NEW DECODING STRATEGY FOR UNDERDETERMINED MIMO TRANSMISSION USING SPARSE DECOMPOSITION

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ABSTRACT

In this paper we address the problem of large dimension decoding in MIMO systems. The complexity of the optimal maximum likelihood detection makes it unfeasible in practice when the number of antennas, the channel impulse response length or the source constellation size become too high. We consider a MIMO system with finite constellation and model it as a system with sparse signal sources. We formulate the decoding problem as an underdetermined sparse source recovering problem and apply the ℓ_1-minimization to solve it. The resulting decoding scheme is applied to large MIMO systems and to frequency selective channel . We also review the computational cost of some ℓ_1-minimization algorithms. Simulation results show significant improvement compared to other existing receivers.

1. INTRODUCTION

Multiple input multiple output (MIMO) systems have known a regain of interest since the mid nineties as a solution to increase the data rate in wireless point-to-point networks at the expense of a multi-antenna interference (MAI) [1]. The performance of the MIMO system is conditioned to a clever management of the MAI. Maximum likelihood joint detection enables to detect at once the signals transmitted in the same time interval [2]. It has been proved to minimize the error probability for medium to high signal to noise ratio (SNR) values. It can be applied even if the decoding problem is underdetermined [3] (for example when the number of transmit antennas is higher than the number of receive antennas). However, its complexity grows exponentially with the constellation size and the antenna dimensions, and makes it unfeasible in practice. Linear equalizers such as minimum mean square error (MMSE) and zero-forcing (ZF) present rather low complexity but achieve poor performance when used in underdetermined MIMO systems. Solutions alternative to ML detection and with reduced complexity have been proposed, among which the sphere decoder performs the best provided the searching sphere is well defined. The sphere decoder suffers from three major weaknesses. First its performance depends on the searching sphere radius value. Secondly it is based on an iterative algorithm and the required iteration number is not bounded. Thirdly, the computation cost keeps high in the low-to-medium SNR region and for large MIMO dimensions [4].

In this paper, we consider an underdetermined MIMO system with finite constellations. We aim to find an efficient decoder with polynomial time complexity . We adapt the detection algorithm that we have proposed in [5] to the underdetermined MIMO decoding problem. The first step is the transformation of the MIMO channel with input belonging to a finite alphabet into an equivalent higher-dimensional MIMO channel with sparse input vector. Then the decoding problem becomes equivalent to the ℓ_0-norm minimization problem subject to some constraints that we define. This problem is usually solved using exhaustive search; also NP-hard. On the other hand, the ℓ_0-norm minimization problem can be relaxed by ℓ_1-norm minimization [6]. Based on this relaxation, the transmitted data can be decoded or recovered using iterative algorithms reviewed in [6, 7]. In addition, we show the performance of the proposed decoding scheme for the following applications **i**) large MIMO systems, and **ii**) frequency selective MIMO channel.

This paper is organized as follows. In Section 2, we describe the system model and we reformulate the MIMO channel with finite input constellation,. In Section 3, we introduce the decoding strategy based on ℓ_1-norm minimization. This decoding strategy is then applied to two different applications described in Section 4 In Section 6, the simulation results enable to evaluate our work. Finally, Section 7 concludes the paper.

Notations: boldface upper case letters and boldface lower case letters denote matrices and vectors, respectively. For the transpose, transpose conjugate and conjugate matrices we use $(.)^T$, $(.)^H$ and $(.)^*$, respectively, $||.||_p$ denotes the norm ℓ_p and $\otimes$ is defined as the Kronecker product.

2. SYSTEM MODEL

We consider a MIMO transmission over a flat fading channel, where the transmitter and the receiver are equipped with N and n antennas, respectively. We assume no prior knowledge of the channel state information (CSI) at the transmitter and a perfect CSI knowledge at the receiver. The received signal for MIMO transmission channel is defined as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z}, \tag{1}$$

where $\boldsymbol{H}$ is an $n \times N$ random channel (or mixing) matrix ($n < N$), $\boldsymbol{x}$ is the $N \times 1$ data vector, and $\boldsymbol{z}$ is the $n \times 1$ circularly symmetric additive Gaussian noise vector with zero mean and covariance $\sigma^2\boldsymbol{I}$. The components of $\boldsymbol{x}$ belong to a finite alphabet constellation defined as $\mathcal{Q} = \{q_1, q_2, \cdots, q_M\}$. For example, assuming 4-QAM constellation yields $\mathcal{Q} = \{\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}\}$ and $M = 4$.

