EXACT TRACKING ANALYSIS OF THE NLMS ALGORITHM FOR CORRELATED GAUSSIAN INPUTS

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ABSTRACT

This work presents an exact tracking analysis of the Normalized Least Mean Square (NLMS) algorithm for circular complex correlated Gaussian inputs. Unlike the existing works, the analysis presented neither uses separation principle nor small step-size assumption. The approach is based on the derivation of a closed form expression for the cumulative distribution function (CDF) of random variables of the form $(||\mathbf{u}||^2_{\mathbf{D}_1})(||\mathbf{u}||^2_{\mathbf{D}_2})^{-1}$ where $\mathbf{u}$ is a white Gaussian vector and $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices and using that to derive the first and second moments of such variables. These moments are then used to evaluate the tracking behavior of the NLMS algorithm in closed form. Thus, both the steady-state mean-square-error (MSE) and mean-square-deviation (MSD)tracking behaviors of the NLMS algorithm are evaluated. The analysis is also used to derive the optimum step-size that minimizes the excess MSE (EMSE). Simulations presented for the steady-state tracking behavior support the theoretical findings for a wide range of step-size and input correlation.

Index Terms — Adaptive filters, NLMS algorithm, Tracking analysis.

1. INTRODUCTION

The NLMS algorithm [1] provides faster convergence compared to the LMS algorithm [2] by utilizing the input power normalization. However, this normalization complicates the performance analysis of the algorithm. In [3], closed form expressions for the transient analysis and the steady-state MSE of the NLMS algorithm are developed which was extended in [4] to the tracking case. However, the expressions developed in [3] and [4] are both in terms of multi-dimensional moments which they falls short of evaluating. Several other works have attempted to evaluate these moments but the corresponding analysis does not result in closed form performance expressions [2, 3, 5, 6], or relies on strong assumptions. Examples of these assumptions include the separation principle [7, 8], approximations [9], white input [6, 9, 10], specific structure of input regressors distribution [7], small step size [7], long filters [9] and approximate solutions using Abelian integrals [7].

Recently, in [11], an exact tracking performance of the ϵ-NLMS for colored circular complex Gaussian data is presented but this approach fails to work in the special case of $\epsilon = 0$ which correspond to the NLMS algorithm. The reason is that the ϵ-NLMS expressions are expressed in terms of the exponential integral function $E_n(x) \triangleq \int_1^\infty t^{-n} e^{-xt} dt$ which becomes infinite when we set $\epsilon = 0$. This is compounded by the fact that the resulting expressions are too involved to resolve indefinite forms like $\infty \pm \infty$ or ∞/∞. Moreover, the approach in [11] employs the separation principle to obtain the steady-state result, which we avoid in this paper. Furthermore, this work evaluates both steady-state tracking EMSE and MSD of the NLMS algorithm, unlike [11] which has ignored the tracking MSD analysis.

The approach of the paper is based on evaluating the CDF and the moments of random variables of the form $\xi(\mathbf{u}, \mathbf{D}_1, \mathbf{D}_2) = (||\mathbf{u}||^2_{\mathbf{D}_1})(||\mathbf{u}||^2_{\mathbf{D}_2})^{-1}$ where $\mathbf{u}$ is a white Gaussian vector and $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices. This is done by expressing these variables as ratios of quadratic forms in isotropic random variables $(||\phi||^2_{\mathbf{D}_1})(||\phi||^2_{\mathbf{D}_2})^{-1}$ where ϕ is an isotropic random vector. Thus, this work deals with isotropic random variables (as opposed to the Gaussian random variables approach used in [11]). The tracking analysis can be performed using various moments of ξ which will be evaluated from the derived CDF.

The paper is organized as follows. Following this introduction, system model is described in Section 2. The tracking performance analysis of the NLMS algorithm is presented in Section 3. Outline of our approach is presented in Section 4. Derivation for the CDF of the random variable of the form $(||\mathbf{u}||^2_{\mathbf{D}_1})(||\mathbf{u}||^2_{\mathbf{D}_2})^{-1}$ is carried out in Section 5. Calculation of the required moments of other random variables are given in Section 6. In Section 7, procedure to evaluate the steady-state tracking EMSE and MSD using the derived moments is presented. The steady-state tracking EMSE is then optimized with respect to algorithm's step-size in Section 8. Simulation results are presented in Section 9 investigating the performance of the derived analytical model. Finally, conclusion is given in Section 10.

