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ERROR FLOOR ELIMINATION FOR DFT-BASED FREQUENCY ESTIMATORS

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ABSTRACT

Frequency estimation for a single-tone complex exponential signal is a classical and fundamental problem in many applications, such as communications, radar, and power. Most DFT-based frequency estimators reported in the literature exhibit an error floor at high SNR, due to their use of approximations to nonlinear equations. This effect is pronounced for short observation windows. In this paper, we propose two simple nonlinear equation solvers for eliminating this error floor. Simulations confirm the effectiveness of this proposed approach. We add to the literature of DFT-based estimators by proposing a frequency estimator for the case where a measured sequence is zero-padded by a factor of two.

1. INTRODUCTION

Frequency estimation for sinusoidal signals is used in many applications, such as communications, radar, sonar, and power. The mean-squared error of low-complexity DFT-based frequency estimators [1–9] essentially tracks the Cramer-Rao lower bound for unbiased estimators, at SNRs above threshold, until they reach an error floor at high SNR. This error floor is the bias-squared attributable to the approximate solution of a nonlinear equation. In this paper we propose two simple nonlinear equation solvers to eliminate this error floor.

DFT-based frequency estimators are usually carried out in two steps. In the first step, a coarse estimate of the frequency is obtained by locating the DFT sample with the maximum magnitude. In the second step, the difference between the coarse estimate and the true frequency is estimated from DFT samples in the neighborhood of the maximum-magnitude sample.

Existing DFT-based estimators all share the same coarse estimate in their first step. Their differences lie in their estimates of a frequency difference in step two [1–9]. In order to estimate the frequency difference, a sequence of insightful approximations is used to estimate this difference as a nonlinear function of DFT samples in the neighborhood of the maximum-magnitude sample. As shown in [9], these estimators are biased at high SNR. As a result, they exhibit an error floor at high SNR. To eliminate the error floor, a sequence of time-domain iterations is introduced in [10]. The method reported in this paper may be considered a frequency-domain alternative to the method of [10]. Its virtue is that it may be used to eliminate the error floor in all of the DFT-based methods reported to date, at low computational complexity.

Each of the methods reported in [1–9] relies on a formula for the difference between a noise-free coarse estimate and the true frequency. This formula is then approximated. We propose to refine the approximations and their solutions by a simple search for the solution to a nonlinear regression equation, using a numerical method such as Newton-Raphson or Secant. Simulations show that this approach eliminates the error floor at high SNR.

The rest of this paper is organized as follows: The problem is formulated in Section 2 and existing work is briefly reviewed in Section 3. Our proposed error floor elimination is formulated as a search problem and solved by two numerical algorithms in Section 4. Simulations are presented in Section 5 to demonstrate performance, with concluding remarks given in Section 6.

2. PROBLEM FORMULATION

A discrete-time single-tone complex exponential signal can be written as

$$x[n] = A e^{j(2\pi \frac{f}{f_s} n + \phi)} + w[n] \;, -f_s/2 < f \le f_s/2 \;,$$

where f is the signal frequency to be estimated, f_s is the sampling frequency, A and ϕ are the unknown magnitude and initial phase respectively, and $w[n]$ is the additive complex noise. In our simulations the noise is taken to be white Gaussian with per sample variance σ^2. Then, we can define the signal to noise ratio (SNR) as $\gamma = \frac{A^2}{\sigma^2}$.

The maximum likelihood (ML) estimator searches for the location of the peak in the discrete-time Fourier transform (DTFT) of the signal [3, 11], which takes a sinc shape centered at f as shown in Fig. 1. However, the discrete Fourier transform (DFT) is utilized in practice, to resolve the DTFT

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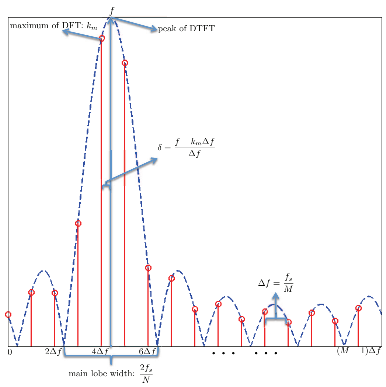


