GENERALIZED COMPLEX TIME-DISTRIBUTION USING MODIFIED ANALYTICAL CONTINUATION

C. Bernard, C. Ioana

GIPSA-Lab/DIS, Grenoble Institute of Technology, Grenoble, France
E-mails: cindy.bernard@gipsa-lab.grenoble-inp.fr, cornel.ioana@gipsa-lab.grenoble-inp.fr

ABSTRACT

The Generalized Complex time distributions have been recently introduced as a way for reducing the auto-terms of any bilinear time-frequency representation that appear when dealing with non-linear time-frequency structures. This concept requires the definition of signal at complex times and this abstract operation is achieved by the analytical continuation principle. In the current version, this principle is efficient only for narrow-band signals, restricting also the application of the complex time distribution to more complicate signals. The purpose of this paper is to propose a method to overcome the limitations of the analytical continuation in the case of signals with a spread time-frequency variation. This method is based on the compression of the signals spectrum to a bandwidth that ensures the efficiency of the analytical continuation technique. Then, the application of generalized complex time distribution will allow an accurate estimation of the instantaneous frequency law. The spectrum expanding will bring this estimation to the correct time-frequency location.

***Index Terms*—** Time-frequency analysis, Signal representation, Analytic continuation

1. INTRODUCTION

Time-frequency representations are very helpful to characterize the richness of the information contained in non-stationary signals. It can help to monitor the appearance of short transient electrical signals and the beat of an heart for example. The subject has already been well covered with the Wigner-Ville representation, spectrogram, wavelet transform, etc... However, it only helps us to characterize the frequency content of a signal, with some limitations like inner interferences, cross-terms, artefacts, trade-off between time and frequency resolutions, etc...

Recently, complex time distribution concept has been introduced in [1] as a way to produce high concentrated distributions along the different phase derivatives of a signal. The main idea is to use the high order moments of the signals calculated for complex-time lags. It has also been shown that it was possible to deal with multi-component signals [2]. This technique has however some drawbacks as it involves the calculation of signal samples at complex coordinates through analytic continuation [3]. This estimation leads to poor representations as it can produce a divergence. Numerical example proves the efficiency of the modified analytical continuation technique extending also the capacity of the complex time distribution to deal with time-frequency structures with larger bandwidth.

In this paper, we show that it is possible to overcome some of the limitations introduced by the analytical continuation. This is achieved by a contraction in the frequency domain of the signal's spectrum.

The paper is organized as follows. In Section 2, the complex time distribution concept is presented. Section 3 describes the limitations of this technique due to the analytical continuation. A new technique is then introduced in Section 4 which permits to overcome some of the limitations. The different algorithms are then compared in Section 5. Section 6 gives a short conclusion and some perspectives for future works.

2. THE COMPLEX TIME DISTRIBUTION CONCEPT

The complex time distribution concept has been introduced a few years ago as a way to provide distributions that are concentrated along the *K-th* derivative of the phase for regular signals [1]. Let consider the signal defined as:

$$s(t) = Ae^{j\Phi(t)} \tag{1}$$

Assuming an analytical signal, and using the Taylor's series expansion of the phase, we can write:

$$s\left(t+\tau\right) = Ae^{j\sum_k \Phi^{(k)}(t)\frac{\tau^k}{k!}} \tag{2}$$

Phase integration in the complex plane using the theory of Cauchy's integral theorem [4] allows the focusing on a particular phase derivative:

$$\Phi^{(K)}(t) = \frac{K!}{2\pi\tau^K}\int_0^{2\pi}\Phi\left(t+\tau e^{j\theta}\right)e^{-jK\theta}d\theta \tag{3}$$

