OPTIMUM DESIGN OF DISCRETE TRANSMIT PHASE ONLY BEAMFORMER

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ABSTRACT

Transmit beamformer design usually results suboptimum beamformers in multicast scenario. In this paper, discrete phase only beamformer design is considered. The design problem is cast in such a form that the solution is always feasible and optimum beamformer can be found using the branch and cut algorithm. Beamformer phase terms and their interrelations are expressed with a set of cosine vectors which lead to linear set of constraint equations that can be solved with mixed integer linear programming. It is shown that the proposed approach is very effective and the number of quantization bits can be increased to obtain results close to optimum continuous phase beamformers.

Index Terms— Transmit beamformer, phase only beamformer, mixed integer linear programming, branch and cut

1. INTRODUCTION

Transmit beamformer design is an important problem which has found widespread applications in different fields including communications, radar, etc. In this paper, "$multicast$" beamforming scenario is considered where the transmitter transmits the same information to several users spread geographically [1]. Therefore simultaneous beams should be directed to users.

Phase is the most critical parameter of a beamformer and it determines the general shape of the beam. Magnitude is also important, especially to reduce the sidelobe level. In this paper, phase only (PO) beamformer design is considered. This is done due to both practical and theoretical concerns. In practice, radar systems use discrete phase transmit beamformers. From a theoretical perspective, dealing with only phase terms decreases the number of variables and in our case allows us to obtain a convenient optimization formulation which leads to optimum solutions.

While PO beamformer design is not a new concept, previous approaches are usually simplistic and deal with only a single beam or a single target [2]. For a single target, optimum beamformer is trivial and it is the array steering vector [3]. When there are more than one target, the problem becomes complex and usually some kind of optimization approach should be followed. Straightforward application of convex optimization with semidefinite relaxation does not give satisfactory results. In fact, relaxation done by dropping the rank condition [1] usually generates higher rank solutions.

In this paper, discrete PO transmit beamformer design is considered and maxmin formulation for the optimum solution is developed. This formulation is based on some cosine vectors which are used to obtain the phase contributions as well as the relations between different phase terms. The advantage of this new formulation is that, we obtain mixed integer linear programming structure which is completely feasible and guaranteed to give the optimum solution. To our knowledge, this method is the only one which guarantees the optimum solution in a multicast transmit beamformer design. As the number of bits for phase quantization increases, the discrete PO beamformer approaches to the continuous PO beamformer. However, it should be noted that there is no guarantee that the optimum continuous phase PO beamformer is obtained by simply increasing the number of bits of optimum discrete phase PO beamformer. Furthermore, quantizing the phase angles of an optimum continuous phase PO beamformer may not result the discrete optimum solution. In this paper, it is shown that even with a moderate number of bits, the quality of the discrete PO beamformer is as good as the continuous PO beamformer. Several experiments are done and the effectiveness of the proposed method is shown.

2. PHASE ONLY BEAMFORMER DESIGN

Phase only beamformers are used in practice in order to transmit power to targets efficiently. While continuous phase transmit beamformer design can be done effectively [4], there is no guarantee for the optimality of the design. In this paper, it is shown that optimum discrete phase PO beamformers can be obtained if the problem is cast in a suitable form for mixed integer linear programming. In this respect, the beamformer phase terms and the relations between these phases are set using cosine vectors. This formulation eliminates the main barrier for optimum solutions, namely the "$rank$" condition [1], [5].

2.1. Continuous Phase Only Beamformer

Consider a base station equipped with M transmitting antennas to transmit the common signal to N receivers, each having a single antenna. Assume that the antennas and path losses are identical. Transmitted signal is narrowband and propagation is nondispersive. The transmitted signal can be written as,

$$\mathbf{x}(t) = s(t)\mathbf{w} \tag{1}$$

where $s(t)$ is the source signal and $\mathbf{w}$ is the Mx1 complex beamformer weight vector. The received signal at k^{th} receiver is given as,

$$y_k(t) = \mathbf{h}_k^H \mathbf{x}(t) + n_k(t) \quad k = 1, \ldots N \tag{2}$$

where $\mathbf{h}_k$ is the Mx1 complex channel vector for the k^{th} receiver and n_k is additive noise with variance σ_k^2. Signal-to-noise ratio (SNR) for the k^{th} receiver is,

$$SNR_k = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{h}_k|^2}{\sigma_k^2} \tag{3}$$

where σ_s^2 is the source signal variance. σ_s^2 = 1 is selected for simplicity in the following part.