2. SYSTEM MODEL

Given a sequence of desired response $\{d_i\}$ and a sequence of regressor (row) vectors $\{\mathbf{u}_i\}$, an adaptive filter generates a weight vector $\mathbf{w}_i$ at each instant so that $\mathbf{u}_i\mathbf{w}_i$ is a good estimate of d_i. In the NLMS algorithm, starting from $\mathbf{w}_{-1} = 0$, the weight vector is updated according to

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \frac{\mathbf{u}_i^*}{||\mathbf{u}_i||^2} e_i, \qquad i \geq 0 \tag{1}$$

where $e_i = d_i - \mathbf{u}_i\mathbf{w}_{i-1}$ is estimation error and μ is the step-size. In tracking analysis, the desired response d_i is generated by the time-variant system identification model as $d_i = \mathbf{u}_i\mathbf{w}_i^o + v_i$ where

$\mathbf{w}_i^o$ varies according to $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$ where $\mathbf{q}_i$ is assumed to be i.i.d with mean zero and covariance matrix $\mathbf{R}_q = E[\mathbf{q}_i\mathbf{q}_i^*]$. Thus, $\mathbf{w}_i$ generated by the adaptive filter attempts to track time varying system $\mathbf{w}_i^o$. Let $\tilde{\mathbf{w}}_i = \mathbf{w}_i^o - \mathbf{w}_i$ denote the weight error vector, then

$$e_i = \mathbf{u}_i\tilde{\mathbf{w}}_{i-1} + v_i + \mathbf{u}_i\mathbf{q}_i, \tag{2}$$

and, by subtracting $\mathbf{w}_i^o$ from both sides of (1), the NLMS update can be equivalently written as

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} - \mu\frac{\mathbf{u}_i^*}{||\mathbf{u}_i||^2}e_i + \mathbf{q}_i, \qquad i \geq 0 \tag{3}$$

We will restrict our attention in this paper to circularly symmetric Gaussian inputs, i.e. $\mathbf{u}_i \sim \mathcal{CN}\ (\mathbf{0}, \mathbf{R})$. Without loss of generality, the autocorrelation matrix $\mathbf{R}$ can be assumed to be diagonal, that is, $\mathbf{R} = \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_M)$ [3, 6]. In addition to the Gaussian assumption on $\mathbf{u}_i$, we assume that the sequences $\{v_i\}$, $\{\mathbf{u}_i\}$, and $\{\mathbf{q}_i\}$ are i.i.d. and mutually independent.

3. TRACKING ANALYSIS OF THE NLMS ALGORITHM

The recursion for the weighted variance of weight error vector $\tilde{\mathbf{w}}_i$ can be set up as [3]

$$E[||\tilde{\mathbf{w}}_i||_\sigma^2] = E[||\tilde{\mathbf{w}}_{i-1}||_{\mathbf{F}\sigma}^2] + \mu^2\sigma_v^2 E\left[\frac{||\mathbf{u}_i||_\sigma^2}{(||\mathbf{u}_i||^2)^2}\right] + E[||\mathbf{q}_i||_{\mathbf{F}\sigma}^2], \tag{4}$$

where $\boldsymbol{\sigma}$ is an $M \times 1$ parameter weight vector that can provide different performance measures by choosing its value appropriately, and $\mathbf{F} = \mathbf{I} - \mu\mathbf{A} + \mu^2\mathbf{B}$ where $\mathbf{I}$ is the identity matrix while $\mathbf{A}$ and $\mathbf{B}$ are the multidimensional input moment matrices defined by

$$\mathbf{A} \triangleq 2E\left[\frac{\mathbf{u}_i^*\mathbf{u}_i}{||\mathbf{u}_i||^2}\right], \quad \mathbf{B} \triangleq E\left[\frac{(\mathbf{u}_i^*\mathbf{u}_i)^T \odot (\mathbf{u}_i^*\mathbf{u}_i)}{(||\mathbf{u}_i||^2)^2}\right], \tag{5}$$

