Conditions for Identifiability in Sparse Spatial Spectrum Sensing

Piya Pal, and P. P. Vaidyanathan
Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: piyapal@caltech.edu, ppvnath@systems.caltech.edu

***Abstract*—Spatial Spectrum estimation is a key technique used in a wide variety of problems arising in signal processing and communication, particularly those employing multiple antennas. In many scenarios such as direction finding using antenna arrays, it is crucial to estimate which directions in space contribute to active sources (indicated by a non zero power). It has been recently shown that if the sources from different directions are statistically uncorrelated, it is possible to identify as many as $O(M^2)$ active sources using only M physical antennas. A sparse representation for the spatial spectrum was further exploited to reconstruct the spectrum using convex optimization techniques. In this paper, we consider the situation when there is non zero cross correlation between the sources impinging from different directions. We investigate if, fundamentally, it still possible to identify more sources than the number of physical sensors and what role the cross correlation terms play. Recovery guarantees are developed to ensure uniqueness of the sparse representation for spectrum sensing. They are further extended to establish conditions under which a greedy heuristic, namely the Orthogonal Matching Pursuit algorithm will successfully recover the sparse spectrum. It is shown that in both cases, it is possible to recover support of larger size provided the correlation terms are small compared to the power of the impinging signals.**
***Keywords* — Sparse Spectrum Estimation, Correlated Sources, Khatri-Rao Product, Kruskal Rank Orthogonal Matching Pursuit.** [1]

I. Introduction

The problem of Spectrum Sensing arises frequently in signal processing and wireless communication. It lies at the core of applications such as cognitive radio [6], direction of arrival (DOA) estimation [7] etc. The spectrum can be either temporal or spatial depending on the application of interest. Instead of reconstructing the signal itself, the goal here is to recover the power spectrum which carries valuable information about the active frequency bands (as in cognitive radio) or the directions of arrival of incoming signals (as in array processing). Hence these methods are largely based on computing the autocorrelation function of the measurements and estimating the parameters of interest from this function. Using the fact that one only needs to sample the autocorrelation function at the Nyquist rate (rather than the signal itself), it has been shown that it is possible to sample the signal at a sub Nyquist rate [11], [12], [16], [18]. In the context of wideband spectrum sensing, this can lead to considerable reduction in the power consumed by A/D converters. For direction finding in array processing [10], this can significantly reduce the number of physical sensors required to estimate the directional sources.

The spectrum sensing problem naturally has a sparse representation in an appropriate basis. The sparse model is obtained by dividing the total spectrum (temporal or spatial) into a large number of candidate bins and arguing that only a few of them are active. The active bins can correspond to the occupied frequency bands/active users in the context of cognitive radio, or they can indicate the directions from which sources are impinging in an antenna array [9]. Given an array of M sensors and D narrow band far-field sources with directions of arrival given by $\{\theta_1, \theta_2, \cdots, \theta_D\}$, the received signals at the M antennas are given by the following sparse approximation:

$$\mathbf{y}[n] \approx \mathbf{A}\mathbf{x}[n] \tag{1}$$

where $\mathbf{A} \in \mathcal{C}^{M\times N_g}$ is a fat matrix ($N_g \gg M$) where each column of $\mathbf{A}$ represents a steering vector corresponding to one among N_g candidate directions between 0° and 180° of elevation angle. The vector $\mathbf{x}[n]$ is a sparse vector for each time sample n, containing exactly D non zero elements. Denote S as the set of indices of non zero elements of $\mathbf{x}$. Then recovering S is essentially the same as recovering the directions of arrival. The DOA estimation problem is essentially a spatial spectrum estimation problem where we want to know which directions correspond to non zero power. Since the support S contain the desired information, the spatial spectrum sensing problem is equivalent to that of sparse support recovery.

The above sparse formulation allows powerful convex optimization tools to be applied to recover the DOAs. There are several advantages of the sparse formulation over traditional subspace based methods like MUSIC [1], such as i) Since the number of active sources is a by product of the recovery algorithm, it need not be known or estimated apriori ii) Fewer number of time snapshots are required (in principle, even one time snapshot can be used [5], [9]). Recently, algorithms which are a hybrid of subspace and sparse formulation [13] have been proposed.

