HIGH RESOLUTION TECHNIQUES FOR RADAR: MYTH OR REALITY ?

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ABSTRACT

We address the problem of effectiveness of the high resolution techniques applied to the conditional model. The rationale is based on a definition of the probability of resolution of maximum likelihood estimators which is computable in the asymptotic region of operation (in SNR and/or in large number of snapshots). The application case is the multiple tones estimation problem (Doppler frequencies estimation in radar).

Index Terms— high resolution techniques, maximum likelihood estimators, Cramer-Rao bound, multiple tones

1. INTRODUCTION

The resolvability of closely spaced signals, in terms of parameter of interest, for a given scenario (e.g., for a given Signal-to-Noise Ratio (SNR), for a given number of snapshots and/or for a given number of sensors) is a former and challenging problem which was recently updated by Smith [1, ref.12], Shahram and Milanfar [1, ref.13], Liu and Nehorai [1, ref.14], Amar and Weiss [1, ref.15] and El Korso et al [1]. Historically, the concept of Statistical Resolution Limit (SRL) has been introduced as the minimum distance between two closely spaced signals embedded in an additive noise that allows a correct resolvability/parameter estimation. The SRL is therefore an important statistical tool to quantify the ultimate performance for parametric estimation problems. Lately authors in [1] have generalized the concept of the SRL to the Multidimensional SRL (MSRL) applied to the multidimensional harmonic retrieval model. In that paper, they derive the SRL for the so-called multidimensional harmonic retrieval model by using a generalization of the previously introduced SRL concepts called Multidimensional SRL (MSRL). They first derive the MSRL using an hypothesis test approach (Liu and Nehorai). This statistical test is shown to be asymptotically an uniformly most powerful test which is the strongest optimality statement that one could expect to obtain. Second, they link the proposed asymptotic MSRL based on the hypothesis test approach to a new extension of the SRL based on the Cramér-Rao Bound approach (Smith). Thus, a closed-form expression of the asymptotic MSRL is given and analyzed in the framework of the multidimensional harmonic retrieval model. In the present paper we propose a different rationale to address the problem of resolvability of closely spaced signals, in terms of parameter of interest. It is based on a definition of the probability of resolution of maximum likelihood estimators (MLEs) which is computable in the asymptotic region of operation (in SNR and/or in large number of snapshots) for the conditional model. The results obtained with the proposed rationale must be regarded as an "upper bound" in terms of resolvability, in the sense it assumes that the number of source is known and that all the sources are present as well. The application case is the multiple tones estimation problem (Doppler frequencies estimation in radar).

This work has been partly funded by the European Network of excellence NEWCOM#

2. PROBABILITY OF RESOLUTION

Throughout the present paper, unless otherwise stated, $\mathbf{x}$ denotes the random observation vector of dimension N, Ω denotes the observations space and $L^2(\Omega)$ denotes the complex Hilbert space of square integrable functions over Ω. The probability density function (p.d.f.) of $\mathbf{x}$ is denoted $p(\mathbf{x};\boldsymbol{\Theta})$ and depends on a vector of $\overline{P}$ real parameters $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_{\overline{P}}) \in \Phi$, where Φ denotes the parameter space. The probability of an event $\mathcal{D} \subset \Omega$ is denoted $\mathcal{P}(\mathcal{D};\boldsymbol{\Theta})$. Let $\boldsymbol{\Theta}^0$ be a selected value of the parameter $\boldsymbol{\Theta}$, and $\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}(\mathbf{x})$ ($\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}$ in abbreviated form) an estimator of $\mathbf{g}(\boldsymbol{\Theta}^0)$ where $\mathbf{g}(\boldsymbol{\Theta}) = \left(g_1(\boldsymbol{\Theta}), \ldots, g_{\overline{Q}}(\boldsymbol{\Theta})\right)^T$ is a vector of $\overline{Q}$ real-valued (for the sake of simplicity) functions of $\boldsymbol{\Theta}$. For any selected value $\boldsymbol{\Theta}^0$, $\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}(\mathbf{x})$ stands for a mapping of the observation space Ω into an estimate of $\mathbf{g}(\boldsymbol{\Theta}^0)$.