We now consider the discrete form of the equation for the *N-th* roots of unity which is used in numerical computation. Using the properties of the roots of unity, $\omega_{N,p} = e^{j2\pi p/N}$

and the variable change $\tau \to \sqrt[K]{\tau \frac{K!}{N}}$ the previous expression becomes:

$$\sum_{p=0}^{N-1} \Phi\left(t + \omega_{N,p} \sqrt[K]{\tau \frac{K!}{N}}\right) \omega_{N,p}^{N-K} = \Phi^{(K)}(t)\tau + Q(t,\tau) \tag{4}$$

where Q is the spread function which contains only the derivatives of order *Nk+K*, defined as:

$$Q(t,\tau) = N \sum_{p=1}^{\infty} \Phi^{(Np+K)}(t) \frac{\tau^{\frac{Np}{K}+1}}{(Np+K)!} \left(\frac{K!}{N}\right)^{\frac{Np}{K}+1} \tag{5}$$

We can now define the generalized complex-time moment (GCM):

$$GCM_N^K[s](t,\tau) = \prod_{p=0}^{N-1} s^{\omega_{N,p}^{N-K}} \left(t + \omega_{N,p} \sqrt[K]{\frac{K!}{N}\tau}\right)$$
$$= e^{j\Phi(K)(t)\tau + jQ(t,\tau)} \tag{6}$$

The Fourier transform of the GCM produces the generalized complex time distribution:

$$GCD_N^K[s](t,\omega) = TF_\tau[GCM_N^K[s](t,\tau)]$$
$$= \delta(\omega - \Phi^{(K)}(t)) \underset{\omega}{*} TF_\tau[Ae^{jQ(t,\tau)}] \tag{7}$$

As stated by this definition, the *K-th* order distribution of the signal, obtained for N complex-lags, highly concentrates the energy around the Kth-order derivate of the phase law. This concentration is optimal if the Φ s derivates of orders greater than *N+K* are 0. Observing Equation 5, it can be noticed that the first term appearing in the spreading function is the phase derivative of order *K+N*, the second one is of ordre *K+2N*,... Thus the parameter N highly affects the spreading function. We can conclude that a high value of N reduces interferences, since Q is reduced and distribution concentration will be less sensitive to higher order phase derivatives. This theory has been well developed in [1].
However, the computation of GCM implies the calculation of signal samples at complex coordinates. The next section will be dedicated to the study of the limitations introduced by this abstract notion.

3. LIMITATIONS OF CLASSICAL ANALYTICAL CONTINUATION

The analytical continuation of a signal $s(t)$ is performed as defined in [3].

$$s(t + jm) = \int_{-\infty}^{\infty} S(f) e^{-2\pi m f} e^{j2\pi f t} df \tag{8}$$

where $S(f)$ is the Fourier transform of the signal $s(t)$. It involves the multiplication of the spectrum by the exponential $e^{-2\pi m f}$ which has different effects on the spectrum. Those are shown in Figure 1 for different values of *m*. When the frequencies are positives, they are strongly attenuated due to the fast decreasing exponential. In the meantime, negative frequencies are strongly amplified, which can lead to a divergence (Figure 1 (c)). It is really important to note that *m* and the bandwidth of the signal have a strong impact on the analytic continuation. The bandwidth is directly attacked by the exponential. As for *m*, a strong value considerably distorts the spectrum. As a matter of consequence, it would be better to use roots of unity weighted by *1/N* that are the closest to the real axis in order to reduce the possible distortion during the computation.

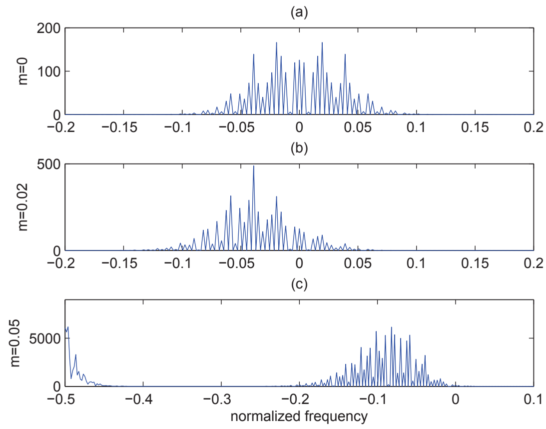

Fig. 1. This Figure represents a spectrum $S(f)$ multiplied by an exponential $e^{-2\pi m f}$ for different value of *m*: (a) $m = 0$, (b) $m = 0.02$, (c) $m = 0.05$.

