WINDOWED ITERATIVE ESTIMATION OF THE PARAMETERS OF A DAMPED COMPLEX EXPONENTIAL IN NOISE

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ABSTRACT

The estimation of the frequency and decay factor of a single decaying exponential in noise is an important problem that has commanded significant attention in the research literature. In this paper, we examine the performance of computationally simple, yet accurate and robust, discrete Fourier transform based estimators. These estimators employ a coarse search followed by an interpolation step to obtain precise estimates of the parameters. Although their performance initially improves as the number of samples is increased, it reaches a minimum before departing significantly from the Cramer Rao Lower Bound (CRLB). We tackle this problem in this paper and propose a windowing strategy that allows the estimation performance to track the CRLB for a large number of samples. Furthermore, a practical implementation of the proposed method is given. Extensive simulations are reported that demonstrate the effectiveness of the proposed strategy.

***Index Terms*—** Damped Exponentials, Frequency Estimation, Parameter Estimation, Additive White Gaussian Noise.

1. INTRODUCTION

The estimation of the parameters of a decaying exponential in additive Gaussian noise is an important research problem with a wide range of applications, [1, 2, 3]. In this work we are interested in robust, efficient, yet computationally simple estimators. The signal model we are concerned with is

$$x[k] = s[k] + w[k], \; k = 0 \ldots N-1, \tag{1}$$

where the signal of interest $s[k]$ is an exponential of the form

$$s[k] = Ae^{(-\eta + j2\pi f)k} = Az^k. \tag{2}$$

Here A is the complex signal amplitude and η the decay factor. The frequency f is normalised by the sampling frequency, i.e. $f \in [-0.5, 0.5]$. The noise terms, $w[k]$, are zero mean, complex additive white Gaussian with variance σ^2, giving a nominal signal to noise ratio (SNR) $\rho_0 = \frac{|A|^2}{\sigma^2}$, [1]. Our goal is to obtain estimates of f and η from a block of N samples. The Cramer Rao Lower Bounds (CRLBs) for the frequency, σ_f^2, and damping factor, σ_η^2, and are given by, [4]

$$\begin{aligned}\sigma_\eta^2 &= 4\pi^2\sigma_f^2 \\ &= \frac{(1-|z|^2)^3(1-|z|^{-2N})}{2\rho_0\left[-N^2|z|^{2N}(1-|z|^2)^2 + |z|^2(1-|z|^{2N})^2\right]}.\end{aligned} \tag{3}$$

A large number of approaches have been proposed to solve this problem and a comprehensive review of these techniques is found in [3]. High resolution methods are usually aimed at the sum of multiple exponentials model, [5, 6, 7, 8]. Although these can be accurate, they are computationally expensive. This makes them at the very least unattractive for a single exponential, and at worst impractical when the number of samples is large. Therefore, the Discrete Fourier Transform (DFT) based estimators should be adopted in the case of a large number of samples due to their robustness and computational efficiency, [3]. In the rest of the paper, we focus on the interpolated DFT approach, [9].

The two-stage DFT-based estimation strategy, consisting of a coarse search followed by interpolation on the DFT coefficients, has proven itself to be effective for the estimation of the frequency and damping factor of a decaying exponential in noise, [10, 11, 1]. Putting $f = \frac{m+\delta}{N}$, where $-N/2 \leq m < N/2$ is an integer and $|\delta| \leq 0.5$ is a frequency residual, the coarse search estimates m before a fine search stage is used to estimate δ and η. The methods of [10] and [11], as well as a number of related estimators proposed in [2], have similar performances and as a result only the Bertocco algorithm, hereafter referred to as the B&Q algorithm (see [9]) will be implemented in this paper. The distinguishing feature of the method proposed in [1], named the A&M estimator, is that it can be implemented iteratively which permits it to achieve the minimum variance for the methods belonging to its class. As we will see, this amenability to iterative implementation is very important to the strategy proposed in this work.

One fundamental problem afflicting these methods in the damped case is that their performance can significantly depart fromt the CRLB as N increases. Whereas the CRLB flattens out as a function of N, the performance of the practical estimators can even exhibit an increase in variance due to the degradation of the effective SNR, [1, 3]. In fact, whereas it was found in [1] that an optimum number of samples exists that gives the best estimation variance, no practical implementation has been put forward to achieve this. Therefore, we study this problem in this paper and propose a windowing strategy that not only improves the estimation performance, allowing it to track the CRLB as it does in the undamped case, [12], but importantly is practically implementable.