2.1. Estimation precision and bounds

The quality (i.e. the precision) of an estimator $\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}$ can be measured using the following canonical objective function:

$$\mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}^-, \boldsymbol{\xi}^+\right) = \mathcal{P}\left(\bigcap_{q=1}^{\overline{Q}}\left(\widehat{g_q(\boldsymbol{\Theta}^0)} \in \left]g_q(\boldsymbol{\Theta}^0) - \xi_q^-, g_q(\boldsymbol{\Theta}^0) + \xi_q^+\right[\right); \boldsymbol{\Theta}^0\right) \quad (1)$$

where $\boldsymbol{\xi}^- = \left(\xi_1^-, \ldots, \xi_Q^-\right)^T$ and $\boldsymbol{\xi}^+ = \left(\xi_1^+, \ldots, \xi_Q^+\right)^T$ define the (left and right) errors on the estimation of $\mathbf{g}(\boldsymbol{\Theta}^0)$ and $\mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}^-, \boldsymbol{\xi}^+\right)$ is a measure of the probability that errors does not exceed $\boldsymbol{\xi}^-$ and $\boldsymbol{\xi}^+$. This objective function is identified as "canonical" since it is deduced naturally from the problem under study: the match between the observations of a random vector and a deterministic vector of interest. We also qualify it as *exhaustive*, in the sense that it incorporates all the available information on the problem, in other words the probabilities. Consequently, we consider that (1) defines the *exhaustive* precision (of estimation). Nevertheless, it is more fruitful practically to consider the *quasi-exhaustive* precision obtained when $\boldsymbol{\xi}^- = \boldsymbol{\xi}^+ = \boldsymbol{\xi}$, then $\mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}^-, \boldsymbol{\xi}^+\right)$ (1) reduces to $\mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}\right)$ defined as:

$$\mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}\right) = \mathcal{P}\left(\bigcap_{q=1}^{\overline{Q}}\left(\left|\widehat{g_q(\boldsymbol{\Theta}^0)} - g_q(\boldsymbol{\Theta}^0)\right| < \xi_q\right); \boldsymbol{\Theta}^0\right) \quad (2)$$

and is bounded by:

$$\mathcal{P}\left(\sum_{q=1}^{\overline{Q}}\frac{\left(\widehat{g_q(\boldsymbol{\Theta}^0)} - g_q(\boldsymbol{\Theta}^0)\right)^2}{\xi_q^2} < 1; \boldsymbol{\Theta}^0\right) \leq \mathcal{O}_{\boldsymbol{\Theta}^0}\left(\widehat{\mathbf{g}(\boldsymbol{\Theta}^0)}, \boldsymbol{\xi}\right) \leq \mathcal{P}\left(\sum_{q=1}^{\overline{Q}}\frac{\left(\widehat{g_q(\boldsymbol{\Theta}^0)} - g_q(\boldsymbol{\Theta}^0)\right)^2}{\xi_q^2} < \overline{Q}; \boldsymbol{\Theta}^0\right) \quad (3)$$

where $\sum_{q=1}^{\overline{Q}}\frac{\left(\widehat{g_q(\boldsymbol{\Theta}^0)} - g_q(\boldsymbol{\Theta}^0)\right)^2}{\xi_q^2}$ is a weighted total square error. In the following, for the sake of legibility, we focus on the case where $\mathbf{g}(\boldsymbol{\Theta}) = \boldsymbol{\Theta}$, and $\boldsymbol{\Theta}^0$ (respectively $\widehat{\boldsymbol{\Theta}^0}$) is denoted $\boldsymbol{\Theta}$ (respectively $\widehat{\boldsymbol{\Theta}}$) wherever it is unambiguous.