A common feature of most existing algorithms for sparse support recovery is that the number of sources that can be resolved obeys $D = O(M)$ [2], [9], [13], [19]. In [14] we showed that this can be improved by orders of magnitude if the source signals from two different directions are statistically uncorrelated. This means,

$$E(\mathbf{x}_i\mathbf{x}_j^*) = 0, i \neq j, i, j \in S \tag{2}$$

In particular, for uncorrelated sources, it has been shown that $D = O(M^2)$ sources can be identified using only M sensors. We proposed a sparse formulation for this problem and showed that two of the recently reported array geometries called the nested [10] and coprime arrays [11] can achieve this number. We also performed statistical analysis [15] of the sparse recovery algorithm in presence of finite snapshots. However, all our results so far have only considered the case when the sources are uncorrelated.

In this paper, we will establish identifiability conditions for the case where the sources can have non zero cross correlation. A dynamic array approach to correlated sources has been recently proposed [17]. However, in this paper, our goal is to establish a set of fundamental conditions on how many parameters can be resolved in the correlated scenario, irrespective of any specific algorithm. The argument is based on finding conditions for *uniqueness of the sparse representation* which should fundamentally hold no matter what algorithm is used to find the solution. For correlated sources, let the correlation between ith and jth sources be denoted by ρ_{ij} where

$$\rho_{ij} \triangleq E(\mathbf{x}_i\mathbf{x}_j^*) \neq 0, i \neq j. \tag{3}$$

[1]Work supported in parts by the ONR grant N00014-11-1-0676, and the California Institute of Technology.

We will investigate what conditions need to be satisfied for identifying D sources in this correlated scenario, especially if D can still be $O(M^2)$. The conditions are based on (i) the relation between the relative magnitudes of ρ_{ij} and the power of the ith source p_i, as well as (ii) the structure of manifold matrix $\mathbf{A}$. The conditions on $\mathbf{A}$ are further simplified in terms of the Kruskal rank [8] and the cumulative coherence function [19] of $\mathbf{A}$. We also study the performance of a greedy approach, namely, Orthogonal Matching Pursuit (OMP) which is popularly used as a fast heuristic for solving the support recovery problem [19]. We establish a similar set of conditions for OMP to successfully recover the sparse support in the noisy setting, building upon the results in [21]. It is shown that if s_0 be the guaranteed sparsity level that traditional formulation of OMP can recover, then OMP applied to the proposed framework can recover a sparsity level s which satisfies $s > s_0$. In fact, both the fundamental condition for uniqueness of sparse support, and the specific guarantee for OMP indicate that it is possible to recover support of much larger size, provided the cross correlation between signals is small enough compared to their power. This is very practical since it quantifies the connection between the degree of correlation and identifiability.

II. Support Recovery from Covariance Matrix: Problem Formulation

Consider the model $\mathbf{y} = \mathbf{A}\mathbf{x}_0$ where $\mathbf{A} \in \mathbb{C}^{M\times N}$ is a fat matrix with $N \gg M$ and $\mathbf{x}$ is a sparse vector with S_0 denoting the set of indices of its non zero elements. The vector $\mathbf{x}_0$ is sparse since $|S_0| < N$. The goal is to recover the set S_0 (also known as the support of $\mathbf{x}_0$) from this measurement model. Unlike most approaches to support recovery which seek to recover the vector $\mathbf{x}_0$ in a deterministic setting (i.e., consider $\mathbf{x}_0$ as an unknown deterministic quantity), we treat $\mathbf{x}$ as a random vector and seek to recover the support from the covariance matrix of

$$\mathbf{R}_{\mathbf{yy}} = \mathrm{E}(\mathbf{y}\mathbf{y}^H)$$

This problem arises in a variety of applications such a spectrum sensing in cognitive radios, Direction of Arrival Estimation, and a variety of spectral estimation techniques that explicitly utilize the covariance matrix.