The rest of the paper is organised as follows: In section 2 we present the windowed algorithms. In Section 3 we address the problem of a practical implementation of the proposed estimators. In particular, we present the iterative strategies for both the B&Q and A&M algorithms and discuss the differences between them. Simulation results are presented in section 4 to assess the estimation performances of the two estimators. Finally, some conclusions are drawn in section 5.

2. WINDOWED INTERPOLATION ON FOURIER COEFFICIENTS

The width of the main lobe of the DFT of a damped exponential is broadened by the presence of the damping factor. In [13] the authors address this by pre-multiplying the signal with a Gaussian window of the form $v[k] = e^{\gamma_0 k - \gamma_1 k^2}, \gamma_0, \gamma_1 \geq 0$. Although they state that the sharpest spectral line is obtained when $\gamma_0 = \eta$, they do not explain how this can be achieved given that η is unknown, and furthermore, do not give any guidance on how to set γ_1. Notwithstanding these issues, multiplying a noisy signal with a rising exponential amplifies the noise at the tail end and makes this strategy useless. On the other hand, the estimation performance was shown in [1] to reach a minimum for a signal length that is dependent on the damping factor ($N_{min} \approx 3/\eta$ for the A&M algorithm). Therefore, one could conceive a strategy for cropping the signal length to N_{min} should it exceed this value. However, it is desirable that any method to improve the performance also tracks the CRLB as N increases. Since it has been established that the performance degradation is caused by the poor SNR of the tail samples, we propose to reduce the emphasis on these samples by pre-multiplying the signal with a decaying window. We suggest that a suitable window is one that matches the envelope of the signal. That is we consider a window of the form $v[k] = e^{-\gamma k}, \gamma \geq 0$. The DFT coefficients of the windowed signal is

$$\begin{aligned} X(n) &= \frac{1}{N}\sum_{k=0}^{N-1} s[k]v[k]e^{-j2\pi\frac{kn}{N}} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} Ae^{(-(\eta+\gamma)+j2\pi f)k+\phi}e^{-j2\pi\frac{kn}{N}}. \end{aligned}$$

Let us put $\alpha = \eta + \gamma$. Thus we have that

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} Ae^{(-\alpha+j2\pi f)k+\phi}e^{-j2\pi\frac{kn}{N}}.$$

We can see that the estimation problem remains the same but with a modified damped damping factor, α. Therefore, the estimators that were developed for the un-windowed case remain applicable with the estimated damping factor being $\hat{\alpha}$. We now present the A&M estimator:

- Let $Y(k)$ denote the FFT of the signal.
- Locate $\hat{m}$, the index of the maximum bin.
- Calculate the DFT coefficients $X(\hat{m}+p)$ for $p = \pm 0.5$.
- Obtain the error discriminant
$$h = \frac{1}{2}\frac{X(\hat{m}+0.5)+X(\hat{m}-0.5)}{X(\hat{m}+0.5)-X(\hat{m}-0.5)}. \quad (4)$$
- Calculate $\hat{z}^{-1} = \cos\left(\frac{\pi}{N}\right) - 2jh\sin\left(\frac{\pi}{N}\right)$.
- Obtain the estimates $\hat{\eta} = \ln\left(\hat{z}^{-1}\right) - \gamma$, and $\hat{\delta} = -\frac{N}{2\pi}\angle\hat{z}^{-1}$.

For the B&Q estimator, the error discriminant and equation for $\hat{z}$ are replaced by their appropriate forms, see [1].

The strategy proposed above was evaluated by simulations and the results are shown in Figs. 1 and 2. The original B&Q and A&M algorithms are shown as well as their windowed counterparts. The CRLB curve is also included for reference. The window damping factor is set equal to the true damping factor (something that of course is not possible in practice). The results clearly show that whereas the original algorithms diverge wildly from the CLRB as N increases, the windowed algorithms track it. Also observe that the curves for the damping factor exhibit exactly the same trend as the frequency. This is true for all results reported here and therefore we will only show the frequency estimation performance in the rest of the paper. Now, in order to validate our choice of $\gamma = \eta$, we show the estimation performance of the frequency against γ in Fig. 3. We see that the variance curves show a minimum around the true value of η.