2.2. Probability of resolution and bounds

Thus we consider a parameter estimation problem where the parameters of interest are the vectors $\{\boldsymbol{\theta}^m\}_{m=1}^M$, where $\boldsymbol{\theta} \in \mathbb{R}^P$ and $\boldsymbol{\theta}^m \neq \boldsymbol{\theta}^l$, $\forall l \neq m \in [1, M]$. Then $\boldsymbol{\Theta}^T = \left(\left(\boldsymbol{\theta}^1\right)^T, \ldots, \left(\boldsymbol{\theta}^M\right)^T\right)$ $(\overline{P} = PM, \overline{Q} = \overline{P})$, $\boldsymbol{\xi}^T = \left(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^M\right)$ where $\boldsymbol{\varepsilon}^m = (\varepsilon_1^m, \ldots, \varepsilon_P^m)$. Let $\mathcal{C}^m$ be the hypercube with centre $\boldsymbol{\theta}^m$ defined by $\mathcal{C}^m(\boldsymbol{\varepsilon}^m) = \left\{\boldsymbol{\theta} : \bigcap_{p=1}^{P}\left|\theta_p - \theta_p^m\right| < \varepsilon_p^m\right\}$. We define the probability of resolvability (of vectors of multiple parameters $\boldsymbol{\theta}$) with precision $\boldsymbol{\xi}$ as the probability $\mathcal{O}_{\boldsymbol{\Theta}}\left(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\xi}\right)$ (2) when $\widehat{\boldsymbol{\theta}^m} \in \mathcal{C}^m(\boldsymbol{\varepsilon}^m), \forall m \in [1, M]$, and the hypercubes are disjoint:

$$\mathcal{C}^m(\boldsymbol{\varepsilon}^m) \cap \mathcal{C}^l\left(\boldsymbol{\varepsilon}^l\right) = \varnothing, \ \forall l \neq m \in [1, M]. \quad (4)$$

In other words, we do not consider as successful a trial leading to at least one $\widehat{\boldsymbol{\theta}^m}$ outside $\mathcal{C}^m(\boldsymbol{\varepsilon}^m)$. The underlying idea is that estimates switch among hypercubes $\mathcal{C}^m(\boldsymbol{\varepsilon}^m)$ is not allowed. Parameters vector $\{\boldsymbol{\theta}^m\}_{m=1}^M$ will be said "resolved" by estimators $\widehat{\boldsymbol{\Theta}}$ if:

$$0.9 \leq \mathcal{O}_{\boldsymbol{\Theta}}\left(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\xi}\right) \text{ s.t. (4)} \leq 0.99 \quad (5)$$

2.3. Gaussian p.d.f.

The (lower and upper) bounds on $\mathcal{O}_{\boldsymbol{\Theta}}\left(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\xi}\right)$ given by (3) are particularly convenient when $\widehat{\boldsymbol{\Theta}}(\mathbf{x}) - \boldsymbol{\Theta} \sim \mathcal{N}(\mathbf{b}(\boldsymbol{\Theta}), \mathbf{C}(\boldsymbol{\Theta}))$, that is $\widehat{\boldsymbol{\Theta}}(\mathbf{x})$ is a Gaussian estimator of $\boldsymbol{\Theta}$ with bias vector $\mathbf{b}(\boldsymbol{\Theta})$ and covariance matrix $\mathbf{C}(\boldsymbol{\Theta})$. Then a straightforward linear transformation of the Gaussian random vector yields that (3) is equivalent to :

$$\mathcal{P}\left(e\chi_Q^2\left(\delta(\boldsymbol{\Theta}), \boldsymbol{\sigma}^2(\boldsymbol{\Theta})\right) < 1\right) \leq \mathcal{O}_{\boldsymbol{\Theta}}\left(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\xi}\right) \leq \mathcal{P}\left(e\chi_Q^2\left(\delta(\boldsymbol{\Theta}), \boldsymbol{\sigma}^2(\boldsymbol{\Theta})\right) < Q\right)$$

where $\delta(\boldsymbol{\Theta}) = \left\|\mathbf{M}^T(\boldsymbol{\Theta})\mathbf{D}_{\boldsymbol{\varepsilon}}^{-1}\mathbf{b}(\boldsymbol{\Theta})\right\|^2$, $\mathbf{D}_{\boldsymbol{\varepsilon}}^{-1}\mathbf{C}(\boldsymbol{\Theta})\mathbf{D}_{\boldsymbol{\varepsilon}}^{-1} = \mathbf{M}(\boldsymbol{\Theta})\mathbf{D}_{\boldsymbol{\sigma}^2(\boldsymbol{\Theta})}\mathbf{M}^T(\boldsymbol{\Theta})$, $(\mathbf{D}_{\boldsymbol{\alpha}})_{q,p} = \alpha_q\delta_p^q$ and $e\chi_Q^2\left(\delta, \boldsymbol{\sigma}^2\right)$ is a non-central quadratic form [3], that is an extension of non-central chi-square with corresponding degrees of freedom in Q and positive noncentrality parameters in δ where the power of each component is not constant:

$$e\chi_Q^2\left(\delta = \sum_{q=1}^{Q}\delta_q, \boldsymbol{\sigma}^2\right) \sim \sum_{q=1}^{Q}\sigma_q^2\left|z_q + \sqrt{\delta_q}\right|^2$$

$$\boldsymbol{\sigma}^2 = \left(\sigma_1^2, \ldots, \sigma_Q^2\right)^T, \mathbf{z} = (z_1, \ldots, z_Q)^T \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