Fig. 1. DTFT (dash line) and M-point DFT of an N-point single-tone signal $x(t) = Ae^{j(2\pi f t + \phi)}$ with sampling frequency f_s.

at resolution $\frac{f_s}{M}$, where M is the zero-padded DFT size. The M-point ($M \geq N$) DFT of $x[n]$ is given by

$$X[k] = \sum_{n=0}^{N-1} A\mathrm{e}^{j\phi}\mathrm{e}^{j2\pi\frac{f}{f_s}n}\mathrm{e}^{-j\frac{2\pi}{M}nk} + W[k] \tag{1}$$

$$= A\mathrm{e}^{j\phi}\mathrm{e}^{j\pi(N-1)\left(\frac{f}{f_s}-\frac{k}{M}\right)}\frac{\sin\!\Big(\pi N\Big(\frac{f}{f_s}-\frac{k}{M}\Big)\Big)}{\sin\!\Big(\pi\Big(\frac{f}{f_s}-\frac{k}{M}\Big)\Big)} + W[k]. \tag{2}$$

where the $W[k]$'s are DFT coefficients of the noise $w[n]$.

When f is not an integer multiple of f_s/M, the DFT does not capture the peak of the DTFT even when noise is absent, as depicted in Fig. 1. However, at high SNR the DFT sample with the maximum magnitude is close to the peak, with high probability, and hence can be used as a coarse estimate. Therefore, the true frequency f can be parameterized in terms of the DFT frequency spacing f_s/M as follows:

$$f = (k_m + \delta)\frac{f_s}{M} , \tag{3}$$

where k_m is the index of the DFT sample with the maximum magnitude, and $\delta \in [-0.5, 0.5]$ is an interpolation parameter. Clearly this parameterization of f will fail at SNRs below a threshold, but for SNRs above this threshold, where $k_m\frac{f_s}{M}$ is, with high probability, within $\frac{1}{2}\frac{f_s}{M}$ of f, this parameterization is expected to effectively interpolate between the coarse frequency estimate of the DFT.

3. EXISTING WORK

In the literature, a two-step frequency estimation approach based on (3) is commonly employed:

1. Take an M-point DFT of the N-point signal, search for the DFT sample with the maximum magnitude, and denote its index as k_m.
2. Estimate the interpolation parameter δ using the maximum DFT sample and its neighbors.

Following this approach, the estimators of [3–6, 8, 9] are briefly summarized in the following subsections.

3.1. Quinn's Estimators [3, 4]

In [3], the fractional frequency deviation δ is estimated by interpolating $X[k_m+1]$, $X[k_m]$ and $X[k_m-1]$, based on the approximation $\mathrm{e}^{j2\pi\left(\frac{f}{f_s}-\frac{k}{N}\right)} - 1 \approx j2\pi\left(\frac{f}{f_s}-\frac{k}{N}\right)$, based on the assumption that the observation window size N is large.

With the approximation, the relationship between δ and these three samples can be derived from (2) and (3) as follows:

$$\frac{X[k_m+i]}{X[k_m]} \approx \frac{\delta}{\delta - i} + \frac{W[k_m+i]}{NC} - \frac{\delta}{\delta - i}\frac{W[k_m]}{NC} , \tag{4}$$

where $C = A\mathrm{e}^{j\phi}(\mathrm{e}^{j2\pi\delta}-1)/(j2\pi\delta)$ and $i \in \{-1,\ 1\}$. Neglecting the noise parts in (4) for N is large, two estimates can be obtained as follows:

$$\hat{\delta}_{\pm1} = \frac{\alpha_{\pm1}}{1-\alpha_{\pm1}} \text{ , where } \alpha_{\pm1} = \mathrm{Re}\left\{\frac{X[k_m \pm 1]}{X[k_m]}\right\} .$$

Quinn then proposed a criterion to select the final frequency estimate between δ_{+1} and δ_{-1} as:

$$\hat{\delta} = \begin{cases} \hat{\delta}_{+1} & \hat{\delta}_{+1} > 0 \text{ and } \hat{\delta}_{-1} > 0 , \\ \hat{\delta}_{-1} & \text{Otherwise} . \end{cases}$$

which choice leads to the one with smaller asymptotic variance.