The transmit beamforming "$maxmin$" problem is to choose beamforming weight vector in order to maximize the minimum power that is transmitted to any target. Considering P_{an} as the antenna power, the PO beamformer problem can be written as follows [4],

$$\begin{aligned} &\max_{\mathbf{w}} t \\ s.t. \quad &\mathbf{w}^H \mathbf{R}_k \mathbf{w} \geq t\gamma_k \sigma_k^2, \quad k = 1, ..., N \\ &(\mathbf{w}\mathbf{w}^H)_{k,k} = P_{an} \end{aligned} \tag{4}$$

where γ_k is the power proportion for the kth target, $\mathbf{R}_k = E\{\mathbf{h}_k \mathbf{h}_k^H\}$. Let us define $\mathbf{W} = \mathbf{w}\mathbf{w}^H$. The problem for PO beamformer can be written as,

$$\begin{aligned} &\max_{\mathbf{W}} t \\ s.t. \quad &Tr\{\mathbf{R}_k \mathbf{W}\} \geq t\gamma_k \sigma_k^2, \quad k = 1, ..., N \\ &\mathbf{W}_{k,k} = P_{an} \\ &\mathbf{W} \succeq 0 \\ &rank(\mathbf{W}) = 1 \end{aligned} \tag{5}$$

The problem can be solved by convex optimization with semidefinite relaxation [1], [5]. Let us denote the solution found by semidefinite relaxation as $\mathbf{V}$ and the principal eigenvector of $\mathbf{V}$ as $\mathbf{v} = \mathcal{P}(\mathbf{V})$. If the solution matrix $\mathbf{V}$ has rank one then $\mathbf{v}\mathbf{v}^H = \mathbf{V}$ and $\mathbf{v}$ is the optimum beamforming weight vector. If $\mathbf{V}$ is not a rank one matrix, magnitude of the beamformer vector elements are taken as $\mu_i = \sqrt{P_{an}}$. Also phase of the principal eigenvector, $\arg v_i$, is fixed as initial values whereas ψ_i is the optimization parameter, i.e., $\hat{w_i} = \mu_i e^{j(\arg v_i + \psi_i)}$.The nonlinear optimization problem for PO beamformer can be written as,

$$\begin{aligned} &\max_{\psi_i} \ t \\ s.t. \quad &\mathbf{w}^H \mathbf{R}_k \mathbf{w} \geq t\gamma_k \sigma_k^2, \quad k = 1, ..., N \\ &|\mathbf{w}_i| = \sqrt{P_{an}} \quad i = 1, ..., M \end{aligned} \tag{6}$$

Above nonlinear optimization problem can be solved by using nonlinear solvers. In our case, "$fmincon$" in MATLAB with interior point algorithm and gradient search is used. This approach finds the local optimum as it is expected in any nonlinear optimization problem in general. Fortunately, the problem has several local optima, which are usually close to the global maximum quality. This phenomenon is known as the "$glass\ break$" in [1].

2.2. Discrete Phase Only Beamformer Design

When the phase angles of the beamformer are selected from a discrete set, it is possible to find an optimum solution for the transmit beamformer. The discrete phase PO transmit beamformer problem can be written as,

$$\max_{\psi_i} \ t$$

$$s.t. \quad \mathbf{w}^H \mathbf{R}_k \mathbf{w} \geq t\gamma_k \sigma_k^2, \quad k = 1, ..., N \tag{7}$$

$$|\mathbf{w}_i| = \sqrt{P_{an}} \quad i = 1, ..., M \tag{8}$$

$$\psi_i \in \{0, \theta, 2\theta, ...(2^n - 1)\theta\} \tag{9}$$

$$\theta = \frac{2\pi}{2^n} \tag{10}$$

where n is the number of bits to represent the discrete phase angles. Since $\mathbf{R}_k$ is a Hermitian symmetric matrix and $|\mathbf{w}_i|$ is constant, the inequality in (7) can be expressed in a cosine form, i.e.,