If $\delta = 0$ (unbiased estimates) then $u \sim e\chi_Q^2\left(0, \boldsymbol{\sigma}^2\right)$ and:

$$p(u) = \int\prod_{q=1}^{Q}\left(1 + j2\pi f\sigma_q^2\right)^{\frac{-1}{2}}e^{j2\pi fu}du = \frac{{}_0F^0\left(-\frac{1}{2}\mathbf{D}_{\boldsymbol{\sigma}^2}^{-1}, u\right)}{2^{\frac{Q}{2}}\Gamma\left(\frac{Q}{2}\right)\left|\mathbf{D}_{\boldsymbol{\sigma}^2}\right|}$$

where ${}_0F^0\,(\,)$ is a generalized hypergeometric function [3].

3. ASYMPTOTIC PERFORMANCE OF CONDITIONAL MODEL

Historically the first MSE lower bound for deterministic parameters to be derived was the Cramér-Rao Bound (CRB),

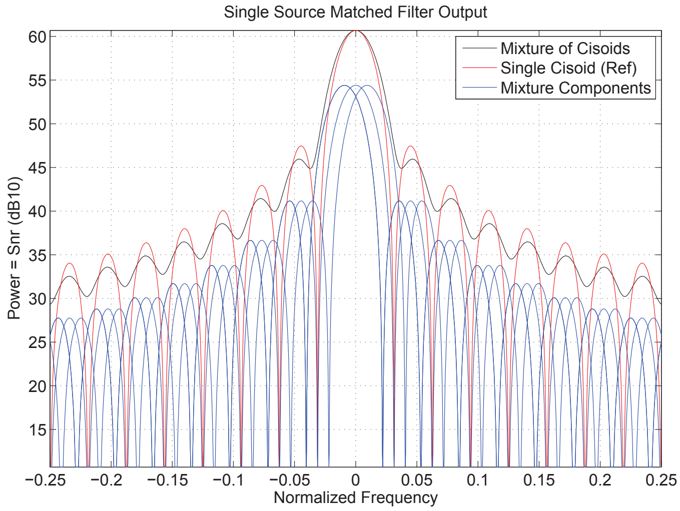

Fig. 1. Single source matched filter output: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.3$

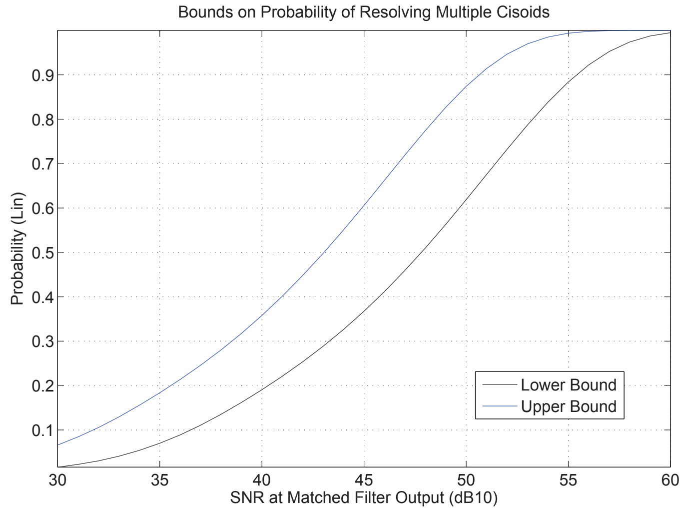

Fig. 2. Bounds on probability of resolving multiple cisoids: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.3, T = 1$