Here the notation $\odot$ denotes an element-by-element (Hadamard) product. Thus, the steady-state tracking EMSE can be evaluated by choosing $\sigma = (\mathbf{I} - \mathbf{F})^{-1}\boldsymbol{\lambda}$ and analyzing the recursion (4) at steady-state (i.e., $i \to \infty$). Consequently, the steady-state tracking EMSE (ζ_{track}) of the NLMS algorithm is found to be [4]

$$\zeta_{track} = \mu^2\sigma_v^2\mathbf{c}(\mathbf{I} - \mathbf{F})^{-1}\boldsymbol{\lambda} + \mathbf{r}_q\mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\boldsymbol{\lambda} \tag{6}$$

where $\mathbf{c}$, $\mathbf{r}_q$, and $\boldsymbol{\lambda}$ are vectors defined as

$$\mathbf{c} \triangleq \text{diag}(\mathbf{C}), \quad \boldsymbol{\lambda} \triangleq \text{diag}(\mathbf{\Lambda}), \quad \text{and} \quad \mathbf{r}_q \triangleq \text{diag}(\mathbf{R}_q) \tag{7}$$

and the moment matrix $\mathbf{C}$ is defined by

$$\mathbf{C} \triangleq E\left[\frac{\mathbf{u}_i^*\mathbf{u}_i}{(||\mathbf{u}_i||^2)^2}\right] \tag{8}$$

Similarly, the steady-state MSD (M_{track}) of the NLMS algorithm can be evaluated by substituting $\sigma = (\mathbf{I} - \mathbf{F})^{-1}\mathbf{1}$ in recursion (4) at $i \to \infty$ and can be shown to be

$$\mathrm{M}_{track} = \mu^2\sigma_v^2\mathbf{c}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{1} + \mathbf{r}_q\mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{1} \tag{9}$$

where $\mathbf{1}$ is an $M \times 1$ vector with all entries equal to 1.

4. OUR METHODOLOGY

The mean-square performance measures of the NLMS algorithm can be evaluated by the moment matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. In order to evaluate these moment matrices, we rewrite them as follows

$$\mathbf{A} = \mathbf{\Lambda}\bar{\mathbf{A}}, \quad \mathbf{B} = \mathbf{\Lambda}\bar{\mathbf{B}}\mathbf{\Lambda}, \quad \text{and} \quad \mathbf{C} = \mathbf{\Lambda}\bar{\mathbf{C}} \tag{10}$$

where $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ are for the "whitened" versions of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively, i.e. the vector $\mathbf{u}_i$ in (5) and (8) is replaced by $\bar{\mathbf{u}}_i$ such that $u_i(k) = \sqrt{\lambda_k}\bar{u}_i(k)$. Now, in order to evaluate the entries of these moment matrices, we define following random variables

$$s_k \triangleq \frac{|\bar{u}(k)|^2}{||\bar{\mathbf{u}}||_\Lambda^2}, \quad s_{k\bar{k}} \triangleq \frac{\sqrt{\frac{\lambda_k}{\lambda_{\bar{k}}}}|\bar{u}(k)|^2 + \sqrt{\frac{\lambda_{\bar{k}}}{\lambda_k}}|\bar{u}(\bar{k})|^2}{||\bar{\mathbf{u}}||_\Lambda^2},$$
$$z_k \triangleq \frac{|\bar{u}(k)|^2 + 1}{||\bar{\mathbf{u}}||_\Lambda^2} \quad \text{and} \quad r \triangleq \frac{1}{||\bar{\mathbf{u}}||_\Lambda^2} \tag{11}$$

where k and $\bar{k}$, respectively, denote the k^{th} and $\bar{k}^{th}$ distinct positions of the elements in input vector $\mathbf{u}_i$. Here we have dropped the subscript i to simplify the notations. With this, the entries of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ can be expressed directly in terms of the moments of these random variables as shown in Table 1. In

Moment Matrix Entries	Required Moment
$\bar{\mathbf{A}}(k,k)$	$2E\left[s_k\right]$
$\bar{\mathbf{A}}(k,\bar{k})$	0
$\bar{\mathbf{B}}(k,k)$	$E\left[s_{k\bar{k}}^2\right]$
$\bar{\mathbf{B}}(k,\bar{k})$	$\frac{1}{2}E\left[s_{k\bar{k}}^2\right] - \frac{1}{2}E\left[s_k^2\right] - \frac{1}{2}E\left[s_{\bar{k}}^2\right]$
$\bar{\mathbf{C}}(k,k)$	$\frac{1}{2}E\left[z_k^2\right] - \frac{1}{2}E\left[s_k^2\right] - \frac{1}{2}E\left[r^2\right]$
$\bar{\mathbf{C}}(k,\bar{k})$	0

Table 1: Entries of $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ in terms of s_k, $s_{k\bar{k}}$, r, and z_k.

the ensuing, CDF and moments of the required random variables defined in (11) are evaluated.