The goal of our work is to find out an efficient decoding scheme of the transmitted data symbols with moderate computational complexity order on the whole SNR region. To this end, we exploit the fact that the transmitted data vector components belong to a finite alphabet, and we transform the MIMO channel above into an equivalent MIMO channel where the input data vector becomes sparse. The sparse source vector is then recovered using iterative algorithms usually employed to resolve ℓ_1-minimization problems.

3. MIMO DECODING SCHEME OF AN EQUIVALENT SPARSE INPUT

In this section we exploit the fact that transmitted data symbols belong to the finite alphabet set $\mathcal{Q}$, and we formulate the vector $\boldsymbol{x}$ as

$$\boldsymbol{x} = \boldsymbol{B_q}\boldsymbol{s}, \tag{2}$$

where

$$\begin{aligned}
\boldsymbol{s} &= [\boldsymbol{s}_1,\ \boldsymbol{s}_2,\ \cdots,\ \boldsymbol{s}_N]^T, \text{ and} \\
\boldsymbol{s}_i &= [I(x_i = q_1),\ I(x_i = q_2),\ \cdots,\ I(x_i = q_M)], \\
I(x_i = q_j) &= \begin{cases} 1 & \text{if } x_i = q_j \\ 0 & \text{otherwise} \end{cases} \\
\boldsymbol{B_q} &= \boldsymbol{I}_N \otimes \boldsymbol{q}_M \\
\boldsymbol{q}_M &= [q_1, q_2, \cdots, q_M].
\end{aligned} \tag{3}$$

$I(x_i = q_j)$ is the indicator function indexed by i, $\mathbf{0}_M$ is the zeros vector of length M, and $\boldsymbol{B_q}$ is a block diagonal matrix with dimension $N \times NM$. Substituting (2) into (1) yields

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{B_q}\boldsymbol{s} + \boldsymbol{z}. \tag{4}$$

The new problem is the decoding of the binary source vector $\boldsymbol{s}$, We poropose to solve it using the following minimization problem

$$\arg\min ||\boldsymbol{s}||_0, \quad \text{subject to } \boldsymbol{s} \in \mathbb{S}, \tag{5}$$

where $\mathbb{S} = \left\{||\boldsymbol{y} - \boldsymbol{H}\boldsymbol{B_q}\boldsymbol{s}||_2^2 < \epsilon, \text{ and } \boldsymbol{B_1}\boldsymbol{s} = \mathbf{1}_N\right\}$. The ℓ_0-minimization problems are usually solved using exhaustive search where all coefficients of $\boldsymbol{s}$ belong to $\{0, 1\}$. The $N \times NM$ matrix $\boldsymbol{B}_1$ is defined as

$$\boldsymbol{B_1} = \boldsymbol{I}_M \otimes \mathbf{1}_M^T \tag{6}$$

where $\mathbf{1}_M$ is the ones vector of length M. The equality constraint $\boldsymbol{B_1}\boldsymbol{s} = \mathbf{1}_N$ ensures that the solution $\boldsymbol{B_q}\boldsymbol{s}$ has N nonzero components, and that the decoded symbols belong to the finite input constellation. The other inequality constraint aims to select a codeword within an euclidean less than a constant ϵ from the received signal. Hence, the problem in (5) is somehow similar to the ML decoding problem. However, in our problem the decoded symbols depend heavily on the choice of ϵ, which in turn depends on the current SNR value. In order to judiciously select ϵ, we define it as follows

$$\epsilon = F^{-1}_{\chi^2_d(\rho^2)}(1 - \gamma), \tag{7}$$

where $F_{\chi^2_d(\rho^2)}$ is the cumulative distribution function of the non-central χ^2 distribution $\chi^2_d(\rho^2)$ with d degrees of freedom and non-central parameter ρ^2. The threshold parameters in our problem are defined as follows: $d = 2n$ due to the complex noise, $\rho^2 = (2\log(n))\sigma^2$ i.e. the universal threshold, and $\gamma \in (0, 1]$. Discussion on the optimality of the chosen ϵ is given in the Appendix.