which was introduced to investigate fundamental limits of a parameter estimation problem or to assess the relative performance of a specific estimator (efficiency) [2]. It has since become the most popular lower bound due to its simplicity of calculation, the fact that in many cases it can be achieved asymptotically (high SNR [4] and/or large number of snapshots [2]) by Maximum Likelihood Estimators (MLE). This initial characterization of locally unbiased estimators has been significantly generalized by Barankin work [5], who established the general form of the highest lower bound on MSE (BB) for uniformly unbiased estimates, but unfortunately with a generally incomputable analytic solution. Therefore, since then, numerous works detailed in [5] have been devoted to deriving computable approximations of the BB and have shown that the CRB and the BB can be regarded as key representatives of two general classes of bounds, respectively the Small-Error bounds and the Large-Error bounds. These works have also shown that in non-linear estimation problems three distinct regions of operation can be observed. In the asymptotic region, the MSE is small and, in many cases, close to the Small-Error bounds. In the *a priori* performance region where the number of independent snapshots and/or the SNR are very low, the observations provide little information and the MSE is close to that obtained from the prior knowledge about the problem. Between these two extremes, there is an additional ambiguity region, also called the transition region. In this region, the MSE of MLEs usually deteriorates rapidly with respect to Small-Error bounds and exhibits a threshold behaviour corresponding to a ”performance breakdown”. The nature of this phenomenon is specified by a complicated non-smooth behaviour of the likelihood function in the ”threshold” area where it tends to generate outliers [2]. Small-Error bound such as the CRB are not able to handle the threshold phenomena, whereas it is revealed by Large-Error bounds that can be used to predict the threshold value. Therefore, provided that one keeps in mind the CRB limitations, that is, to become an excessively optimistic lower bound when the observation conditions degrade (low SNR and/or low number of snapshots), the CRB is still a lower bound of great interest for system analysis and design in the asymptotic region.

3.1. Asymptotic performance of radar conditional model

The choice of focusing on the (Gaussian) conditional model comes from our primary interest for active systems such as radar (or sonar) where a known waveform is transmitted, and the signals scattered from the targets of interest are used to estimate their parameters. Typically, the received signals are modelled as scaled, delayed, and Doppler-shifted versions of the transmitted signal. Estimation of the time delay and Doppler shift provides information about the range and radial velocity of the targets. The use of spatial diversity, i.e. antenna arrays, compared with a single sensor, guarantees more accurate range and velocity estimation and allows estimation of the targets direction. Last, but no least, waveform diversity may be used to improve the estimation of all targets parameters. In an active system, as the waveform parametric model is known and deterministic (in opposition with a passive system where a probabilistic modelling of the waveform is generally considered), the most accurate statistical prediction for an observation will be obtained when considering the signal amplitudes as deterministic (since it is well known that the complex Gaussian amplitude modelling provide an average unconditional CRB higher that the corresponding conditional CRB [2]). The asymptotic (in SNR and/or in large number of

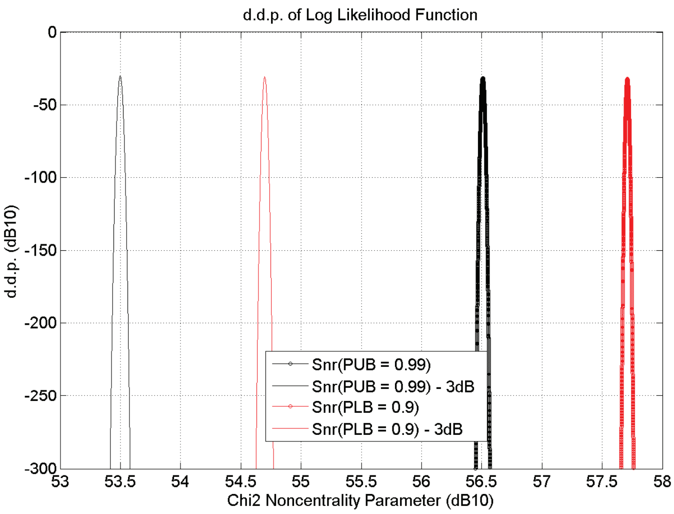

Fig. 3. Loglikelihood p.d.f. at limits of SNR interval allowing resolution of: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.3, T = 1$