In order to illustrate using an example, let consider two signals defined as:

$$s_1(t) = e^{j(6\cos(\pi t) + \frac{4}{3}\cos(3\pi t) + \frac{4}{3}\cos(5\pi t))} \tag{9}$$

$$s_2(t) = e^{j(18\cos(\pi t) + 4\cos(12\pi t) + 4\cos(40\pi t))} \tag{10}$$

Figure 2 shows the theoretical instanteneous frequency laws for s_1 and s_2, as well as the results of the complex time distribution using $N = 6$ and $K = 1$. We notice that if the DGTC gives good results for s_1 we can no longer estimate the first phase derivative for s_2. This is due to the computation of the analytical continuation and the large bandwidth of s_2.

4. NEW METHOD

In this section, a way for reducing the effect of analytical continuation is introduced. It consists in modifying the frequency support of the analyzed signal, s, in order to reduce the attenuation of the analytical continuation term.
Let consider a signal $B(t)$ defined as:

$$B(t) = s(\alpha t) \tag{11}$$

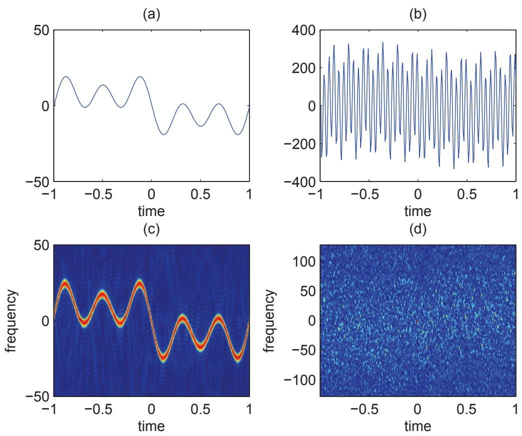

Fig. 2. This Figure represents the theoretical instantenous frequency laws for (a) $s_1(t)$ and (b) $s_2(t)$ and the classical GCD associated for (c) $s_1(t)$ and (d) $s_2(t)$.

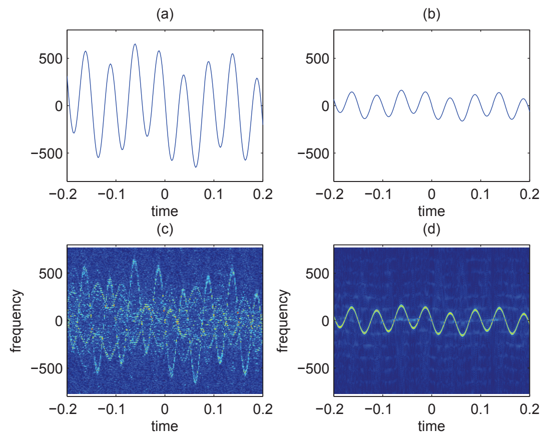

Fig. 3. (a) and (b) represent the theoretical instantaneous frequency laws of s_2 and its dilated version. (c) and (d) are the respective GCD.

with $\alpha > 1$ a dilatation coefficient and s a signal defined as Equation 1. The dilatation of the temporal signal leads in the frequency domain to a contraction of the bandwidth. This is actually the concept of the time-scale representation that we use at this point [5].
Considering the complex-time moment of B, we have:

$$GCM_N^K[B](t,\tau) = \prod_{p=0}^{N-1} s^{\omega_{N,p}^{N-K}}\left(\alpha t + \alpha\omega_{N,p}\sqrt[K]{\frac{K!}{N}}\tau\right) \quad (12)$$

We can clearly notice that the main impact directly concerns the analytical continuation. We then focus on its calculation. According to the Taylor serie expansion (Equation 2), we have:

$$s\left(\alpha t + j\alpha m\right) = \sum_{k=0}^{\infty} \frac{s^{(k)}\left(\alpha t\right)}{k!}\left(j\alpha m\right)^k \quad (13)$$