In [4], by forcing the asymptotic variance to the Cramer-Rao lower bound (CRLB), Quinn proposed a new estimator incorporating both δ_{+1} and δ_{-1} as follows:

$$\hat{\delta} = \frac{(\hat{\delta}_{+1} + \hat{\delta}_{-1})}{2} + k(\hat{\delta}_{+1}) - k(\hat{\delta}_{-1}) ,$$

where $k(x) = \frac{1}{4}\log(3x^4 + 6x^2 + 1) - \frac{\sqrt{6}}{24}\log\left(\frac{x^2+1-\sqrt{\frac{2}{3}}}{x^2+1+\sqrt{\frac{2}{3}}}\right)$.

3.2. Macleod's Estimator [5]

Two approximations, $\frac{N-1}{N} \approx 1$ and $\sin\left(\frac{\pi}{N}(\delta - i)\right) \approx \frac{\pi}{N}(\delta - i)$, i=$\{0, \pm1\}$, were made for large N. With these approximations, the noiseless $X[k_m$+$i]$ could be expressed as follows by substituting (3) into (2):

$$X[k_m + i] = A\mathrm{e}^{j(\phi+\pi\delta)}\frac{N\sin(\pi\delta)}{\pi(\delta - i)} . \tag{5}$$

Out of a number of low-complexity numerator and denominator combinations of $X[k_m]$, $X[k_m+1]$ and $X[k_m-1]$, Macleod selected the following:

$$d = \frac{\mathrm{Re}\{R(-1) - R(1)\}}{\mathrm{Re}\{2R(0) + R(-1) + R(1)\}} ,$$

where $R(i) = X[k_m + i]X^*[k_m]$. From (5), $d = \frac{-\delta}{2\delta^2-1}$. Accordingly, the estimate of δ could be easily obtained by solving the quadratic equation $2d\delta^2 + \delta - d = 0$ which results in

$$\hat{\delta} = \frac{\sqrt{1+8d^2} - 1}{4d} .$$

3.3. Jacobsen's [6] and Candan's [8] Estimators

The estimator proposed by Jacobsen in [6] also utilized the N-point DFT and was based on empirical observations as follows,

$$\hat{\delta} = \mathrm{Re}\left\{ \frac{X[k_m-1] - X[k_m+1]}{2X[k_m] - X[k_m-1] - X[k_m+1]} \right\} . \tag{6}$$

In [8], Candan analyzed this estimator by Taylor series expansion on the expressions of $X[k_m]$, $X[k_m + 1]$ and $X[k_m - 1]$. The estimator in (6) is related to the true δ as:

$$\mathrm{Re}\left\{ \frac{X[k_m-1] - X[k_m+1]}{2X[k_m] - X[k_m-1] - X[k_m+1]} \right\} = \frac{\pi \cot(\pi/N)}{N(1+\pi^2\delta^2)}\delta .$$

By neglecting $\pi^2\delta^2$ in the denominator on the right hand side of the above equation since δ^2 is small, a new estimator can be obtained as:

$$\hat{\delta} = \frac{\tan(\pi/N)}{\pi/N} \mathrm{Re}\left\{ \frac{X[k_m-1] - X[k_m+1]}{2X[k_m] - X[k_m-1] - X[k_m+1]} \right\} . \tag{7}$$

Candan's estimator is an improved version of Jacobsen's with a bias-correction term $\frac{\tan(\pi/N)}{\pi/N}$. However, it is still biased due to the approximations in the estimator development.