$$\begin{aligned} &\max_{\psi_i} \ t \\ s.t. \quad &\sum_{i=1}^{M-1} \sum_{j=i+1}^{M} 2P_{an} cos(\arg \mathbf{R}_k^{ij} + \psi_j - \psi_i) + MP_{an} \geq t\gamma_k \sigma_k^2 \\ &k = 1, ..., N \\ &\psi_i \in \{0, \theta, 2\theta, ...(2^n - 1)\theta\} \end{aligned} \tag{11}$$

where $\mathbf{R}_k^{ij}$ is the i^{th} row and j^{th} column element of the $\mathbf{R}_k$ matrix. Since the first element of the beamformer vector can be normalized without loss of generality, $\psi_i = 0$ is selected. Then the problem becomes,

$$\begin{aligned} &\max_{\psi_i} \ t \\ s.t. \quad &\sum_{m=2}^{M} 2P_{an} cos(\arg \mathbf{R}_k^{1m} + \psi_m) \\ &+ \sum_{i=2}^{M-1} \sum_{j=i+1}^{M} 2P_{an} cos(\arg \mathbf{R}_k^{ij} + \psi_j - \psi_i) + MP_{an} \geq t\gamma_k \sigma_k^2 \\ &k = 1, ..., N \\ &\psi_i \in \{0, \theta, 2\theta, ...(2^n - 1)\theta\} \end{aligned} \tag{12}$$

Note that the setting in (12) is more than a simplification since in the following parts, the angular differences, $\psi_j - \psi_i$, are expressed in terms of only the phase terms. In order to do this, $\beta_{ij} = \psi_j - \psi_i$, is defined where $i = 2, ..., M-1$ and $j = i+1, ..., M$. If the cosine terms in (12) are decomposed using the trigonometric identities, the following problem setting is obtained,

$$\begin{gathered}
\max_{\psi_m, \beta_{ij}} \; t \\
s.t. \sum_{m=2}^{M} 2P_{an}(cos(\arg \mathbf{R}_k^{1m})cos(\psi_m) - sin(\arg \mathbf{R}_k^{1m})sin(\psi_m)) \\
+ \sum_{i=2}^{M-1} \sum_{j=i+1}^{M} 2P_{an}(cos(\arg \mathbf{R}_k^{ij})cos(\beta_{ij}) - sin(\arg \mathbf{R}_k^{ij})sin(\beta_{ij})) \\
+MP_{an} \geq t\gamma_k \sigma_k^2 \qquad (13) \\
k = 1, ..., N \\
\psi_i \in \{0, \theta, 2\theta, ...(2^n - 1)\theta\} \\
\beta_{ij} = +\psi_j - \psi_i, \; i = 2, ..., M-1 \; and \; j = i+1, ..., M \\
\psi_m \in \{0, \theta, 2\theta, ..., (2^n - 1)\theta\} \;\; m = 2, ..., M \\
\beta_{ij} \in \{-(2^n - 1)\theta, ..., -\theta, 0, \theta, ..., (2^n - 1)\theta\} \\
i = 2, ..., M-1, \; j = i+1, ..., M \qquad (14)
\end{gathered}$$

In order to convert the problem in (13-14) into a form suitable for integer programming, the following row vectors are defined,

$$\begin{gathered}
\mathbf{c} = [\; 1 \; cos\theta \; cos2\theta \; ... \; cos(2^n - 1)\theta \;] \\
\tilde{\mathbf{c}} = [\; cos(-(2^n - 1)\theta) \; ... \; cos(-\theta) \; 1 \; cos\theta \; ... \cos(2^n - 1)\theta \;] \\
\mathbf{s} = [\; 0 \; sin\theta \; sin2\theta \; ... \; sin(2^n - 1)\theta \;] \\
\tilde{\mathbf{s}} = [\; sin(-(2^n - 1)\theta) \; ... \; sin(-\theta) \; 0 \; sin\theta \; ... \sin(2^n - 1)\theta \;]
\end{gathered} \qquad (15)$$