When the minimization problem in (5) is solved via an exhaustive search, the complexity grows exponentially with the number of transmitted symbols per channel use; resulting in an NP-hard problem. That is why the authors in [6] have proposed to replace the ℓ_0-minimization problem by the ℓ_1-minimization, and have shown that the equivalence holds when the restricted isometry property (RIP) is less than a small constant. On the other hand, as mentioned in [8], in our case, the RIP condition is sufficient and not necessary. It is also shown that for binary alphabet, when the random mixing matrix satisfies $n \geq N/2$ the convergence to the optimal solution obtained by the problem in (5) is guaranteed for large N using ℓ_1 minimization. Thus, more transmitted symbols per channel use results in a robuster decoding scheme.

4. APPLICATIONS

In this section, we apply the decoding scheme based on ℓ_1-minimization to the following applications: **i)** Large MIMO systems, **ii)** Frequency selective MIMO channel.

4.1. large MIMO systems

Large MIMO systems, also known as large-scale antenna systems, is a research topic proposed recently in communication theory, antenna systems, and some other fields [9,10]. The increased attention is mainly due to their high spectral efficiencies [1]. Large MIMO scheme involves tens to hundreds of antennas at the transmitter and the receiver sides. The model is similar to the standard MIMO scheme except that the dimensions are larger. Assuming a point-to-point MIMO transmission in flat fading channel, the received signal is expressed as:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z} = \boldsymbol{H}\boldsymbol{B}_q\boldsymbol{s} + \boldsymbol{z}, \tag{8}$$

where $\boldsymbol{H}$ is the $n \times N$ channel matrix, and $\boldsymbol{x}$ is the $N \times 1$ data vector. The equivalence between (8) and (1) is obvious. The decoding problem can be solved using the following minimization problem

$$\arg\min ||\boldsymbol{s}||_1, \quad \text{subject to } \boldsymbol{s} \in \mathbb{S}. \tag{9}$$

where $\mathbb{S}$ is given in (5).

4.2. Frequency Selective MIMO channel

Another application is the frequency selective MIMO channel where $L+1$ multipaths interfere at every channel use. The transmitter and the receiver are equipped with N and n antennas, respectively. We consider a transmission over a frame, and for every transmitted symbols $L+1$ copies are received through $L+1$ different paths. A guard interval is inserted between consecutive frames to avoid inter-symbol interference from one frame to another. The frequency selective channel output at instant n is expressed as

$$\boldsymbol{y}(n) = \sum_{l=0}^{L} \boldsymbol{H}_l \boldsymbol{x}(n-l) + \boldsymbol{z} = \sum_{l=0}^{L} \boldsymbol{H}_l \boldsymbol{B}_a \boldsymbol{s}(n-l) + \boldsymbol{z}, \tag{10}$$

where $\boldsymbol{H}_l$ represents the l^{th} path in the frequency selective channel. In order to decode the original symbol vector, we propose a joint detection over the whole frame. Thus, instead of applying the decoding process T times, one time for each data vector, we only decode at once a concatenated vector which consists of all transmitted vectors. The channel model is reformulated as

$$\boldsymbol{y}_{T_1} = \left(\boldsymbol{I}_{N.T} \otimes \left[\boldsymbol{H}_0^T \cdots \boldsymbol{H}_L^T\right]\right)^T .\boldsymbol{x} + \boldsymbol{z}, \tag{11}$$

where $T_1 = T + L$, $\boldsymbol{y}_{T_1} = \left[\boldsymbol{y}^T(1), \cdots, \boldsymbol{y}^T(T_1)\right]^T$ is the $n.T_1 \times 1$ vector with all received vectors stacked over one frame observation, $\boldsymbol{x} = \left[\boldsymbol{x}^T(1), \cdots, \boldsymbol{x}^T(T)\right]^T$ and $\boldsymbol{z} = \left[\boldsymbol{z}^T(1), \cdots, \boldsymbol{z}^T(T_1)\right]^T$.