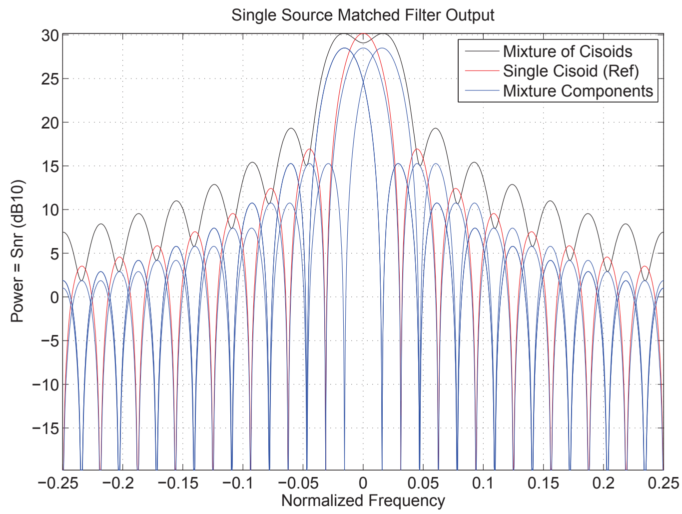

Fig. 4. Single source matched filter output: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.5$

snapshots) Gaussianity and efficiency of CMLEs (conditional MLEs) in the multiple parameters case has been proved under the assumption that the maximum of the (reduced) log likelihood function belongs to its main lobe. As an example, let us consider the general linear observation model:

$$\mathbf{x}_t\left(\boldsymbol{\Theta}^0\right) = \mathbf{A}\left(\boldsymbol{\Theta}^0\right)\mathbf{s}_t + \mathbf{n}_t, \quad t \in [1, T]$$

where T is the number of independent observation, M is the number of signal sources, $\mathbf{s}_t = \left(s_{t,1}, \ldots, s_{t,M}\right)^T$ is the vector of complex amplitudes of the M sources for the t^{th} observation, $\mathbf{A}\left(\boldsymbol{\Theta}\right) = \left[\mathbf{a}\left(\theta_1\right), \ldots, \mathbf{a}\left(\theta_M\right)\right]$ where $\boldsymbol{\Theta} = \left(\theta_1, \ldots, \theta_M\right)^T$ and $\mathbf{a}\left(\,\right)$ is a vector of N parametric functions depending on a single parameter θ (for sake of simplicity), $\mathbf{n}_t$ are Gaussian complex circular independent noises with spatially white covariance matrix: $\mathbf{C}_\mathbf{n} = \sigma_\mathbf{n}^2 \mathbf{I}_N$, independent from the M sources. Then the reduced log likelihood function $L\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)$ is given by [2]:

$$L\left(\boldsymbol{\Theta};\boldsymbol{\Theta}^0\right) = \frac{\sum_{t=1}^{T} \left\|\boldsymbol{\Pi}_{\mathbf{A}(\boldsymbol{\Theta})}\mathbf{x}_t\left(\boldsymbol{\Theta}^0\right)\right\|^2}{TM} \sim \mathcal{CX}^2_{MT}\left(F\left(\boldsymbol{\Theta};\boldsymbol{\Theta}^0\right), \frac{\sigma^2}{TM}\right)$$

$$F\left(\boldsymbol{\Theta};\boldsymbol{\Theta}^0\right) = \frac{\sum_{t=1}^{T} \left\|\boldsymbol{\Pi}_{\mathbf{A}(\boldsymbol{\Theta})}\mathbf{A}\left(\boldsymbol{\Theta}^0\right)\mathbf{s}_t\right\|^2}{TM}$$

where $F\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)$ is a generalized correlation function (aka generalized matched filter) and $\mathcal{CX}^2_K\left(\delta, \sigma^2\right)$ denotes a noncentral complex (circular) chi-square with corresponding degrees of freedom in K and positive noncentrality parameters in δ. Let $\widehat{\boldsymbol{\Theta}} \triangleq \widehat{\boldsymbol{\Theta}}\left(\mathbf{x}\right) = \arg\max\left\{L\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)\right\}$ denote the CMLE of $\boldsymbol{\Theta}$ and let $\Upsilon_{\boldsymbol{\Theta}^0}\left(\alpha\right) = \left\{\boldsymbol{\Theta} : \frac{F\left(\boldsymbol{\Theta};\boldsymbol{\Theta}^0\right)}{F\left(\boldsymbol{\Theta}^0;\boldsymbol{\Theta}^0\right)} > \alpha\right\}$ denote the main lobe at α $(0 < \alpha < 1)$. Then a condition of asymptotic region of operation for CMLE can be :