Although the above results show the effectiveness in the proposed strategy and its ability to solve the problem of the original algorithms, the damping factor of the signal is unknown (to be estimated) and the question of how to implement the estimators remains. This problem will be addressed in the following section.

3. ITERATIVE ESTIMATION

We have so far shown that the estimation algorithms perform significantly better if the signal is pre-multiplied by an exponentially decaying window. Furthermore, we have determined that the damping factor of the exponential window should be matched to the true damping factor of the signal. However, the signal's damping factor is not known a-priori and is to be estimated. Therefore, we propose to estimate the decay factor first and then use this estimate to construct the

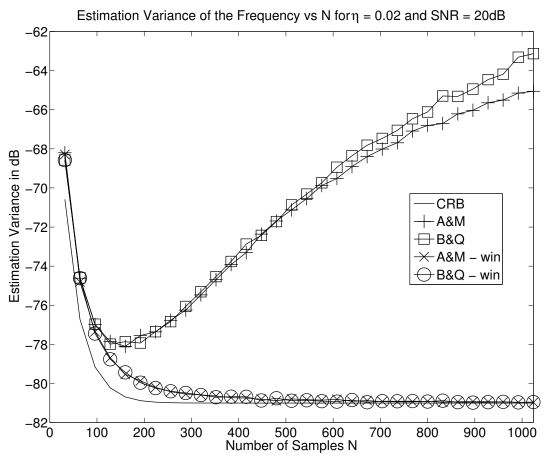

Fig. 1. Frequency estimation comparison between the windowed and original algorithms. 2000 Monte Carlo runs were used.

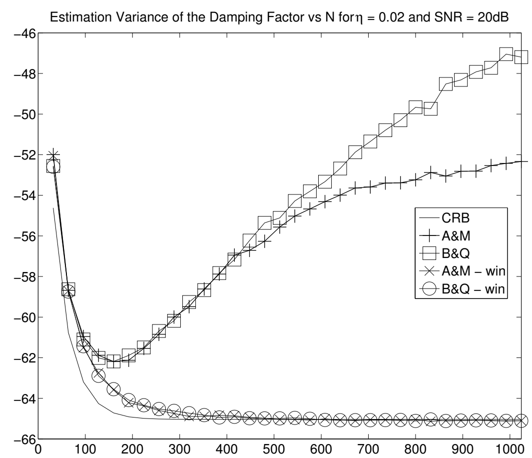

Fig. 2. Damping factor estimation comparison between the windowed and original algorithms. 2000 Monte Carlo runs were used.

Table 1. Windowed B&Q Estimator

Let $h_p = \frac{X(\hat{m}+p)}{X(\hat{m})}$, and $\hat{z}_p = \frac{1-h_p}{1-h_p e^{-j2\pi p/N}}$, for $p = \pm 1$
Get $\hat{\eta}_p = -\ln\left\|\hat{z}_p\right\|$, and $\hat{\delta}_p = \frac{N}{2\pi}\angle\hat{z}_p$
Set $\hat{\eta} = \hat{\eta}_1$, if $\hat{\delta}_1 \leq 0$ and $\hat{\delta}_{-1} \leq 0$,
$\hat{\eta} = \hat{\eta}_{-1}$, otherwise
Let $\hat{\eta}^{(1)} = \hat{\eta}$ and $v(k) = e^{-\hat{\eta}k}$, $k = 0 \ldots N-1$
Apply window to signal to give $x(k) \leftarrow x(k)v(k)$
Recalculate the DFT coefficients $X(\hat{m}+p)$, $p = 0, \pm 1$
Repeat steps 1 to 3 to obtain new estimates $\hat{\eta}$ and $\hat{\delta}$
Finally adjust $\hat{\eta}$ using $\hat{\eta} \leftarrow \hat{\eta} - \hat{\eta}^{(1)}$