5. THE CDF OF S_K AND $S_{K\bar{K}}$

First we evaluate the CDF of random variable $s_{k\bar{k}}$. The key to evaluating the CDF of $s_{k\bar{k}}$ is to first define $s_{k\bar{k}}$ in terms of isotropic random variable. By using the definition $\phi = \left[\frac{\bar{u}(1)}{\|\bar{u}\|}, \cdots, \frac{\bar{u}(M)}{\|\bar{u}\|}\right]$, we can rewrite $s_{k\bar{k}}$ as

$$s_{k\bar{k}} = \frac{\sqrt{\frac{\lambda_k}{\lambda_{\bar{k}}}}|\bar{u}(k)|^2 + \sqrt{\frac{\lambda_{\bar{k}}}{\lambda_k}}|\bar{u}(\bar{k})|^2}{\sum_{i=1}^M \lambda_i|\bar{u}(i)|^2} = \frac{\|\phi\|_{\mathbf{\Sigma_{k\bar{k}}}}^2}{\|\phi\|_{\mathbf{\Lambda}}^2}, \tag{12}$$

where $\mathbf{\Sigma_{k\bar{k}}}$ is an $M \times M$ diagonal matrix with diagonal elements $\sigma_i^{k\bar{k}}$ such that $\sigma_i^{k\bar{k}}$ is equal to $\sqrt{\frac{\lambda_k}{\lambda_{\bar{k}}}}$ for $i = k$, $\sqrt{\frac{\lambda_{\bar{k}}}{\lambda_k}}$ for $i = \bar{k}$, and 0 otherwise. The random vector ϕ in (12) is known as the isotropic random vector [12] and has the pdf $p(\phi) = \frac{\Gamma(M)}{\pi^M}\ \delta(\|\phi\|^2 - 1)$. We can now express the CDF of $s_{k\bar{k}}$ using the isotropic random variable ϕ. Specifically,

$$F_{s_{k\bar{k}}}(x) = Pr\{\|\phi\|_{x\mathbf{\Lambda} - \mathbf{\Sigma_{k\bar{k}}}}^2 \geq 0\} = \int_{\|\phi\|_{x\mathbf{\Lambda} - \mathbf{\Sigma_{k\bar{k}}}}^2 \geq 0} p(\phi)d\phi \tag{13}$$

This is an M-dimensional integral over the region defined by the inequality $\|\phi\|_{x\mathbf{\Lambda} - \mathbf{\Sigma_{k\bar{k}}}}^2 \geq 0$ which is difficult to evaluate. We can write (13) as an unconstrained integral by using the unit step function as

$$F_{s_{k\bar{k}}}(x) = \int p(\phi)\tilde{u}(\|\phi\|^2_{x\mathbf{\Lambda}-\mathbf{\Sigma}_{\mathbf{k\bar{k}}}})d\phi, \quad (14)$$

In order to evaluate the above integral, we use the following integral representations of the step and the delta functions [12]

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{x(j\omega+\beta)}}{(j\omega+\beta)}d\omega \text{ and } \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x(\alpha+j\omega)}d\omega, \quad (15)$$

where $\alpha,\ \beta > 0$ and are free parameters that we can conveniently choose. After replacing delta and step functions with their equivalent integral representations, integrals appearing in the above equation can be evaluated using the methodology of [13, 14]. We skip these derivations due to space limitations. The CDF of $s_{k\bar{k}}$ can be shown to be

$$F_{s_{k\bar{k}}}(x) = \sum_{i=1}^{M} \frac{(\lambda_i x - \sigma_i^{k\bar{k}})^{M-1}\tilde{u}(\lambda_i x - \sigma_i^{k\bar{k}})}{\Pi_{j=1,j\neq i}^{M}[(\sigma_j^{k\bar{k}} - \sigma_i^{k\bar{k}}) - (\lambda_j - \lambda_i)x]}, \quad (16)$$

where $\sigma_i^{k\bar{k}}$ is the i^{th} diagonal element in matrix $\mathbf{\Sigma}_{\mathbf{k\bar{k}}}$. Similarly, the CDF of s_k, denoted by $F_{s_k}(x)$, can be found using the above mentioned approach and is found to be the same expression as given in (16) with $\sigma_i^{k\bar{k}}$ equal to 1 for $i = k$ only and 0 otherwise.