Knowing the following Fourier 's formula:

$$s^{(k)}(t) = \int_{-\infty}^{\infty} \left(j2\pi f\right)^k S(f) e^{j2\pi f t} df \quad (14)$$

we then deduce:

$$s^{(k)}\left(\alpha t\right) = \int_{-\infty}^{\infty} (j2\pi f)^k S(f) e^{j2\pi\alpha f t} df$$

Considering the variable change $f \leftarrow \alpha f$, we obtain:

$$s^{(k)}\left(\alpha t\right) = |a|^k \int_{-\infty}^{\infty} \left(j2\pi\frac{f}{\alpha}\right)^k S\left(\frac{f}{\alpha}\right) e^{j2\pi f t} \frac{df}{\alpha} \quad (15)$$

Taking into account Equations 13 and 15, we deduce:

$$s\left(\alpha t + j\alpha m\right) = \frac{1}{\alpha}\int_{-\infty}^{\infty} S\left(\frac{f}{\alpha}\right) \sum_{k=0}^{\infty} \frac{(-2\pi m f)^k}{k!} e^{j2\pi f t} df \quad (16)$$

Finally, we obtain the contracted analytical continuation:

$$s\left(\alpha t + j\alpha m\right) = \frac{1}{\alpha}\int_{-\infty}^{\infty} S\left(\frac{f}{\alpha}\right) e^{-2\pi m f} e^{j2\pi f t} df \quad (17)$$

We can see that this leads to a contraction of the spectrum, and as a matter of fact, it will be less affected by the attenuation term $e^{-2\pi m f}$. We defined s_α as the signal s whose frequency support is contracted by the dilatation coefficient $alpha$, ie $S_\alpha(f) = S\left(\frac{f}{\alpha}\right)$. We obtain:

$$s\left(\alpha t + j\alpha m\right) = \frac{1}{\alpha} s_\alpha(t + jm) \quad (18)$$

As we can notice, the two signals are related. The GCM then becomes:

$$GCM_N^K[B](t,\tau) = \prod_{p=0}^{N-1}\left(\frac{1}{\alpha}s_\alpha\left(t + \omega_{N,p}\sqrt[K]{\frac{K!}{N}}\tau\right)\right)^{\omega_{N,p}^{N-K}} \quad (19)$$

$$= \left(\frac{1}{\alpha}\right)^{\sum_{p=0}^{N-1}\omega_{N,p}^{N-K}} GCM_N^K[s_\alpha](t,\tau) \quad (20)$$

Two scenario then need to be studied:

- When $N = K(moduloN)$, ie when the number of roots of unity is equal to the order of phase derivative. Then we have:

$$GCM_N^K[B](t,\tau) = \frac{1}{\alpha^N} GCM_N^K[s_\alpha](t,\tau) \quad (21)$$

- Otherwise:

$$GCM_N^K[B](t,\tau) = GCM_N^K[s_\alpha](t,\tau) \quad (22)$$

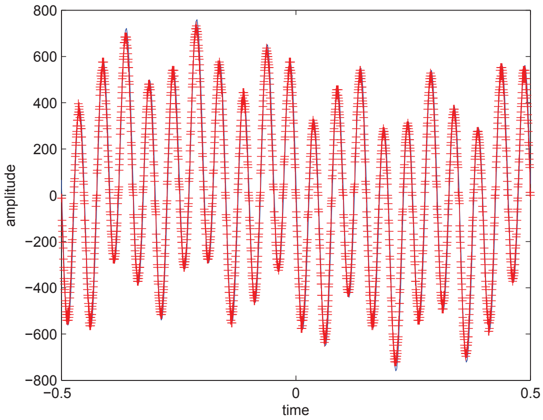

Fig. 4. The blue curve represents the theoretical instanteneous frequency law of s_2 meanwhile the red cruces represent its contracted version

