf{F}\mathbf{C}_n = \bar{\mathbf{R}}^T\bar{\mathbf{S}}$,

where $\bar{\mathbf{F}}$ is a $m \times (n-1)$ matrix with entries $\bar{f}_{ij} = f_{i,j+1}^2 - f_{i1}^2 - 2f_{i,j+1}o_{j+1} + 2f_{i1}o_1$, $\bar{\mathbf{R}}_i = \begin{bmatrix} 1 & \mathbf{r}_i^T \end{bmatrix}^T$ and $\bar{\mathbf{S}}_j = \begin{bmatrix} \mathbf{s}_{j+1}^T\mathbf{s}_{j+1} - o_{j+1}^2 - (\mathbf{s}_1^T\mathbf{s}_1 - o_1^2) & -2(\mathbf{s}_{j+1} - \mathbf{s}_1)^T \end{bmatrix}^T$. This effectively gives a constraint that the matrix $\bar{\mathbf{F}}$ is at most of rank $K + 1$. Let $\mathbf{A} = \{f_{ij}^2 - f_{i1}^2\}_{j\geq 2}$, $\mathbf{B} = \{-2f_{ij}\}_{j\geq 2}$, $\mathbf{c} = \{2f_{i1}\}_{i\geq 1}$ and $\mathbf{e}$ is a $(n-1) \times 1$ vector $[o_1, \ldots, o_1]^T$, $\mathbf{T}$ is the diagonal matrix with $\{o_j\}_{j\geq 2}$ as entries. We have

$$\bar{\mathbf{F}} = \bar{\mathbf{R}}^T\bar{\mathbf{S}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{T} \\ \mathbf{e}^T \end{bmatrix}. \quad (1)$$

Given the first column of $\bar{\mathbf{R}}^T$ are all ones, there exist $(K+1)$ columns of $\bar{\mathbf{F}}$ whose linear combination forms a column of ones. If we choose $m = 2K + 3$ ($\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{c} \end{bmatrix}$ is of full rank) and $n = K + 2$ ($\bar{\mathbf{F}}$ is of rank $K + 1$), we can find a unique solution for $\mathbf{u}$ to the following system:

$$\bar{\mathbf{F}}\mathbf{w} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{c} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{T} \\ \mathbf{e}^T \end{bmatrix} \mathbf{w}}_{\mathbf{u}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{c} \end{bmatrix}\mathbf{u} = \mathbb{1}_{2K+3}. \quad (2)$$

Then we can recover the offsets $\{o_j\}$ as $o_1 = \frac{u_{2K+3}}{\sum_{j=1}^{K+1} u_j}$ and $o_j = \frac{u_{j+K}}{u_{j-1}}$ for $j = 2, ..., K + 2$. For cases in 3D ($K = 3$), we need only 9 receivers and 5 transmitters.

3.1.2. Non-linear Rank Constraints

We further utilize the similar structure in $\mathbf{R}$. By multiplying $\bar{\mathbf{F}}$ from the left with $\mathbf{C}_m = \begin{bmatrix} -\mathbb{1}_{m-1} & \mathbf{I}_{m-1} \end{bmatrix}^T$, this gives correspondingly $\tilde{\mathbf{F}} = \mathbf{C}_m^T\mathbf{F}\mathbf{C}_n = \tilde{\mathbf{R}}^T\tilde{\mathbf{S}}$. Here $\tilde{\mathbf{R}} = \mathbf{R}\mathbf{C}_m$ and $\tilde{\mathbf{S}} = \mathbf{S}\mathbf{C}_n$ and $\tilde{\mathbf{F}}$ is of size $(m-1) \times (n-1)$. Given that the first row of $\tilde{\mathbf{R}}$ and the last row of $\tilde{\mathbf{S}}$ are all zeros, the equality $\tilde{\mathbf{F}} = \tilde{\mathbf{R}}^T\tilde{\mathbf{S}}$ is preserved after removing the last row of $\tilde{\mathbf{R}}$ and the first row of $\tilde{\mathbf{S}}$. We then have $\tilde{\mathbf{R}}_i = \begin{bmatrix} (\mathbf{r}_{i+1} - \mathbf{r}_1)^T \end{bmatrix}$, $\tilde{\mathbf{S}}_j = \begin{bmatrix} -2(\mathbf{s}_{j+1} - \mathbf{s}_1)^T \end{bmatrix}$ and $\tilde{\mathbf{F}} = \tilde{\mathbf{R}}^T\tilde{\mathbf{S}} = \{\tilde{f}_{ij}\}$ with $\tilde{f}_{ij} = g_{ij} - g_{0j} - g_{i0} + g_{00}$, where $g_{ij} = f_{i+1,j+1}^2 - 2f_{i+1,j+1}o_{j+1}$. It is clear that the matrix $\tilde{\mathbf{F}}$ is at most of rank K. Therefore, given that each entry of $\tilde{\mathbf{F}}$ is a function of the unknown offsets $\{o_1, \ldots, o_n\}$, we can solve for the offsets by enforcing