5.1. The moments of s_k and $s_{k\bar{k}}$

It is easy to see that s_k is a positive random variable whose support is the interval $[0, 1/\lambda_k]$. Thus, by employing integration by parts and using (16), the first moment of s_k can be evaluated as

$$E[s_k] = \int_0^{\infty} (1 - F_{s_k}(x))\, dx = \int_0^{\frac{1}{\lambda_k}} (1 - F_{s_k}(x))\, dx \quad (17)$$

Using partial fraction expansion, we can show after tedious but straight forward algebraic manipulations that

$$E[s_k] = \frac{1}{\lambda_k} - \sum_{i=1,i\neq k}^{M} p_i \left(\frac{1}{\lambda_k} + \frac{\ln(\lambda_{ik})}{(\lambda_i - \lambda_k)}\right) \quad (18)$$

where ln(.) represents the natural logarithm while λ_{ij} and p_i are defined as

$$\lambda_{ij} \triangleq \frac{\lambda_i}{\lambda_j}, \forall\, i, j \quad \text{and} \quad p_i \triangleq \frac{\lambda_i^{M-1}}{\Pi_{j=1,j\neq i}^{M}(\lambda_i - \lambda_j)}, \forall\, i. \quad (19)$$

The second moment of $s_{k\bar{k}}$ can be similarly evaluated. Using the CDF from (16) with $[0, 1/\lambda_k]$ as support of $s_{k\bar{k}}$ and by employing the partial fraction expansion, the second moment of $s_{k\bar{k}}$ can be shown to be

$$E[s_{k\bar{k}}^2] = \frac{1}{\lambda_k\lambda_{\bar{k}}} - \sum_{i=1,i\neq k,\bar{k}}^{M} p_i \left(\frac{1}{\lambda_k\lambda_{\bar{k}}} + \frac{2c_{ki} + 2c_{\bar{k}i}}{\sqrt{\lambda_k\lambda_{\bar{k}}}} + \ln(\lambda_{ki})^{2c_{ki}\bar{\sigma}_{ki}} + \ln(\lambda_{\bar{k}i})^{2c_{\bar{k}i}\bar{\sigma}_{\bar{k}i}}\right) \quad (20)$$

where $\bar{\sigma}_{ji}$, c_{ki}, and $c_{\bar{k}i}$ are defined as

$$\bar{\sigma}_{ji} = \frac{\sigma_j^{k\bar{k}}}{(\lambda_i - \lambda_j)},\ \ c_{ki} = \frac{(-\bar{\sigma}_{ki})^{M-1}}{\Pi_{l=1,l\neq k,i}^{M}(\bar{\sigma}_{li} - \bar{\sigma}_{ki})} \quad (21)$$

$$c_{\bar{k}i} = \frac{(-\bar{\sigma}_{\bar{k}i})^{M-1}}{\Pi_{l=1,l\neq\bar{k},i}^{M}(\bar{\sigma}_{li} - \bar{\sigma}_{\bar{k}i})} \quad (22)$$

Following the same approach, and using the CDF of s_k, the second moment of s_k is found to

$$E[s_k^2] = \frac{1}{\lambda_k^2} - \sum_{i=1,i\neq k}^{M} p_i \left(\frac{1}{\lambda_k^2} - \frac{2}{\lambda_k(\lambda_i - \lambda_k)} + \frac{2\ln(\lambda_{ik})}{(\lambda_i - \lambda_k)^2}\right) \quad (23)$$

6. MOMENTS OF r AND z_k

In this section, the moments of random variables r and z_k are evaluated using their respective CDFs.

6.1. The CDF and the pdf of r

Consider the random variable r defined in (11) for circular complex Gaussian input. Thus, the CDF of r, denoted by $F_r(x)$, can be evaluated using the technique of [15] which is found to be:

$$F_r(x) = \sum_{m=1}^{M} \frac{\lambda_m^M}{|\mathbf{\Lambda}|\Pi_{i=1,i\neq m}^{M}(\lambda_{mi} - 1)}\, e^{\frac{-1}{\lambda_m x}}\, \tilde{u}(x) \quad (24)$$