3.4. Special Sized DFT-Based Estimator [9]

Reference [9] reports an estimator based on the DFT with special zero-padding size $M = 2N$, which gives rise to this relationship between δ and $X[k_m \pm 1]$:

$$\tan\left(\frac{\pi}{2N}\delta\right) = \tan\left(\frac{\pi}{2N}\right) \frac{\rho - 1}{\rho + 1}. \tag{8}$$

Here $\rho = |X[k_m + 1]|/|X[k_m - 1]|$. The approximation $\tan(\frac{\pi}{2N}\delta) \approx \frac{\pi}{2N}\delta$, produces an estimate,

$$\hat{\delta} = \frac{2N}{\pi} \cdot \tan\left(\frac{\pi}{2N}\right) \frac{\rho - 1}{\rho + 1} . \tag{9}$$

3.5. Summary and Preview of Error Floor Elimination

In summary, all aforementioned estimators involve approximations, and as a consequence they exhibit error floors at high SNR. The accuracy of the approximations depends on the value of N, and as a result, error floors are higher at small values of N than at large values. But, importantly, at high SNR (really, noise-free) each of these estimators is invariant to amplitude A and phase ϕ. So our methodology will be to treat each of the estimators as a *statistic*, to which the signal component of (2) is to be fitted, with f parameterized as in (3).

4. ERROR FLOOR ELIMINATION

We propose an error floor elimination scheme to improve the high-SNR performance of existing estimators. We will start with the special sized DFT-based estimator in [9] and then generalize the proposed scheme to other estimators.

From (8), we see that the problem is to solve a nonlinear regression equation $g(\delta) - G(X) = 0$ for δ, where $g(\delta) = \tan\left(\frac{\pi}{2N}\delta\right)$, $G(X) = \tan\left(\frac{\pi}{2N}\right)\frac{\rho-1}{\rho+1}$, and X denotes DFT samples. A numerical search algorithm may be applied, using the approximation $\hat{\delta}$ to initialize the search.

Therefore, the error floor elimination algorithm based on the Newton-Raphson method (EFE-Newton) is summarized as follows:

Algorithm [EFE]:

1. *Determine the regression equation* $g(\delta) - G(X) = 0$.
2. *Determine a good starting value for* δ*, typically the value proposed by the authors of the cited papers.*
3. *Run a numerical search, such as Newton or Secant, for the solution to the regression equation.*
4. *Reconstruct the frequency based on* (3)*, using the value of* δ *returned by the numerical search.*

This procedure may be used to eliminate the error floor for other DFT-based estimators, such as those reported in [3–6, 8]. The key is to treat a proposed frequency estimator as a statistic $G(X)$, compute its noise-free value $g(\delta)$ by replacing the $X[k]$'s in the estimator with their noise-free values, form the regression equation $g(\delta) - G(X) = 0$, choose a good starting value for δ (namely, the proposed estimator $\hat{\delta}$), and run a search for the refined interpolation parameter δ. That is, the function $g(\delta)$ is obtained by substituting (3) into (2) and expressing the noiseless DFT samples of interest as

$$X[k_m + i] = A e^{j\phi} e^{j\pi\frac{N-1}{M}(\delta - i)} \frac{\sin\left(\frac{\pi N}{M}(\delta - i)\right)}{\sin\left(\frac{\pi}{M}(\delta - i)\right)} \tag{10}$$

where $i = \{-1, 0, 1\}$. Accordingly, $g(\delta)$ for each estimator is obtained by substituting (10) into an estimator expression, such as (6), (7) and so on. As noted previously, $g(\delta)$ in each case only contains the unknown δ, since the other two unknown parameters in (10), amplitude A and phase angle ϕ, are eliminated in the expressions of the original estimators. That is, their noise-free versions are invariant to A and ϕ. Hence, the error floor elimination of all existing estimators can be framed as the root search problem on $g(\delta) - G(X) = 0$, where $g(\delta)$ and $G(X)$ are determined by the specific DFT-based algorithm.