It has been recently shown [14] that under the assumption that $\mathbf{x}_0$ is a zero mean random vector with a diagonal correlation matrix, (i.e., the entries of $\mathbf{x}_0$ are statistically uncorrelated), it is possible to design suitable matrix $\mathbf{A}$ such that one can recover sparse support of size upto $|S_0| = O(M^2)$. Previously existing results on sparse recovery which do not consider such a correlation structure for $\mathbf{x}_0$ could only recover sparsity levels of $|S_0| = O(M)$. In this paper, we consider the situation when the non zero elements of $\mathbf{x}_0$ can be actually correlated, violating the condition of a diagonal correlation matrix. We would like to investigate how large the support can be so that it is still possible to uniquely identify it from the covariance vector. We would also like to explicitly characterize the role played by the correlation in this regard.

To address these questions, let us set up a model based on the correlation matrix $\mathbf{R}_{\mathbf{yy}}$. Define $p_i = \mathrm{E}([\mathbf{x}]_i[\mathbf{x}]_i^*)$ and $\rho_{ij} = \mathrm{E}([\mathbf{x}]_i[\mathbf{x}]_j^*)$. Representing $\mathbf{R}_{\mathbf{yy}}$ in a vectorized form, we obtain

$$\mathbf{z} = \mathrm{vec}\mathbf{R}_{\mathbf{yy}} = (\mathbf{A}^* \otimes \mathbf{A})\mathrm{vec}\mathbf{R}_{\mathbf{xx}} \quad (4)$$

$$= \underbrace{(\mathbf{A}^* \odot \mathbf{A})}_{\mathbf{A}_{KR}}\mathbf{p} + \underbrace{\mathbf{A}_{\text{cross}}\mathbf{q}}_{\mathbf{e}} \quad (5)$$

Here, the symbol $\odot$ denotes the Khatri Rao product whereas $\otimes$ denotes the Kronecker product. The vector $\mathbf{q} \in \mathbb{C}^{(N^2-N)\times 1}$ is a sparse vector consisting of elements $\rho_{ij}, 1 \le i \ne j \le N$. If we represent the ith column of $\mathbf{A}$ as $\mathbf{a}_i$, then the columns of $\mathbf{A}_{\text{cross}} \in \mathbb{C}^{M^2\times(N^2-N)}$ are given by $\mathbf{a}_i^* \otimes \mathbf{a}_j, i \ne j, i, j \in \{1, 2, \cdots, N\}$. Note that $\mathbf{q}$ is also a sparse vector with sparsity $|S_0|^2 - |S_0|$. The vector $\mathbf{p} \in \mathbb{C}^{N\times 1}$ is a sparse non-negative vector with the *same support* S_0 and the elements in the non zero locations given by $p_i, i \in S_0$. Hence, if we can recover the vector $\mathbf{p}$ from (5), we will be able to recover S_0. It is to be noted that the non zero cross correlation between the elements of $\mathbf{x}$ manifest themselves in the form of the additive noise like term $\mathbf{e}$. Hence the model (5) is reminiscent of a sparse observation model (with $\mathbf{A}^* \odot \mathbf{A}$ playing the role of the measurement matrix) in a noisy setting. We will like to investigate the conditions under which S_0 can be uniquely recovered, first for a generic support recovery problem with noise and then specifically for $\mathbf{z}$ satisfying (5).

III. Uniqueness of Support Recovery from Correlation Matrix

Consider a noisy observation with bounded noise

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{n} \quad (6)$$

where $\mathrm{Supp}(\mathbf{x}_0) = S_0$ and $\| \mathbf{n} \|_2 < \epsilon$. We would be interested in knowing the condition under which the following (non convex) support recovery problem will yield the true support:

$$\min_{\mathbf{x}} \| \mathbf{x} \|_0 \qquad (P0)^\epsilon \quad (7)$$

$$\text{subject to} \quad \| \mathbf{y} - \mathbf{A}\mathbf{x} \|_2 < \epsilon \quad (8)$$

Since ϵ gives a measure of the noise power, we define the Signal-to-Noise-Ratio (SNR) as

$$\mathrm{SNR} \triangleq \frac{|\mathbf{x}_{0min}|}{\epsilon} \quad (9)$$

where $\mathbf{a}_{min}$ represent the minimum non zero element of a vector $\mathbf{a}$. We also define parameter $\sigma_s^{min}(\mathbf{A})$ and $\sigma_s^{max}(\mathbf{A})$ of $\mathbf{A}$ which will be useful to provide guarantees for recovery of S_0. These quantities are closely related to the RIP constant as proposed by Candes [4]. Given a set of integers S, we denote $\mathbf{A}_S$ as the sub matrix of A consisting of the columns of $\mathbf{A}$ indexed by the elements of S.