Table 2. Windowed A&M Estimator

Initialise $\hat{\eta}_0 = 0$ and $\hat{\delta}_0 = 0$
For $q = 1$ to Q
Construct window $v(k) = e^{(-\hat{\eta}_{q-1} - j2\pi\frac{\hat{\delta}_{q-1}}{N})k}$, $k = 0 \ldots N-1$
Apply window to signal to give $x_w(k) \leftarrow x(k)v(k)$
Calculate the DFT coefficients $X_w(\hat{m}+p)$, $p = \pm 0.5$
Let $h = \frac{1}{2}\frac{X_w(\hat{m}+0.5)+X_w(\hat{m}-0.5)}{X_w(\hat{m}+0.5)-X_w(\hat{m}-0.5)}$
Calculate $\hat{z}^{-1} = \cos\left(\frac{\pi}{N}\right) - 2jh\sin\left(\frac{\pi}{N}\right)$
Get $\hat{\eta} = \ln\left\|\hat{z}^{-1}\right\|$, and $\hat{\delta} = -\frac{N}{2\pi}\angle\hat{z}^{-1}$
Obtain estimates are $\hat{\eta}_q \leftarrow \hat{\eta} - \hat{\eta}_{q-1}$ and $\hat{\delta}_q \leftarrow \hat{\delta} + \hat{\delta}_{q-1}$

window before repeating the estimation process. We apply this method to the B&Q as well as the A&M estimators. However, as the A&M estimator is already implemented iteratively, see [1], we can incorporate the estimation into the iterative steps. The resulting estimators are shown in tables 1 and 2 (Note that the coarse estimation stage, giving $\hat{m}$ is as before and is not shown here).

The B&Q estimator, shown in table 1, involves an initial estimation of the damping factor in order to construct the window. Although only the damping factor is needed for the window and the re-application of the estimator, the decision rule that we adopt (to choose between the estimates corresponding to $p = \pm 1$) is based on the signs of the estimated frequency residuals. This decision rule has been shown to outperform the rule based on a comparison between $X(\hat{m}-1)$ and $X(\hat{m}+1)$, [14].

Table 2 shows the procedure of the A&M algorithm. It is important to emphasise the fundamental differences between it and the B&Q estimator. The A&M estimator can be applied iteratively, which permits us to incorporate the estimation of the damping factor and windowing directly into the iteration. Therefore, we iterate Q times, and at each iteration we estimate both $\hat{\eta}$ and $\hat{\delta}$. Note that whereas the frequency estimate accumulates from one iteration to the next, the damping factor estimate is adjusted by compensating for the estimate of the previous iteration. This is due to the fact that the damping factor of the previous iteration plays the role of the window damping factor, γ, in the current iteration.

The theoretical analysis of the algorithms is beyond the scope of this paper. Therefore, we proceed in the following section to validate the performance of the algorithms using simulations.

4. SIMULATION RESULTS

The estimators detailed in the previous section were implemented and their performances verified. As previously mentioned, we only show the plots of the frequency estimates. The estimation performance for the damping factor shows identical trends.

First we obtain the estimation performance as a function

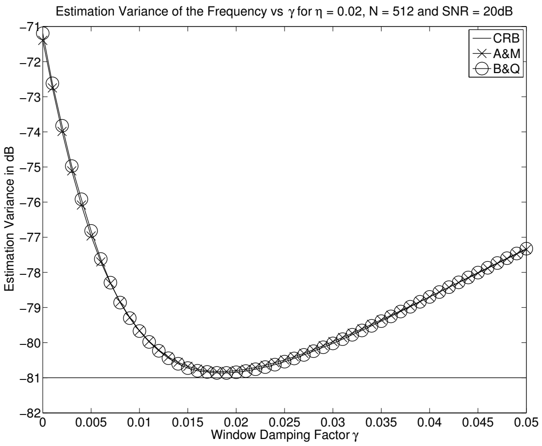

Fig. 3. Frequency estimation as a function of the window damping factor γ. 2000 Monte Carlo runs were used.

of the number of samples. The results are given in Fig. 4. Included are the curves for the original estimators as well as the CRLB. Observe that although the original iterative A&M algorithm improves on the B&Q estimator, especially for small N, both diverge from the CRLB for large N. The windowed estimators, on the other hand, track the CRLB well as N increases. Fig. 5 shows a close up of the performance curves of the windowed estimators. The B&Q estimator clearly has the worst performance. The curves of the A&M estimator corresponding to two and three iterations coincide for $N \leq 400$, but the performance after two iterations deteriorates slightly after that. The third iteration, however, brings the algorithm back in line with the CRLB curve. Note that the loss with respect to the CRLB is quite small at around 0.1 dB.