One can notice that the performance of the proposed decoding scheme is heavily dependent on the number of multipaths. So when there are no multipaths, the channel matrix $\boldsymbol{H}$ becomes block diagonal, and the problem of (10) is then equivalent to decoding each $N \times 1$ data vector independently, and the joint decoding over the received frame is no longer exploited. In contrast, more multipaths results in higher achievable diversity order implying more reliable joint decoding via ℓ_1-minimization. This explanation is better clarified in section 6.

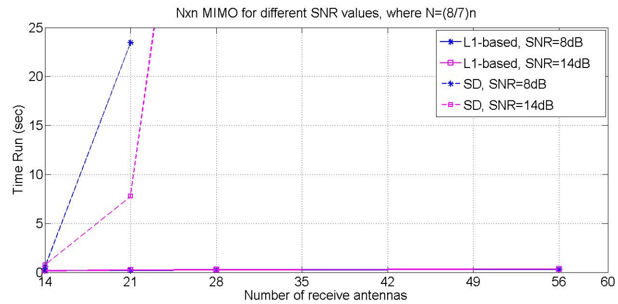

Fig. 1: Time-run comparison of the ℓ_1-minimization versus the Sphere Decoder for different SNR values.

5. COMPUTATIONAL REQUIREMENTS OF THE ℓ_1-MINIMIZATION

In order to solve ℓ_1-minimization problems, different algorithms have been proposed [11]. Herein, we review the complexity order of some of one of these algorithms.

The ℓ_1 minimization algorithms are solvable in polynomial time, however, their computational complexity orders are dissimilar. The authors in [11] have discussed the performance of the primal-dual interior point (PDIP) method widely used for such problems. The PDIP method requires an average running time that increases linearly with the projection dimensions. The number of required iterations is $\mathcal{O}(\sqrt{(N)})$, and each iteration can be executed in $\mathcal{O}(N^3)$ operations. Thus, it requires much less complexity order than other algorithms (e.g. Maximum likelyhood, sphere decoder) that requires a number of operations growing exponentially with the dimensions n and N.

6. SIMULATION RESULTS

In this section we evaluate the Bit Error Rate (BER) and the computational complexity of the proposed decoding scheme based on ℓ_1-minimization. We consider $N \times n$ underdetermined MIMO systems ($n < N$), where N and n are the number of transmit and receive antennas, respectively. The channel coefficients are i.i.d. circular symmetric complex Gaussian distributed with zero mean and unit variance, and the data symbols belong to a finite constellation. The number of transmitted symbols is equal to N at one channel use. For our proposed decoding scheme, we use the cvx toolbox which is a Matlab-based modeling system for convex optimization. Cvx employs the SeDuMi solver to solve the ℓ_1 minimization problem under convex constraints [12, 13]. We simulate this system using Matlab 2009b on a CPU Intel Xeon E5620 at 2.40GHz and memory 6GB RAM.

Fig. 1 compares the time-run of the sphere decoder (SD), described in [14], to the ℓ_1-based decoder. The time-run represents the average duration that needs a processor to decode

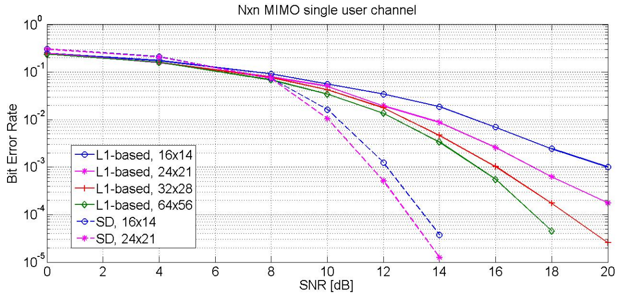

Fig. 2: Performance of the ℓ_1-minimization versus the Sphere decoder for different antenna dimensions in large MIMO systems