Thus, the pdf of r, denoted by $p_r(x)$, can be obtained via differentiation of the above CDF and can be shown to be:

$$p_r(x) = \sum_{m=1}^{M} \frac{\lambda_m^{M-1}}{x^2|\mathbf{\Lambda}|\Pi_{i=1,i\neq m}^{M}(\lambda_{mi} - 1)}\, e^{\frac{-1}{\lambda_m x}} \quad (25)$$

6.2. The second moment of r

Since, we need only the second moment of r, we evaluate this moment using the pdf of r derived in (25). Upon substituting the value of $p_r(x)$, the second moment of r can be evaluated as follows:

$$\begin{aligned} E[r^2] &= \sum_{m=1}^{M} \frac{\lambda_m^{M-1}}{|\mathbf{\Lambda}|\Pi_{i=1,i\neq m}^{M}(\zeta_{mi} - 1)} \int_0^{\infty} \frac{e^{\frac{-1}{\lambda_m x}}}{x^2}dx \\ &= \sum_{m=1}^{M} \frac{\lambda_m^M}{|\mathbf{\Lambda}|\Pi_{i=1,i\neq m}^{M}(\zeta_{mi} - 1)} \end{aligned} \quad (26)$$

6.3. The second moment of z_k

We can evaluate the second moment of z_k using the conditional moment $E[z_k^2|a_k]$ by knowing that $E[z_k^2] = E[E[z_k^2|a_k]]$ where $a_k = \sum_{m=1,m\neq k}^{M} |u(m)|^2$. The conditional moment $E[z_k^2|a_k]$ can be calculated using the conditional pdf $f_{z_k|a_k}(x)$ and is found to be

$$\begin{aligned} E[z_k^2|a_k] &= \frac{1}{\lambda_k^2} + \frac{2(\lambda_k - a_k)}{\lambda_k^3}\, e^{\frac{a_k}{\lambda_k}}\, \Gamma\left(0, \frac{a_k}{\lambda_k}\right) \\ &+ \frac{(\lambda_k - a_k)^2}{\lambda_k^4}\, e^{\frac{a_k}{\lambda_k}}\, \Gamma\left(-1, \frac{a_k}{\lambda_k}\right) \end{aligned} \quad (27)$$

where $\Gamma(\alpha, x)$ is the Incomplete Gamma function. Now, the pdf of a_k is derived using the approach of [15] and is found to be

$$f_{a_k}(a_k) = \frac{\lambda_k}{|\mathbf{\Lambda}|} \sum_{m=1,m\neq k}^{M} c_m\, e^{-\frac{a_k}{\lambda_m}} \quad (28)$$

where $c_m = \left(\Pi_{l=1,l\neq k,m}^{M}\left(\frac{1}{\lambda_l}-\frac{1}{\lambda_m}\right)\right)^{-1}$. Thus, finally, the second moment of z_k, $E[z_k^2]$, is calculated by averaging the conditional moment $E[z_k^2|a_k]$ given in (27) over $f_{a_k}(a_k)$. By defining $\eta_{mk} = 1-\lambda_{mk}$, the moment $E[z_k^2]$ is found to be:

$$E[z_k^2] = \sum_{m=1,m\neq k}^{M} \frac{c_m}{|\mathbf{\Lambda}|}\ln(\lambda_m) + \tag{29}$$

$$\sum_{m=1,m\neq k}^{M} \frac{c_m\lambda_m}{\lambda_k|\mathbf{\Lambda}|}\left(1 +_2 F_1\left(1,1;2;\eta_{mk}\right) - {}_2F_1\left(1,1;3;\eta_{mk}\right)\right)$$

$$+ \sum_{m=1,m\neq k}^{M} \frac{c_m\lambda_m^2}{\lambda_k^2|\mathbf{\Lambda}|}\left(\frac{1}{3}\, {}_2F_1\left(1,2;4;\eta_{mk}\right) -_2 F_1\left(1,2;3;\eta_{mk}\right)\right)$$

7. STEADY-STATE TRACKING EMSE AND MSD EVALUATION

Equations (18), (20), (23), (26), and (29) provide closed form expressions for $E[s_k]$, $E[s_{k\bar{k}}]$, $E[s_k^2]$, $E[r^2]$, and $E[z_k^2]$, respectively, which can be used to evaluate the entries of $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ and hence of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ given in (10). Finally, upon substituting in (6) and (9), the steady-state tracking EMSE and MSD of the NLMS algorithm can be evaluated, respectively.