these rank constraints on the sub-matrices of $\tilde{\mathbf{F}}$. Specifically, all $(K+1) \times (K+1)$ sub-matrices of $\tilde{\mathbf{F}}$ (if there exist) will be rank-deficient and have rank K. This gives equivalently constraints on the determinants of the set of $(K+1) \times (K+1)$ sub-matrices $\Lambda_{K+1} : \det \mathbf{Q} = 0, \ \forall \mathbf{Q} \in \Lambda_{K+1}$.

For a $(m-1) \times (n-1)$ matrix $\tilde{\mathbf{F}}$ of rank K, the number of constraints $N_c = |\Lambda_{K+1}| = \binom{m-1}{K+1}\binom{n-1}{K+1}$ among which $(m-1-K)(n-1-K)$ constraints are linearly independent. Each constraint is a polynomial equation of degree $K+1$ in $\{o_1, \ldots, o_n\}$. For different choices of m and n, this system of polynomials equations can either be well-defined, over-determined or under-determined. To resolve this, we rely on algebraic geometry tools and make use of *Macaulay2* [15]. It turns out that there are several choices for m and n that produce well-defined and solvable polynomial systems. We summarize those cases and the number of solutions of the related polynomial systems for $K = 3$ and $K = 2$ in Table 1. In the following discussion, we denote the case with m receivers and n transmitters as mr/ns. Note that the two cases with only 1 solution: 9r/5s in 3D and 7r/4s in 2D correspond to the linearly solvable cases discussed in Section 3.1.1. Given these solvable cases, we can apply numerically stable polynomial solvers based on methods described in [16] to solve for the unknown offsets.

One could say that we are using necessary constraints on the corrected matrix $\mathbf{D}$ with entries $d_{ij}^2 = (f_{ij} - o_j)^2$ to determine the offsets. Notice, however, this constraint is a necessary, but not sufficient condition on $\mathbf{D}$ coming from TOA measurements. For instance, although 7r/6s is a minimal case for determining the offsets from the $\text{rank}(\mathbf{C}_m^T \mathbf{D} \mathbf{C}_n) = 3$, the resulting TOA problem is actually over-determined [8].

3.2. Solving TOA Calibration

Once we have calibrated the measurement matrix with the offsets $\{o_j\}$ estimated as in Section 3.1, we can proceed to solve the locations of $\{\mathbf{r}_i\}$ and $\{\mathbf{s}_j\}$ as a TOA problem. We follow the two-step technique in [8] that reduce the TOA problem to solving a system of polynomials with $N = K + (K+1)K/2$ unknowns e.g. for K = 3, N = 9. Due to the limited space here, we here briefly discuss the related modification and we refer to [8] for more technical details. Specifically, after a factorization step using singular value decomposition (requiring $m \geq 4$ and $n \geq 4$), for m receivers and n transmitters, one obtains $m-1$ linear equations and n polynomial equations. Then the linear equations are utilized to reduce the number of unknowns further. There are two minimal configurations i.e. $m = 6, n = 4$ (equivalently $m = 4, n = 6$) and $m = 5, n = 5$, which are difficult to solve. For all our solvable cases for unknown offsets in 3D, we can actually utilize the extra measurements to form as many linear equations as possible. For instances, for the case 9r/5s, one can eliminate 8 out of the 9 unknowns utilizing the 8 linear equations. Then we can solve for the only remaining unknown with one of

non-linear equations (among the 5) with companion matrix. For other cases, we just need to solve corresponding polynomial systems which are much easier to solve than the minimal cases.

3.3. Solving TDOA Self-Calibration

We can combine the steps for unknown offsets and the TOA problem to solve the full TDOA problem. To this end, we have devised a set of close-to-minimal solvers for both 3D and 2D TDOA self-calibration problem. We discuss here schemes for solving the problem with over-determined measurements.