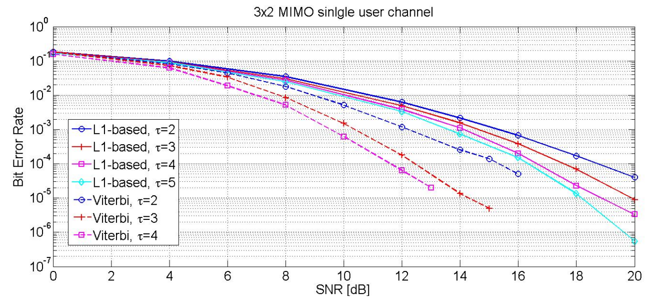

Fig. 4: Performance of the ℓ_1-minimization versus the ML-Viterbi decoder assuming BPSK modulation. τ represents the number of multipaths.

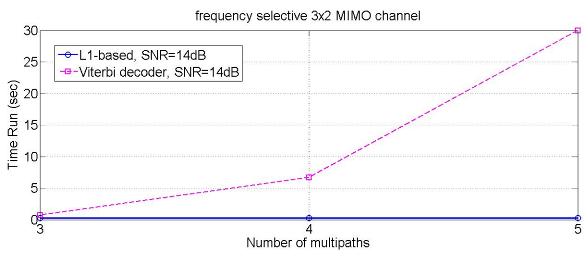

Fig. 3: Time-run comparison of the ℓ_1-minimization versus the Viterbi-ML for SNR = 14dB, assuming BPSK modulation

the received signal. We assume a 4-QAM constellation mapping known at both the transmitter and the receiver. One can notice that using the ℓ_1-based decoder, the computational complexity keeps almost invariant with the system dimensions and the SNR level, whereas the SD time-run increases exponentially with these two factors. For the SD, we do not go beyond the 24×21 due to a too high computational complexity, which is upper-bounded by $\mathcal{O}(M^{\gamma N})$, where $\gamma \in (0, 1]$ [15]. On the other hand, Fig. 2 illustrates a slight performance gain of our scheme in the low SNR region. Otherwise, i.e. beyond 8dB, the SD outperforms the proposed scheme. For example at BER 10^{-2}, we observe a gain of about 5dB with a 16×14 and of 4dB with a 24×21.

Next, Fig. 3 compares the time-run of our proposed decoder to the Viterbi decoder one. The Viterbi algorithm exploits the channel multi-paths, and achieves optimal performance over a frequency selective channel. The frame length is equal to $T = 20$, and the constellation mapping is BPSK. The computational complexity of the Viterbi decoder is known to grow exponentially with the number of transmit antennas and the number of channel multi-paths τ as $\mathcal{O}(2^{N.\tau})$. This is consistent with the obtained results in Fig. 3, where the Viterbi time-run increases from 0.73 seconds for $\tau = 3$ to 6.77 seconds for $\tau = 4$, whereas the time-run of our proposed scheme remains constant and equal to around 0.25 seconds. However, the Viterbi decoder leads to a BER improvement as illustrated in Fig. 4. For example, at BER 10^{-2}, the gain fluctuates between 1dB and 2.5dB, and at BER 10^{-3} the gain fluctuates between 3dB and 4dB, for different multi-path number.

Remarks: 1) It is important to mention that despite the non-optimality of the current Matlab solver, it can always show the increase of the computational complexity order of the above decoding schemes.
2) The gain obtained over the proposed scheme can be reduced using an error-correcting code e.g. turbocode.

7. CONCLUSION

In this paper we have addressed the problem of decoding in high dimension MIMO systems with finite constellation. We have modeled the transmission as a higher-dimension MIMO channel with sparse input vector. We have then defined an ℓ_1-minimization problem to detect the sparse vector. An iterative algorithm of polynomial moderate complexity has been used to solve the problem. Simulations carried out in the cases of large MIMO systems on flat fading channel and MIMO systems on frequency selective channels, where the efficiency of the proposed decoding scheme has been proved.