8. OPTIMUM STEP-SIZE

To evaluate the optimum step-size (μ_o), we need to differentiate the expression of ζ_{track} in (6) with respect to step-size (μ) and set it zero. This is difficult to do due to the inverse of $(\mathbf{I}-\mathbf{F})$ appearing in (6). To go around this, we make the following approximation:

$$E[s_{k\bar{k}}] \approx \sqrt{E\left[s_k^2\right].E\left[s_{\bar{k}}^2\right]} \tag{30}$$

which allow us to approximate $\bar{\mathbf{B}}$ as a rank one matrix

$$\bar{\mathbf{B}} \approx \bar{\mathbf{b}}_r\bar{\mathbf{b}}_r^T \tag{31}$$

where $\bar{\mathbf{b}}_r = \left[\sqrt{E\left[s_1^2\right]}, \sqrt{E\left[s_2^2\right]}, \cdots, \sqrt{E\left[s_M^2\right]}\right]$. With this bound, we can approximate the matrix $\mathbf{F}$ as

$$\mathbf{F} \approx \mathbf{I} - \mu\mathbf{\Lambda}\bar{\mathbf{A}} + \mu^2\mathbf{\Lambda}\bar{\mathbf{b}}_r\bar{\mathbf{b}}_r^T\mathbf{\Lambda} \tag{32}$$

We can use matrix inversion lemma to evaluate $(\mathbf{I}-\mathbf{F})^{-1}$ as

$$(\mathbf{I}-\mathbf{F})^{-1} \approx \frac{1}{\mu}\bar{\mathbf{A}}^{-1}\mathbf{\Lambda}^{-1} + \frac{\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}_r\bar{\mathbf{b}}_r^T\mathbf{\Lambda}\bar{\mathbf{A}}^{-1}\mathbf{\Lambda}^{-1}}{1-\mu\bar{\mathbf{b}}_r^T\mathbf{\Lambda}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}_r} \tag{33}$$

With this approximation, differentiate both sides of (6) with respect to μ and set the result to zero to get

$$\left(2\mu\sigma_v^2\mathbf{c} + \mathbf{r}_q(2\mu\mathbf{B}-\mathbf{A})\right)\left(\frac{\mathbf{A}^{-1}}{\mu} + \frac{\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}\mathbf{\Lambda}\mathbf{A}^{-1}}{1-\mu\bar{\mathbf{b}}_r^T\mathbf{\Lambda}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}_r}\right)\boldsymbol{\lambda}$$
$$= \left(\mu^2\sigma_v^2\mathbf{c} + \mathbf{r}_q\mathbf{F}\right)\left(\frac{\mathbf{A}^{-1}}{\mu^2} - \frac{\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}\mathbf{\Lambda}\mathbf{A}^{-1}\bar{\mathbf{b}}_r^T\mathbf{\Lambda}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}_r}{\left(1-\mu\bar{\mathbf{b}}_r^T\mathbf{\Lambda}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}_r\right)^2}\right)\boldsymbol{\lambda} \tag{34}$$

Note that (34) is a fourth order polynomial equation in μ which can be easily solved to find its roots and hence the optimum μ.

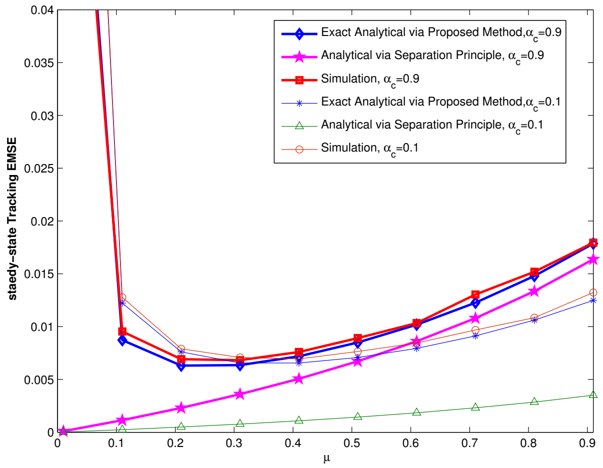

Figure 1: Steady-state tracking EMSE versus μ for $\alpha_c = 0.1$ and $\alpha_c = 0.9$ with $\sigma_q^2 = 10^{-4}$.