$$P\left(\boldsymbol{\Theta} \in image\left(\widehat{\boldsymbol{\Theta}}\right) \mid \boldsymbol{\Theta} \notin \Upsilon_{\boldsymbol{\Theta}^0}\left(\frac{1}{2}\right)\right) \approx 0 \qquad (6)$$

where $\Upsilon_{\boldsymbol{\Theta}^0}\left(\frac{1}{2}\right)$ is the usual main lobe at $-3dB$. The quasi-nullity of the probability of an outlier (6) can be demonstrated by computing the p.d.f. of $L\left(\boldsymbol{\Theta}^0; \boldsymbol{\Theta}^0\right)$ and $L\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)$ where $F\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right) = \frac{1}{2}$ and by checking that their supports do not overlap above a certain p.d.f. threshold value, as small as possible (10^{-30} in the present paper). As the p.d.f. of $\mathcal{CX}^2_{MT}\left(F\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right), \frac{\sigma^2}{TM}\right)$ is an increasing function in $F\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)$, it is sufficient to check that p.d.f. of $L\left(\boldsymbol{\Theta}^0; \boldsymbol{\Theta}^0\right)$ and $L\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right)$ where $F\left(\boldsymbol{\Theta}; \boldsymbol{\Theta}^0\right) = \frac{1}{2}$ do not overlap to ensure that this property is valid for any $\boldsymbol{\Theta} \notin \Upsilon_{\boldsymbol{\Theta}^0}\left(\frac{1}{2}\right)$, what proves that $image\left(\widehat{\boldsymbol{\Theta}}\right) \subset \Upsilon_{\boldsymbol{\Theta}^0}\left(\frac{1}{2}\right)$. Then, in the asymptotic region [6]:

$$\widehat{\boldsymbol{\Theta}}\left(\mathbf{x}\right) \sim \mathcal{N}\left(\boldsymbol{\Theta}, \frac{\sigma_\mathbf{n}^2}{2T} \operatorname{Re}\left\{\mathbf{H}\left(\boldsymbol{\Theta}\right) \odot \widehat{\mathbf{R}}_\mathbf{s}^T\right\}^{-1}\right)$$

$$\widehat{\mathbf{R}}_\mathbf{s} = \sum_{t=1}^{T} \frac{\mathbf{s}_t \mathbf{s}_t^H}{T}, \mathbf{H}\left(\boldsymbol{\Theta}\right) = \frac{\partial \mathbf{A}(\boldsymbol{\Theta})}{\partial \theta}^H \boldsymbol{\Pi}^\perp_{\mathbf{A}(\boldsymbol{\Theta})} \frac{\partial \mathbf{A}(\boldsymbol{\Theta})}{\partial \theta}$$

and it has been proved that for each source [6]: the highest (worst) variance is obtained when the sources amplitudes are fully correlated and the lowest (best) variance is obtained when the sources amplitudes are uncorrelated.

3.2. Doppler frequency (multiple tones) estimation

As an example for radar, we consider the problem of Doppler frequency estimation which is a particular application case of the very general multiple tones estimation problem where :

$$\mathbf{a}\left(\theta\right) = \left[1, \ldots, e^{j2\pi n\theta}, \ldots, e^{j2\pi(N-1)\theta}\right]^T \qquad (7)$$

For sake of simplicity but without loss of generality, we consider only scenarios where the Doppler frequencies are equispaced with a step $d\theta = \Delta\theta_{3dB} \times \beta$ in order to take into

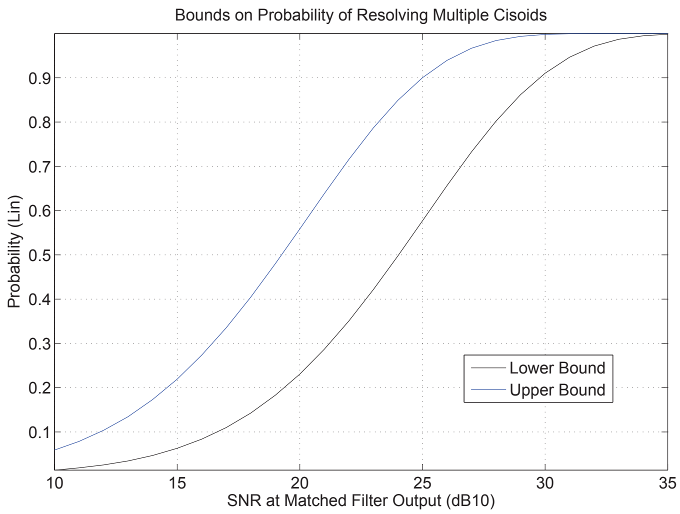

Fig. 5. Bounds on probability of resolving multiple cisoids: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.5, T = 2$