Definition 1: Given a matrix $\mathbf{A}$ and integer s, the parameters $\sigma_s^{min}(\mathbf{A})$ and $\sigma_s^{max}(\mathbf{A})$ are defined as

$$\sigma_s^{min}(\mathbf{A}) = \min_{S\subset[N]:|S|=s} \sqrt{\lambda_s(\mathbf{A}_S^H\mathbf{A}_S)} \quad (10)$$

$$\sigma_s^{max}(\mathbf{A}) = \max_{S\subset[N]:|S|=s} \sqrt{\lambda_1(\mathbf{A}_S^H\mathbf{A}_S)} \quad (11)$$

where for a matrix $\mathbf{B}$, $\lambda_i(\mathbf{B})$ denotes the ith largest eigenvalue. The following theorem states the sufficient condition under which $(P0)^\epsilon$ yield S_0. A result of similar flavor is available regarding the stability of the solution to $(P_0)^\epsilon$ (see, for example, [3]) but to the extent of our knowledge, such results do not establish uniqueness of the support.

Theorem 1: Consider the model given by (6). The solution $\mathbf{x}^*$ to $(P0)^\epsilon$ satisfies $\mathrm{Supp}(\mathbf{x}^*) = S_0$ if

$$\mathrm{Krank}(\mathbf{A}) > 2 \mid S_0 \mid \quad (12)$$

and

$$\mid \mathbf{x}_{0min} \mid > \frac{2\epsilon}{\sigma_{2|S_0|}^{min}(\mathbf{A})}. \quad (13)$$

Proof: The solution $\mathbf{x}^*$ to $(P0)^\epsilon$ will recover the true support S_0 if and only if there is only one vector $\mathbf{x}_0$ with support S_0 that satisfies $\| \mathbf{y} - \mathbf{A}\mathbf{x}_0 \|_2 < \epsilon$. We will show that under the conditions $\text{Krank}(\mathbf{A}) > 2 \mid S_0 \mid$ and $\mid \mathbf{x}_{0min} \mid > \frac{2\epsilon}{\sigma^{min}_{2|S_0|}(\mathbf{A})}$, this is indeed true. We will prove this by contradiction. Let us assume that the solution $\mathbf{x}^*$ to $(P0)^\epsilon$ does not have the support S_0. Let $S_x = \text{Supp}(\mathbf{x}^*)$. Since S_x is the minimizing support, $\mid S_x \mid \leq \mid S_0 \mid$. Since both $\mathbf{x}_0$ and $\mathbf{x}^*$ are feasible, we can write

$$\|\mathbf{A}_{S_x}[\mathbf{x}^*]_{S_x} - \mathbf{A}_{S_0}[\mathbf{x_0}]_{S_0}\|_2 < 2\epsilon \quad (14)$$

$$\Rightarrow \|\mathbf{A}_{S_0 \cup S_x}\mathbf{x}_T\|_2 < 2\epsilon \quad (15)$$

where

$\mathbf{x}_T = \Big[[\mathbf{x}^*]_{S_x \setminus S_0} \quad | \quad [\mathbf{x}^*]_{S_x \cap S_0} - [\mathbf{x}_0]_{S_x \cap S_0} \quad | \quad -[\mathbf{x}_0]_{S_0 \setminus S_x}\Big]^T$. Since $\mid S_x \mid < \mid S_0 \mid$, the number of columns in $\mathbf{A}_{S_0 \cup S_x}$ is at most $2 \mid S_0 \mid$ and we can say (15) implies

$$\sigma^{min}_{2|S_0|}(\mathbf{A})\| \mathbf{x}_T \|_2 < 2\epsilon \quad (16)$$

Since $\text{Krank}(\mathbf{A}) > 2 \mid S_0 \mid$, $\sigma^{min}_{2|S_0|}(\mathbf{A}) > 0$ and we can say (16) implies

$$\| \mathbf{x}_T \|_2 < \frac{2\epsilon}{\sigma^{min}_{2|S_0|}(\mathbf{A})} \quad (17)$$