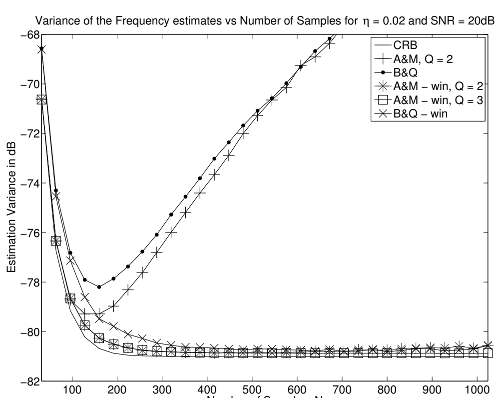

Fig. 4. Frequency estimation performance of the iterative windowed algorithms as a function of the number of samples. 2000 Monte Carlo runs were used.

Next we show in Figs. 6 and 7 the performance as a function of the nominal SNR, ρ_0. The estimation variance is presented in the first plot and the curves for the original estimators and the CRLB are included for comparison. Observe that the original estimators are grouped together and are about 9dB worse than the windowed algorithms. Also note that in both cases, original and windowed, the B&Q algorithm has slightly worse breakdown threshold than the A&M estimators. In order to appreciate the differences between the windowed B&Q algorithm and the A&M estimator (after $Q = 2$ and 3 iterations), we show their ratio of the estimation variance to the CRLB in Fig. 7. It is clear that the best performance both in terms of estimation variance and breakdown threshold belongs to the A&M estimator after 3 iterations.

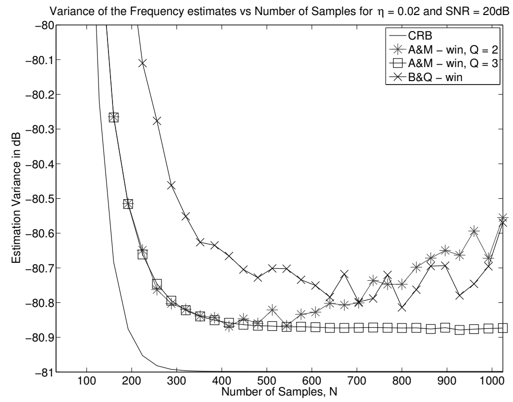

Fig. 5. Frequency estimation performance of the iterative windowed algorithms as a function of the number of samples. Zoomed plot showing only the windowed estimators. 2000 Monte Carlo runs were used.

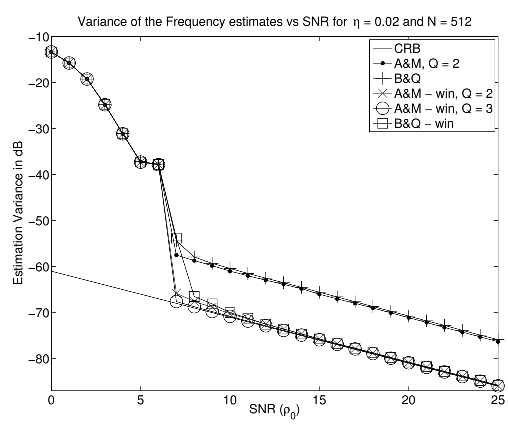

Fig. 6. Frequency estimation performance using the iterative windowed algorithms as a function of SNR. 2000 Monte Carlo runs were used.

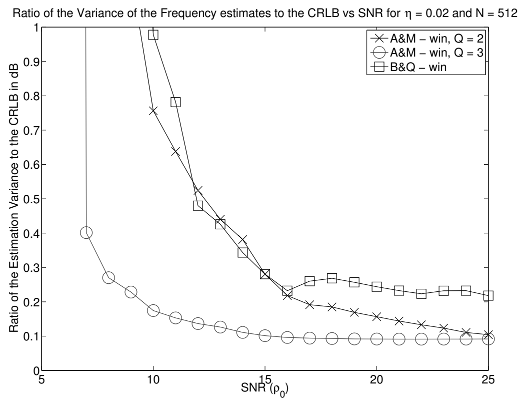

Fig. 7. Plot of the ration of the estimation variance of the iterative windowed algorithms to the CRLB. 2000 Monte Carlo runs were used.

5. CONCLUSIONS

In this paper we examined the problem of estimating the frequency and damping factor of a single decaying exponential in noise. We have proposed and evaluated a set of estimators that employ an exponentially decaying window to prevent the performance from deteriorating as the number of samples increases. We also presented practical implementations of the proposed strategy for both the A&M and B&Q algorithms. Extensive simulation results were given to verify the large improvement in the estimation performance.

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