However, sometimes the function $g(\delta)$ is too complicated and the closed-form expression for the derivative is unavailable, such as for the estimator developed in [3]. In this case, the Newton-Raphson method cannot be directly

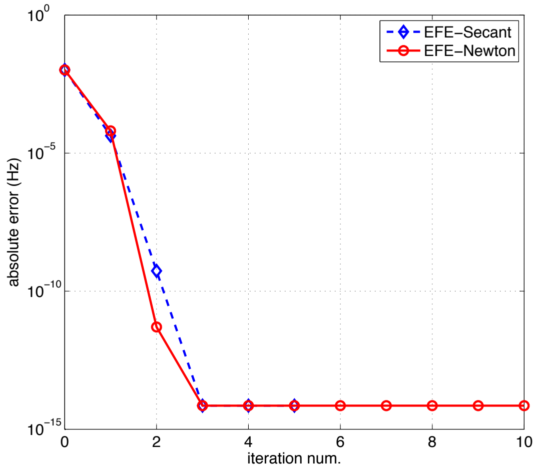


Fig. 2. Noiseless absolute bias comparisons between EFE-Newton and EFE-Secant on Candan11 with different iteration number. The true frequency $f = 55$ Hz.

applied. Then the Secant method [12] can be used to replace the derivative $g'(\cdot)$ by a finite difference as:

$$g'(\hat{\delta}_i) \approx \frac{g(\hat{\delta}_i) - g(\hat{\delta}_{i-1})}{\hat{\delta}_i - \hat{\delta}_{i-1}} \,. \tag{11}$$

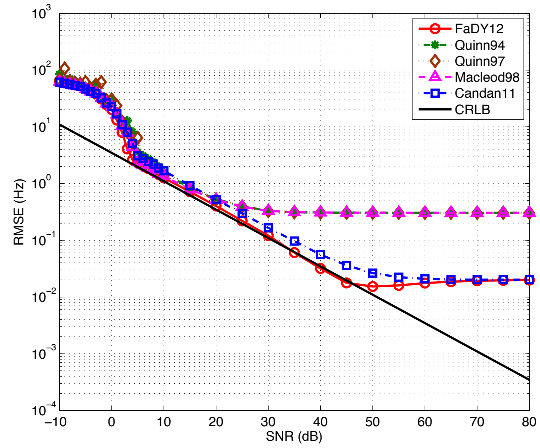
where subscript i denotes iteration index here.

5. SIMULATIONS

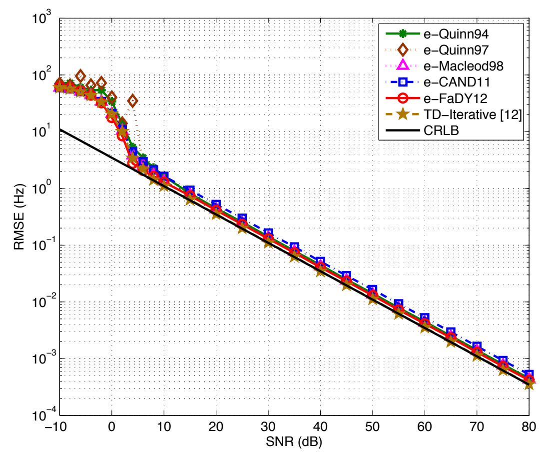
In this section, we first verify the effectiveness of EFE-Newton and EFE-Secant algorithms in terms of the absolute noiseless frequency error $|\hat{f} - f|$, and then compare the existing estimators in their original and refined versions in terms of mean-squared error, estimated from Monte-Carlo experiments. In all simulations, the observation window size is $N = 8$, the sampling frequency is $f_s = 200$ Hz, and the unknown frequency f lies within the Nyquist band. Of course only the ratio $\frac{f}{f_s}$ matters.

Fig. 2 shows the absolute noiseless bias of Candan's frequency estimator [8] refined by EFE-Newton and EFE-Secant with various iteration numbers. Clearly, the absolute noiseless bias can be treated as zero when the iteration number is greater than 3 for both EFE-Newton and EFE-Secant. In addition, the Newton-Raphson method converges slightly faster than the Secant method, for the Secant method is based on the approximation of the Newton method.