We can apply similar strategy as incremental structure from motion in computer vision. One starts by choosing m^* receivers that have largest number of correspondences from n^* transmitters and solves for the offsets. Then the offsets of remaining transmitters can then be solved incrementally with least square followed by also a non-linear optimization to refine the solution. We can then recovered the positions of the chosen receivers and transmitters. The positions of remaining receivers and transmitters are calculated by trilateration e.g. [17]. In the presence of outliers, our proposed solvers can be utilized for robust fitting technique e.g. RANSAC.

The parameters obtained can then be used as initial estimates to local optimization of the non-linear least squares $\min_{\mathbf{r}_i, \mathbf{s}_j, o_j} \sum_{ij} \left(f_{ij} - (||\mathbf{r}_i - \mathbf{s}_j||_2 + o_j) \right)^2$ using standard techniques (Levenberg-Marquart) in order to obtain the maximal likelihood estimate of the parameters.

4. EXPERIMENTS

4.1. Synthetic Data

In this section, we study the numerical behaviors of the TDOA solvers on synthetic data. We simulate the positions of microphones and sounds as 3D points with independent Gaussian distribution of zero mean and identity covariance matrix. As for the offsets, we choose them from independent Gaussian distribution with zero mean and standard deviation 10. We study the effects of zero-mean Gaussian noise on the solvers, where we vary the standard deviation of the Gaussian noise added to the TDOA measurements. When solving TOA problem, we have used the scheme discussed in Section 3.2 for over-determined cases. To compare the reconstructed positions of microphones and sounds with the true positions, we rotate and translate the coordinate system accordingly. We can see that from Fig. 2, our proposed solvers $9r/5s$, $7r/6s$ and $6r/8s$ give numerically similar results as the $10r/5s$ case in [9] for both minimal settings and over-determined cases. In Fig.3, random initialization of the time offsets resulted in poor convergence in the non-linear optimization, while our method provides with a much better starting point. On the other hand, we have also compared our solver with the iterative method proposed in [12] for estimating the offsets. The method in [12] converges very slowly ($5sec. - 1min.$ on
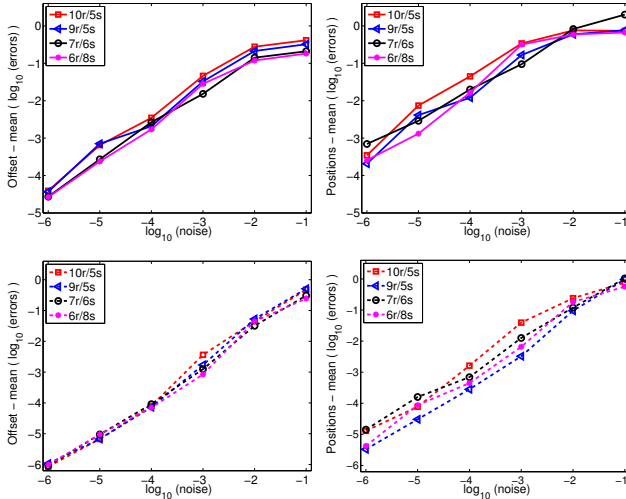
Fig. 2. Synthetic experiments for TDOA solvers on 3D under Gaussian noise. The errors of estimated time offsets (left) and reconstructed positions of microphones and sounds (right) are shown. (Top) Performance of different solvers (10r/5s [9], 9r/5s, 7r/6s and 6r/8s) with their corresponding minimal settings for solving offsets; (Bottom) with 20 receivers and 20 microphones.

a MacBook Air with i5 1.8GHz core, especially for (near) minimal settings) and tends to converge to the wrong local minima. While our solvers perform consistently well for all cases, they are also much faster (approximately $0.5s$ for the unoptimized codes). This suggests the usability of our proposed solvers in RANSAC as well as for practical settings with with limited availability of the receivers.

4.2. Real Data

To collect real TDOA data, we work with sound signals and microphones. We placed 8 synchronized microphones (Shure SV100) recorded at 44.1kHz in an office. They are approximately 0.3-1.5 meters away from each other in a non-planar fashion. We connected them to an audio interface (M-Audio Fast Track Ultra 8R), which is connected to a computer. We generated sounds by moving around in the room and clapping approximately 1-2 meters from the microphones. We collected 5 independent recordings of approximately 20s. Each recording contained roughly 30 claps (transmitters).