where θ is given in (10). Above vectors are used to select each discrete phase angle contribution in a convenient form. Note that the contributions for $cos\psi_m$, $sin\psi_m$, $cos\beta_{ij}$, and $sin\beta_{ij}$ are now represented by $\mathbf{c}$, $\mathbf{s}$, $\tilde{\mathbf{c}}$, and $\tilde{\mathbf{s}}$ respectively. Let $\hat{\mathbf{v}}_p$ be the vector of zeros of length 2^n with the p^{th} element equal to one. Let $\hat{\mathbf{u}}_p$ be the vector of zeros of length $(2^{n+1} - 1)$ with the p^{th} element equal to one, i.e.,

$$\hat{\mathbf{v}}_p = [\; 0 \; ... \; \underbrace{1}_{p^{th} \text{ element}} \; ... \; 0 \;]^T$$

$$\hat{\mathbf{u}}_p = [\; 0 \; ... \; 0 \; \underbrace{1}_{p^{th} \text{ element}} \; 0 \; ... \; 0 \;]^T$$

These vectors become the variables of the final optimization setting. The multiplication of these vectors with $\mathbf{c}$, $\mathbf{s}$, $\tilde{\mathbf{c}}$, and $\tilde{\mathbf{s}}$ generate the inequality expression in (13). Therefore the problem in (13-14) can be written as,

$$\begin{gathered}
\max_{\mathbf{v}_m, \mathbf{u}_{ij}} \; t \\
s.t. \sum_{m=2}^{M} \mathbf{Q}_m \mathbf{v}_m + \sum_{i=2}^{M-1} \sum_{j=i+1}^{M} \mathbf{Z}_{ij} \mathbf{u}_{ij} + MP_{an} \geq t\gamma_k \sigma_k^2 \\
k = 1, ..., N \\
[\; 0 \; 1 \; ... \; (2^n - 1) \;](-\mathbf{v}_i + \mathbf{v}_j) = \\
[\; -(2^n - 1) \; ... \; -1 \; 0 \; 1 \; ... \; (2^n - 1) \;]\mathbf{u}_{ij} \qquad (16) \\
i = 2, ..., M-1, j = i+1, ..., M \\
\mathbf{v}_m \in \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, ..., \hat{\mathbf{v}}_{2^n}\} \\
\mathbf{u}_{ij} \in \{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, ..., \hat{\mathbf{u}}_{2^{n+1}-1}\} \qquad (17)
\end{gathered}$$

where

$$\mathbf{Q}_m = 2P_{an}cos(\arg \mathbf{R}_k^{1m})\mathbf{c} - 2P_{an}sin(\arg \mathbf{R}_k^{1m})\mathbf{s} \qquad (18)$$

$$\mathbf{Z}_{ij} = 2P_{an}cos(\arg \mathbf{R}_k^{ij})\tilde{\mathbf{c}} - 2P_{an}sin(\arg \mathbf{R}_k^{ij})\tilde{\mathbf{s}} \qquad (19)$$

In the above formulation $\mathbf{Q}_m$ and $\mathbf{Z}_{ij}$ vectors are known and generate a linear inequality. In addition, there is no rank condition which is ignored during semidefinite relaxation [1], [5].

In the above formulation, $\mathbf{v}_m$ is the primary variable and $\mathbf{u}_{ij}$ is the secondary variable to generate the constraints between the phase terms and their differences as in (16). The above problem setting generates always a feasible solution and is suitable for mixed integer linear programming. Note that a solution for both the inequality and equality constraints can always be found from the given discrete set.

Once the solution for $\mathbf{v_m}$ is found, the phase angles for the beamformer vector are obtained as,

$$\psi_m = \mathbf{d}_\psi^T \mathbf{v}_m$$

where $\mathbf{d}_\psi = [\; 0 \; \theta \; 2\theta \; ... \; (2^n - 1)\theta]^T$ and θ as in (10).