For fair comparisons, we adopt the EFE-Secant to eliminate the error floor of the estimators for all the remaining simulations with 3 iterations. Here, we use δ^N to denote the interpolation parameter for estimators taking an N-point DFT, and δ^{2N} to denote this parameter when taking a $2N$-point DFT, as δ's for different DFT sizes will differ. In ad-



(a) Original Estimators

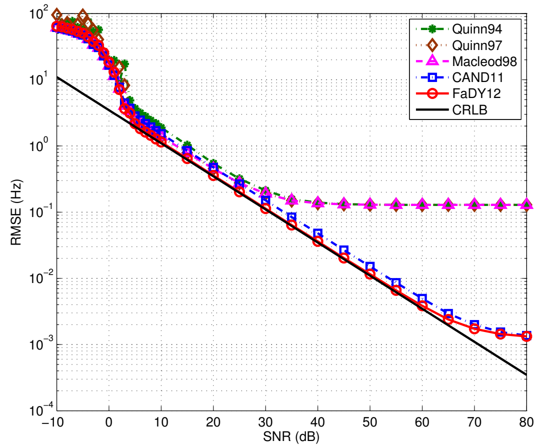


(b) Enhanced Estimators

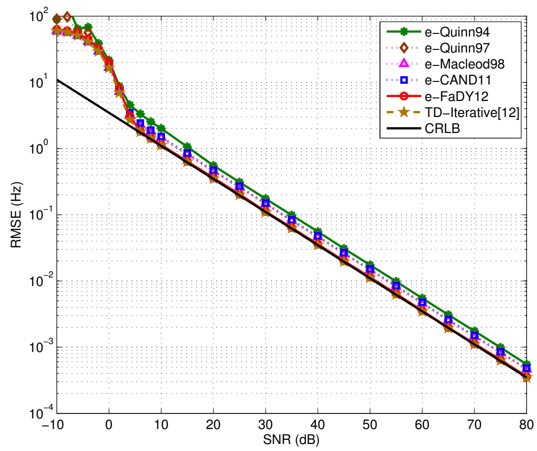
Fig. 3. RMSE performance of interpolation algorithms when the true frequency $f = 56.25$ Hz, where $\delta^{2N} = \pm 0.5$, $\delta^{N} = 0.25$.

dition, we also compare our method with the time-domain iterative (TD-Iterative) method proposed in [10], which iteratively adjusts the original time-domain signal and computes a new DFT samples at each iteration.

Fig. 3 shows the results when $f = 56.25$ Hz, where $\delta^N = 0.25$ and $\delta^{2N} = \pm 0.5$. Fig. 4 shows the results, when $f = 52.5$ Hz, where $\delta^N = 0.1$ and $\delta^{2N} = 0.2$. The Cramer-Rao lower bound (CRLB) is calculated numerically according to the expressions given in [5]. Compared with the original estimators, the refined estimators eliminate the error floor and approach the CRLB at high SNR, which confirms the effectiveness of our proposed scheme. Furthermore, the performance of refined estimators is not impaired at medium SNR. As a tradeoff, the performance of the refined estimators is very slightly compromised at very low SNR. However, in this SNR range, frequency estimators are already far off the



(a) Original Estimators



(b) Enhanced Estimators

Fig. 4. RMSE performance of interpolation algorithms when the true frequency $f = 52.5$ Hz, where $\delta^{2\mathrm{N}} = 0.2$, $\delta^{\mathrm{N}} = 0.1$.

CRLB and the minor performance loss is insignificant. In Fig. 3(b), the TD-Iterative estimator [10] has a slight performance gain over the refined estimators, when $f = 56.25$ Hz. However, this gain disappears for $f = 52.5$ Hz in Fig. 4(b). These results seem to indicate that the refined version of the estimator reported in [9] slightly outperforms other refined DFT-based estimators, and has roughly the same performance as the TD-Iterative method of [10], at lower computational complexity.

6. CONCLUSIONS

In this paper, we have reviewed several DFT-based estimators that have been reported in the literature. Each of them exhibits an error floor at high SNR, due to the bias introduced by approximations. By determining a regression equation that governs the nonlinear dependence of a frequency difference on a nonlinear function of DFT samples, we have developed a methodology for eliminating this error floor. Simulations confirm that the proposed methodology removes the error floor of all of the estimators we have studied, so that estimators track the Cramer-Rao lower bound at high SNR.

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