8. APPENDIX

From a general point of view, let Y, Θ, X be three d-dimensional real random vectors where $X \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$ when Θ and X are independent and $Y = \Theta + X$. Given tolerance $\tau \geqslant 0$, Random Distortion Testing (RDT) [16] is the problem of testing whether $\|\Theta(\omega) - \theta_0\| \leqslant \tau$ or not, when we are given Y and the probability distribution of Θ is unknown. By analogy with standard terminology in statistical inference, we say that this problem is the testing of the null event $\left[\, \|\Theta - \theta_0\| \leqslant \tau \,\right]$ against the alternative event $\left[\, \|\Theta - \theta_0\| > \tau \,\right]$ on the basis of observation Y. The RDT problem [16] is summarized as follows:

$$\text{RDT:}\begin{cases}\textbf{Observation:}\, Y = \Theta + X \begin{cases}\Theta \text{ and } X \text{ independent,} \\ X \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d),\end{cases} \\ \textbf{Null event:}\left[\, \|\Theta - \theta_0\| \leqslant \tau \,\right], \\ \textbf{Alternative event:}\left[\, \|\Theta - \theta_0\| > \tau \,\right].\end{cases} \tag{12}$$

Given any $\eta \geqslant 0$, let $\mathcal{T}_\eta$ be any thresholding test with threshold height η defined for any $y \in \mathbb{R}^d$ by

$$\mathcal{T}_\eta(y) = \begin{cases} 1 & \text{if} \quad \|y - \theta_0\| > \eta \\ 0 & \text{if} \quad \|y - \theta_0\| \leqslant \eta. \end{cases} \tag{13}$$

Given $\gamma \in (0, 1]$ and $\rho \geqslant 0$, there exists a unique solution $\lambda_\gamma(\rho) \geqslant 0$ in η to $1 - F_{\chi_d^2(\rho^2)}(\eta^2) = \gamma$, where $F_{\chi_d^2(\rho^2)}$ is the cumulative distribution function of the non-central χ^2 distribution $\chi_d^2(\rho^2)$ with d degrees of freedom and non-central parameter ρ^2. In [17], it is then proved that the thresholding test $\mathcal{T}_{\lambda_\gamma(\tau)}$ with threshold height $\lambda_\gamma(\tau)$ is such that the conditional probability values $\mathrm{P}\left[\, \mathcal{T}_{\lambda_\gamma(\tau)}(\Theta + X) = 1 \,\middle|\, \|\Theta - \theta_0\| \leqslant \tau \,\right]$ have supremum equal to γ, whatever Θ such that $\mathrm{P}\left[\, \|\Theta - \theta_0\| \leqslant \tau \,\right] \neq 0$. We thus say that $\mathcal{T}_{\lambda_\gamma(\tau)}$ has size γ for RDT. Moreover, it turns out that $\mathcal{T}_{\lambda_\gamma(\tau)}$ is optimal for RDT among all tests with same size in the following sense: 1) Save for values of ρ in some subset $\mathcal{D} \subset (\tau, \infty)$ such that $\mathrm{P}\left[\|\Theta - \theta_0\| \in \mathcal{D}\right] = 0$, the conditional probability $\mathrm{P}\left[\mathcal{T}_{\lambda_\gamma(\tau)}(\Theta + X) = 1 \,\middle|\, \|\Theta - \theta_0\| = \rho\right]$ does not depend on the distribution of Θ for every $\rho \in (\tau, \infty) \setminus \mathcal{D}$ and 2) $\mathrm{P}\left[\mathcal{T}_{\lambda_\gamma(\tau)}(\Theta + X) = 1 \,\middle|\, \|\Theta - \theta_0\| = \rho\right] \geqslant \mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta - \theta_0\| = \rho\right]$ for all test $\mathcal{T}$ with level γ and such that $\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta - \theta_0\| = \rho\right]$ does not depend on the distribution of Θ either. In other words, with respect to some criterion suitable for the natural invariance exhibited by RDT on the spheres centered at θ_0 in $\mathbb{R}^d$, thresholding tests $\mathcal{T}_{\lambda_\gamma(\tau)}$ are optimal.

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