9. SIMULATION RESULTS

In simulations, the tracking performance of the NLMS algorithm for a random walk channel is investigated for two different values of σ_q^2, that is 10^{-4} and 10^{-6} with input correlation $\mathbf{R}(i,j) = \alpha_c^{|i-j|}$ with correlation factor α_c $(0 < \alpha_c < 1)$ and noise variance $\sigma_v^2 = 0.01$. In Figure 1, the tracking performance of the NLMS algorithm is reported for σ_q^2 equal to 10^{-4}. In this experiment, two different values of correlation factor α_c, that is, 0.1 and 0.9 showing smaller and larger correlation, respectively. The tracking results of the proposed exact analytical solution are compared with the analytical results via separation principle [2] and the one obtained from simulation. It can be easily depicted from the figure that the proposed exact analytical result accurately evaluated the tracking EMSE for a wide range of step-size while the one from separation principle gives under estimate of the actual values. Moreover, unlike the stationary case, the tracking EMSE is not a monotonically increasing function of step-size. The optimum values of step-size found from equation (34) are found to be .28 and 0.4, respectively, for α_c equal to 0.9 and 0.1 which are verified from the simulation results given in Figure 1. In order to investigate the effect of σ_q^2, simulation is carried for σ_q^2 equal to 10^{-6} while correlation factor α_c is kept at 0.5 in Figure 2. Finally, simulation result for the steady-state MSD of the NLMS algorithm is presented for σ_q^2 equal to 10^{-6} and α_c equal to 0.5 in Figure 3. The results show that the derived analytical expressions are valid for both large and small values of σ_q^2.

10. CONCLUSION

This work has evaluated certain expectations of complex Gaussian functionals using a novel approach which is needed for recursions for the tracking analysis of the NLMS algorithm. Our approach then showed that these moments can be evaluated by first evaluating the CDF of a variable of the form $(||\mathbf{u}||^2_{\mathbf{D}_1})(||\mathbf{u}||^2_{\mathbf{D}_2})^{-1}$ where $\mathbf{u}$ is a white Gaussian vector and $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices and using that to derive the first and second moments of such variables. The advantage of this approach is its ability to evaluate per-

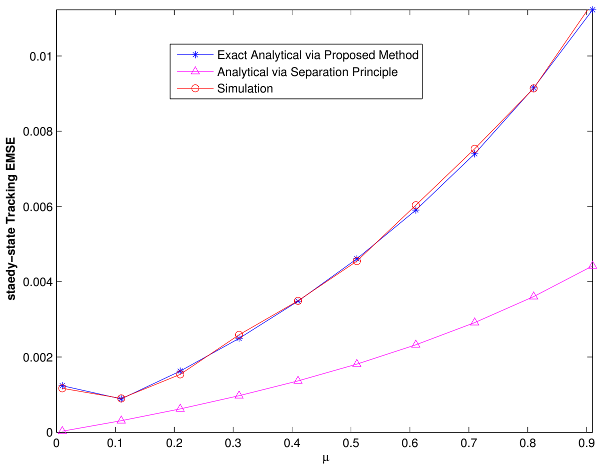

Figure 2: Steady-state tracking EMSE versus μ for $\alpha_c = 0.5$ and $\sigma_q^2 = 10^{-6}$.

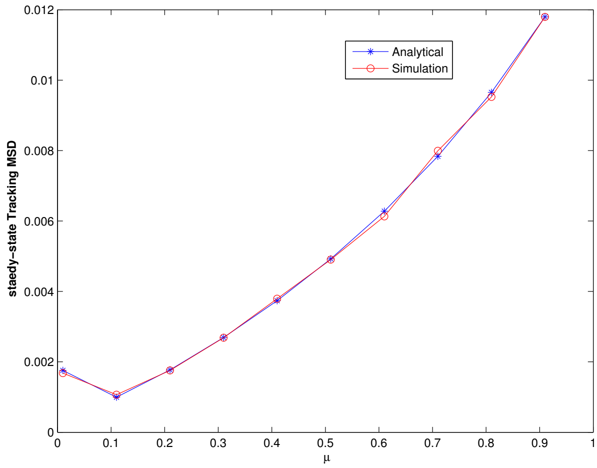

Figure 3: Steady-state tracking MSD versus μ for $\alpha_c = 0.5$ and $\sigma_q^2 = 10^{-6}$.

formance of the NLMS algorithm in closed form without imposing separation principle or small step-size assumption. The theoretical results are shown to yield excellent agreement with Monte Carlo simulations.

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