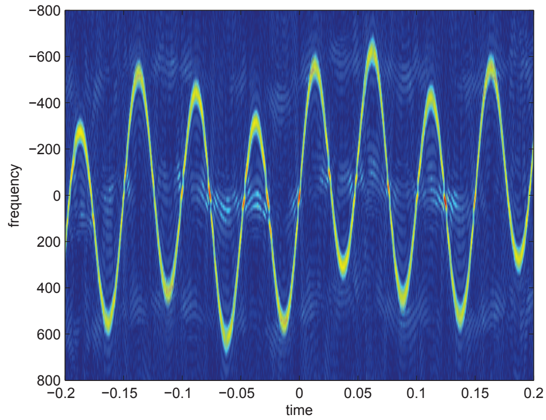

Fig. 5. This Figure represents the GCD of s_3 after dilatation. It is equal to the one of s_2

In both cases, we can conclude that the GCD gives the same results with a different intensity when the factor $\frac{1}{\alpha^N}$ appears. Equation 22 shows that it is possible to extract the *K-th* phase derivative order distribution of a signal by using its dilated version. We then need to expand the distribution to obtain the real distribution for signal s.
Next Section is then dedicated to the study of an example.

5. RESULTS

In this section, we apply the GCD algorithm to $s_2(t)$ and $s_3(t) = s_2(t/\alpha)$ with $\alpha = 4$ for $K = 1$ using 6 roots of unity. Figure 3 represents the different theoretical instantaneous frequency laws of the two signals, as the results of the GCD implementations. It is easy to note that the dilatation of the temporal support leads to a contraction of the bandwidth of s_3. The GCD fails to represent s_2, this is due to its large bandwidth and the analytic prolongation. Meanwhile, the dilated version of s_2 shows very good results as its bandwidth has been reduced, the continuation remains possible.
The frequency law obtained for s_3 is as stated by Equation 22 a contraction of the one of s_2, to obtain the last one,it is necessary to dilate the frequency law obtained with the dilatation coefficient α.
Figure 4 shows the comparaison between the theoretical intantaneous frequency law of s_2 and the dilated frequency law of s_3. We can notice that they match almost perfectly.Figure 5 shows the DGTC of s_3 after dilatation.
We have seen that it was possible to overcome one of the limitation of the analytic continuation for the GCD method using a dilation coefficient.

6. CONCLUSION

This paper proposes a new analytical continuation technique that will allow the generalized complex time distribution to deal with time-frequency structures having larger frequency variation. This technique is based on the compression of the signals bandwidth and, then, the application of the generalized complex distribution. This transformation allows accurately estimating of the IFL. In the future, works will propose an adaptive approach for the selection of the optimal scale parameter with respect of the bandwidth variation of the analyzed signal. Another future work direction will focus on the application of this analytical continuation technique for transient signals. The combination of the time-scale theory and the complex-time distribution will be also generalized providing a new way to analyze high speed time-frequency varying signals.

7. REFERENCES

[1] Cedric Cornu, Srdjan Stankovic, Cornel Ioana, André Quinquis, and Ljubisa Stankovic, "Generalized Representation of Phase Derivatives for Regular Signals," *IEEE Transactions on Signal Processing*, vol. 55, no. 10, pp. 4831–4838, Oct. 2007.

[2] Bertrand Gottin, Cornel Ioana, Srdjan Stankovic, Ljubisa Stankovic, and Jocelyn Chanussot, "On the concept of time-frequency distributions based on complex-lag moments," in *16th European Signal Processing Conference EUSIPCO-2008 EURASIP*, 2008.

[3] S. Stankovic and L. Stankovic, "Introducing time-frequency distribution with a complex-time argument," *Electronics Letters*, vol. 32, no. 14, pp. 1265, 1996.

[4] W. Rudin, *Real and Complex Analysis*, McGraw Hil, Boston, Mass, USA, 1987.

[5] S. G. Mallat, *A wavelet tour of signal processing*, Academic Press, San Diego, 2nd edition, 1999.