To obtain TDOA measurements, we coarsely matched sounds of the claps to sound flanks (edges between periods with low energy and periods with high energy) recorded from different microphones. For the experiment we used only those claps that were detected in all 8 channels. We run both the 7r/6s and 6r/8s solvers to determine the offsets followed by an alternating optimization that refines the offset estimation. After solving the unknown transformation and translation, we recover an initial euclidean reconstruction for
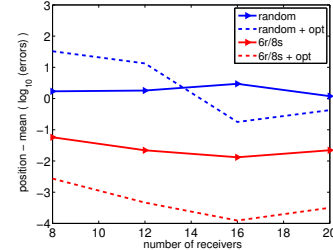

Fig. 3. Initialization with random offsets and our 6r/8s solver with varying number receivers (8 to 20) and 8 transmitters (noise level 10^{-4}) for non-linear optimization.

the locations of microphones and claps. Finally we refine the reconstruction with non-linear optimization. The result of one of these 5 reconstructions are shown in Figure 4 (middle). The reconstructed microphone positions from these 5 independent multi-channel recordings were put in a common coordinate system and compared to each other. The average distance from each microphone to the its corresponding mean position (estimated from corresponding reconstruction of the 5 recordings) is 2.60 cm. It is important to point out that without proper initialization using our methods, the solutions we get converge poorly (with large reconstruction errors). Previous solvers do not work here due to either insufficient number of receivers (10 receivers needed in [9]) or violating the assumption that one of the microphones collocates with one of the claps [6] .

As an additional evaluation, we have also reconstructed the locations of the microphones based on computer vision techniques. We took 11 images of the experimental setup. Figure 4 (left) shows one of the 11 images used. We manually detected the 8 microphone center positions in these 11 images and used standard structure from motion algorithms to estimate the positions of the 8 microphones. The resulting reconstruction is also compared to that of the five structure from sound reconstructions. The comparison is shown in Figure 4 (right). We can see the TDOA-based reconstructions are consistent with the vision-based reconstruction.

5. CONCLUSIONS

In this paper we have studied the sensor network calibration problem in the time-difference-of-arrival (TDOA) setting, where only relative distances between the transmitters and receivers are given. We have formulated the problems utilizing stricter non-linear constraints on the measurements, which is the key to reduce the required number of measurements to solve these problems. We have shown that our non-iterative solvers are fast and numerically stable[1].

There are several interesting avenues of future research. Although, the presented technique improve on the state-of-the-art, the minimal cases for TDOA structure from sound

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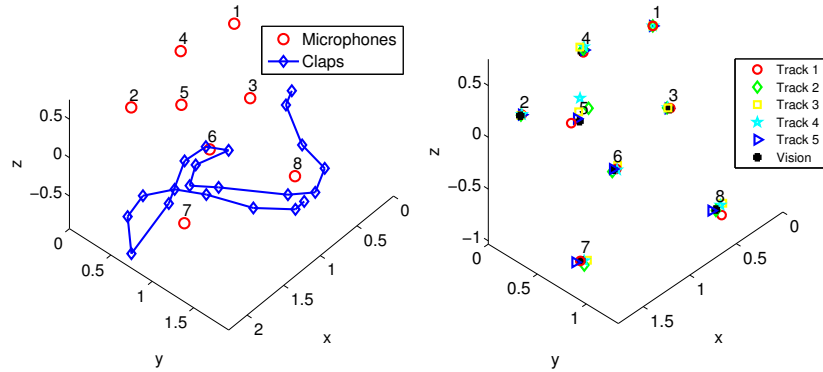[1]Codes are available at http://www2.maths.lth.se/vision/downloads/.

Fig. 4. Results on TDOA with microphones and sounds. Left : Reconstruction of microphone (8, red - 'o') and sound (21, blue - '◇') positions for one the 5 recordings; Right : Reconstructed microphone positions estimated from 5 different tracks of TDOA measurements and the corresponding reconstruction from computer vision (black - '+')

have not been solved. It would be interesting to solve these cases, to study the failure modes, both generic failure modes (or critical configurations) for the generic problem, and also if there are additional failure modes of the presented algorithms. One commonly encountered critical configuration is when the transmitters or receivers span a lower dimensional space, e.g. if the receivers or transmitters are all on a plane or on a line.

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