The optimization problem in (16-19) is solved using the branch and cut algorithm [6] with mixed integer linear programming. The problem in (16-19) matches with the linear model given in [6](p.145). It is known that mixed integer linear programming with branch and cut gives the optimum solution for such an optimization problem [6](Theorem 1, p.147).

3. SIMULATIONS

Proposed mixed integer linear programming for discrete phase PO design is implemented by using the optimization solver "Gurobi" [7].

M=6 transmit antennas are used in a 3x2 planar array. The distance between each antenna is 0.5λ where λ is the wavelength. There are N=2 receivers which are considered as the targets. The first target is a fixed target at (60°, 90°), which correspond to the azimuth and elevation angles (ϕ, θ), respectively. The second target is variable. Its elevation angle is fixed at 90° degrees and its azimuth angle is varied between 0° to 180° degrees. Different powers are assigned for the two sources and γ_k coefficients are selected as γ_1=1, γ_2=2 respectively for the targets. For simplicity, $P_{an} = 1$ is selected. Noise variance for each channel is the same and σ_k^2=1, k=1,2.

In Fig. 1, optimization parameter "t" in (16) is presented using $n = 3$ bits for phase quantization. Note that $t = \min\{\frac{SNR_1}{\gamma_1}, \frac{SNR_2}{\gamma_2}\}$. In this respect, "$t$" shows the quality of the solution. The larger it is, the better is the solution. Results of both brute force search and mixed integer linear programming are compared. In case of brute force approach, all the possible beamformer solutions are searched and the optimum is found. As it is seen in Fig. 1, both approaches have the same solution, which is expected due to the fact that proposed approach is optimum. In our case, the computational complexity is significantly lower than the brute force.

In Fig. 2, SNR for the first and second targets are shown for $n = 3$ bits of phase quantization. The first target has a staircase characteristic due to the discrete nature. The beamformer does not change for Target 1 for a certain period. While the same beamformer is in action for Target 2, its angular position changes at each azimuth angle scan and therefore it does have different power at each angle. So, Target 2 shows a more smooth characteristic.

In Fig. 3, discrete and continuous optimum PO beamformers are compared and mean square error (MSE) between their beampatterns is presented.

$$MSE = \frac{1}{180} \sum_{\phi=1^\circ}^{180^\circ} (P_C(\phi) - P_D(\phi))^2$$

where $P_C(\phi)$ and $P_D(\phi)$ are the transmitted power to the direction (ϕ, 90°) by continuous phase PO beamformer and discrete phase PO beamformer, respectively. Continuous optimum PO beamformer is obtained using convex optimization for the second target fixed at (95°, 90°). At this specific case, convex optimization results rank one solution and it is the optimum solution. The number of bits, n, for the discrete PO beamformer is increased from 3 to 8 to see the difference for the beampatterns. As it is seen from this figure, $n = 6$ is sufficient to get a good approximation with discrete PO beamformer.

In Fig. 4, the beampatterns for both discrete and continuous optimum PO beamformers are shown. Discrete PO beamformer uses $n = 6$ bits. The beampatterns are very close to each other indicating the good performance of the discrete PO beamformer.

As it is pointed before, continuous PO beamformer design is not guaranteed to give the optimum solution. One of the best approaches for continuous PO beamformer design is given in [4]. The structure of this beamformer is discussed in section 2.1. In Fig. 5, this beamformer is compared with the discrete optimum beamformer. The optimization parameter "t" is shown in this figure. In this case, Target 1 is at the same angular position, i.e., (60°, 90°) and Target 2 elevation is fixed at 90°, but its azimuth angle is varied between 0° and 180°. $n = 6$ is selected for the discrete phase PO beamformer. As it is seen from this figure, continuous phase beamformer becomes suboptimum at several points and discrete PO solution is either better or very close to the continuous phase solution.

4. CONCLUSION

In this paper, discrete phase PO transmit beamformer design is considered. The problem is cast as a maxmin design suitable for mixed integer linear programming. The formulation is based on some cosine and sine vectors of discrete phase angles. Such a formulation generates a feasible solution and can be solved effectively using branch and cut algorithm. It is shown that the proposed method results optimum solution and as the number of quantization bits are increased, the quality of the beamshape approaches to the continuous phase solution. In this respect, only 6 bits are sufficient to get good results. On the other hand, there is no guarantee that quantizing the phase angles of an optimum continuous PO beamformer results the optimum discrete PO beamformer. This fact can be easily checked using single user or target solutions.