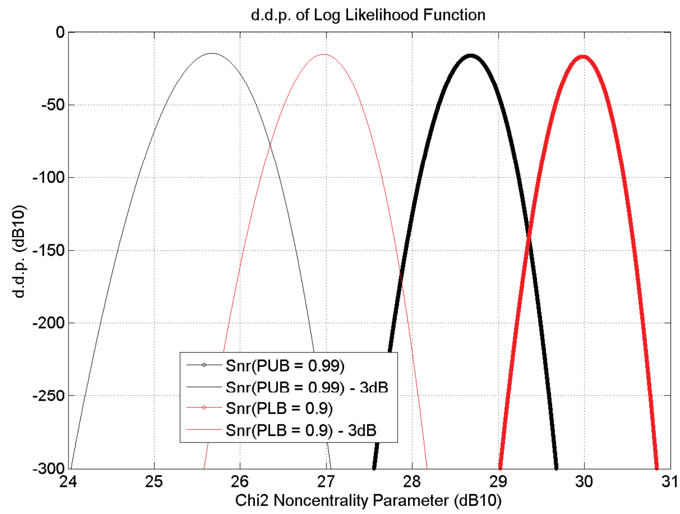

Fig. 6. Loglikelihood p.d.f. at limits of SNR interval allowing resolution of: 3 cisoids separated by $\Delta\theta_{3dB} \times 0.5, T = 2$

account an isotropic estimation error $\boldsymbol{\xi} = \frac{d\theta}{2}\mathbf{1}_M$ in the definition of probability of resolution (5), where $\mathbf{1}_M$ is a M-dimensional vector with components equal to 1. Additionally in all scenarios: $N = 32$ and the target amplitude are equal and therefore fully correlated $\mathbf{s}_t = \sqrt{\frac{SNR}{N}}\mathbf{1}_M$ (but it may not be the worst correlation case [6]). The main resolution features of each scenario are described with 3 figures:

• the output of the single source matched filter $\left(\frac{1}{N}\left\|\mathbf{a}\left(\boldsymbol{\Theta}\right)^H \mathbf{x}_t\left(\boldsymbol{\Theta}^0\right)\right\|^2\right)$ which could be the first step in a practical implementation of the CMLE (Clean algorithm, Alternating Projection algorithm).

• the probability (lower and upper) bounds (PLB and PUB) defined by (3) under (4) where $\boldsymbol{\xi} = \frac{d\theta}{2}\mathbf{1}_M$, as a function of the SNR computed at output of the single source matched filter. These bounds allow to determine the SNR interval containing the SNR from which the sources are resolved according to (5): SNR_{res}. Indeed : $SNR\left(PUB = 0.99\right) \leq SNR_{res} \leq SNR\left(PLB = 0.9\right)$.

• the p.d.f. of $L\left(\boldsymbol{\Theta}^0; \boldsymbol{\Theta}^0\right)$ and $L\left(\boldsymbol{\Theta}^0_{3dB}; \boldsymbol{\Theta}^0\right)$ for $SNR\left(PUB = 0.99\right)$ and $SNR\left(PLB = 0.9\right)$ to prove that within $\left[SNR\left(PUB = 0.99\right), SNR\left(PLB = 0.9\right)\right]$ the condition of asymptotic region of operation for CMLE is valid.

4. CONCLUSION

In the first scenario $SNR_{res} \in [54.4, 55.6]$ dB, which is a quite high required value to resolved a non demanding high resolution scenario of 3 targets ($d\theta = \Delta\theta_{3dB} \times 0.3$). This result suggests that high resolution techniques in operational radar system with a limited transmitted power will be rather a myth. On the other hand the addition of a second observation ($T = 2$) in the second scenario coupled with more spaced parameters values ($d\theta = \Delta\theta_{3dB} \times 0.5$) allows to decrease $SNR_{res} \in [28.5, 29.8]$. This result suggests that high resolution techniques in operational radar system can be a reality in some not too demanding scenarios provided a relevant waveform is transmitted.

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