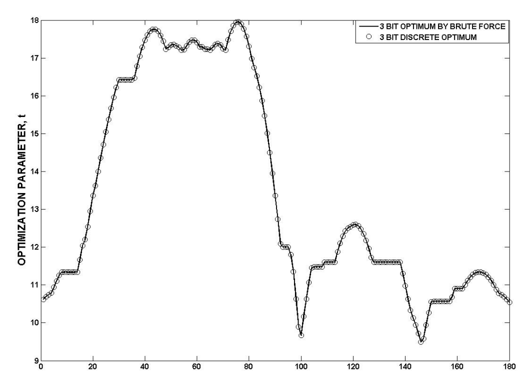

Fig. 1. Optimization parameter "t" using $n = 3$ bits for brute force search and mixed integer linear programming.

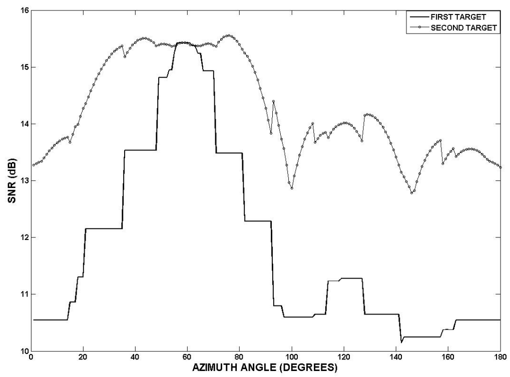

Fig. 2. SNR for first target and second target using $n = 6$ bits for discrete solution.

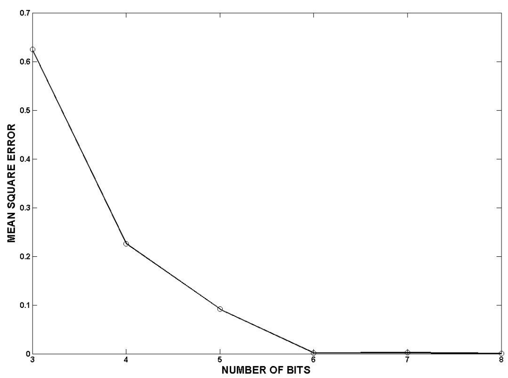

Fig. 3. MSE between beampatterns of continuous and discrete phase solution versus number of bits.

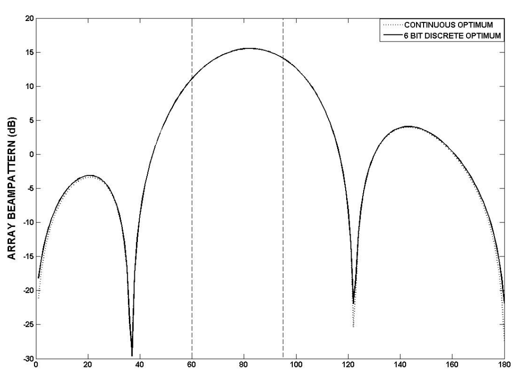

Fig. 4. Beampatterns for continuous and discrete solution using $n = 6$ bits.

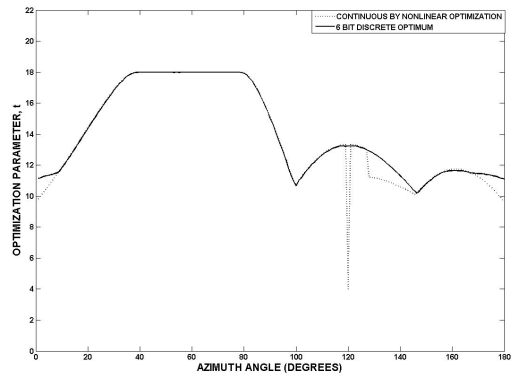

Fig. 5. Optimization parameter "t" versus azimuth angle for continuous and discrete solution.

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