Since $S_0 \backslash S_x \neq \emptyset$, it follows that $\| \mathbf{x}_T \|_2 > \| [\mathbf{x}_0]_{S_0 \setminus S_x} \|_2 > \mid \mathbf{x}_{0min}. \mid$ Therefore, (17) implies

$$\mid \mathbf{x}_{0min} \mid < \frac{2\epsilon}{\sigma^{min}_{2|S_0|}(\mathbf{A})}$$

which is a contradiction to the given condition, and this concludes our proof. ■

Theorem 1 states that even in the noisy scenario, the following are true:

1) The solution of $(P0)^\epsilon$ can yield the true support S_0 of size upto $|S_0| < \frac{\text{Krank}(\mathbf{A})}{2}$. Since $\text{Krank}(\mathbf{A}) \leq M$, this indicates that the maximum size of the recoverable support is $|S_0| = O(M)$.
2) The SNR should exceed the threshold $t = \frac{2}{\sigma^{min}_{2|S_0|}(\mathbf{A})}$.

Comparing (6) and (5), we find that for recovering S_0 from (5), the matrix of interest is $\mathbf{A}_{KR}$. The following lemma establishes the relation between the singular values of $\mathbf{A}$ and $\mathbf{A}_{KR}$

Lemma 1: Given a subset $S \subset \{1, 2, \cdots, N\}$, the minimum singular values of $\mathbf{A}_S$ and $\mathbf{A}_{KRS}$ satisfy

$$\sigma_{min}(\mathbf{A}_{KRS}) \geq \sigma^2_{min}(\mathbf{A}_S) \quad (18)$$

Proof: From the definition of column-wise Kronecker products, it can be verified that

$$\begin{aligned} \mathbf{A}_{KR}{}^H_S \mathbf{A}_{KRS} &= \left(\mathbf{A}^*_S \odot \mathbf{A}_S\right)^H \left(\mathbf{A}^*_S \odot \mathbf{A}_S\right) \\ &= \left(\mathbf{A}^T_S \mathbf{A}^*_S\right) \circ \left(\mathbf{A}^H_S \mathbf{A}_S\right) \end{aligned}$$

Here $\circ$ denotes the Hadamard Product. From properties of Hadamard Product, for two positive semidefinite matrices $\mathbf{M}, \mathbf{N}$, the minimum eigenvalues λ_{min} satisfy [22, p. 312]

$$\lambda_{min}(\mathbf{M} \circ \mathbf{N}) \geq \lambda_{min}(\mathbf{M})\lambda_{min}(\mathbf{N}) \quad (19)$$

from which the desired condition follows. ■

The problem of recovering the sparsest $\mathbf{p}$ satisfying (5) can be cast as

$$\min_{\mathbf{r}} \| \mathbf{r} \|_0 \qquad (P0)^\epsilon_{Co-SMV} \quad (20)$$

$$\text{subject to} \quad \| \mathbf{z} - \mathbf{A}_{KR}\mathbf{r} \|_2 \leq \epsilon \quad (21)$$

where ϵ is an upper bound on $\mathbf{e}$.

Denote $p^{min} = \min_{i \in S_0} p_i$. Then, applying Theorem 1, we get the sufficient condition for the solution of $(P0)^\epsilon_{Co-SMV}$ to uniquely recover the support S_0:

Theorem 2: Consider the model (5) derived from the correlation matrix of the observed vector $\mathbf{y}$. The solution $\mathbf{r}^*$ to $(P0)^\epsilon_{Co-SMV}$ satisfies $\text{Supp}(\mathbf{r}^*) = S_0$ if

$$\text{Krank}(\mathbf{A}_{KR}) > 2 \mid S_0 \mid \quad (22)$$

and

$$p^{min} > \frac{2\epsilon}{\sigma^{min}_{2|S_0|}(\mathbf{A}_{KR})} \quad (23)$$

Remarks:

We can make the following important observations from Theorem 2

1) **Size of the identifiable support** : It is fundamentally possible to uniquely identify sparse support of size as large as $\text{Krank}(\mathbf{A}_{KR})$. Since $2\text{Krank}(\mathbf{A}) \leq \text{Krank}(\mathbf{A}_{KR}) \leq M^2$, it is possible to recover sparsity levels as large as $O(M^2)$. In [14], we give examples of constructing $\mathbf{A}$ such that $\mathbf{A}_{KR}$ has Kruskal rank as large as $O(M^2)$. It can further be proved that if $\mathbf{A}$ is a random matrix with i.i.d entries, then $\mathbf{A}_{KR}$ has Kruskal rank $O(M^2)$
2) **Role of non zero cross correlation**: The above claim regarding size of the recoverable support is true if the condition condition $p^{min} > \frac{2\epsilon}{\sigma_{2|S_0|}(\mathbf{A}_{KR})}$ holds. Denote $\rho^{max} \triangleq \max_{i \neq j} \mid \rho_{ij} \mid$. Then,

$$\begin{aligned} \|\mathbf{e}\|_2 &= \|\mathbf{A}\text{cross}\mathbf{q}\|_2 \\ &\leq \|\mathbf{A}\text{cross}\|_2 \|\mathbf{q}\|_2 \\ &\leq \|\mathbf{A}^*_{S_0} \otimes \mathbf{A}_{S_0}\|_2 \|\mathbf{q}\|_2 \\ &= \left(\sigma^{max}_{|S_0|}(\mathbf{A})\right)^2 \|\mathbf{q}\|_2 \\ &\leq \left(\sigma^{max}_{|S_0|}(\mathbf{A})\right)^2 \sqrt{\mid S_0 \mid^2 - \mid S_0 \mid}\rho^{max} \end{aligned} \quad (24)$$

Therefore the condition (23) is satisfied if

$$\frac{p^{min}}{\rho^{max}} > 2\sqrt{\mid S_0 \mid^2 - \mid S_0 \mid} \frac{\left(\sigma^{max}_{|S_0|}(\mathbf{A})\right)^2}{\sigma^{min}_{2|S_0|}(\mathbf{A}_{KR})} \quad (25)$$

This relation explicitly shows that, if $(P0)^\epsilon_{Co-SMV}$ can be solved, it will be possible to uniquely identify support as large as $O(M^2)$ (for suitable $\mathbf{A}$ such as those corresponding to nested and cop rime arrays) *provided the correlation terms* $\rho_{i,j}, \forall i \neq j \in S_0$ *are small enough compared to the power* p_i *of the non zero entries of* $\mathbf{x}_0$. This leads to the following corollary

Corollary 1: It is possible to recover support S_0 of size $\mid S_0 \mid < \frac{\text{Krank}(\mathbf{A}_{KR})}{2}$ by solving $(P0)^\epsilon_{Co-SMV}$ provided the cross correlation terms $\rho_{i,j}$ are small compared to the power p_i of the non zero elements of $\mathbf{x}_0$, satisfying

$$\frac{p^{min}}{\rho^{max}} > 2\sqrt{\mid S_0 \mid^2 - \mid S_0 \mid} \frac{\left(\sigma^{max}_{|S_0|}(\mathbf{A})\right)^2}{\sigma^{min}_{2|S_0|}(\mathbf{A}_{KR})} \quad (26)$$

Since $(P0)^{\epsilon}_{Co-SMV}$ is NP-hard, we next analyze the performance of a greedy heuristic which seeks to solve $(P0)^{\epsilon}_{Co-SMV}$

IV. Greedy Support Recovery Under Noise

In this section, we consider the performance of a specific greedy algorithm called the Orthogonal Matching Pursuit (OMP) which seeks to find the support S_0 from $\mathbf{z}$ in (5). The performance of OMP in the noisy setting was recently analyzed in [21]. To obtain the condition under which perfect support recovery is possible, we will need the following definition [19]

Definition 2: [19] Given an integer s, the cumulative coherence function of a matrix $\mathbf{A}$ is defined as

$$\mu_1^{\mathbf{A}}(s) = \max_{S:|S|=s} \max_{j \notin S} \|\mathbf{A}_S^H \mathbf{a}_j\|_1 \tag{27}$$

Consider the noisy observation model (6). Define $M_{S_0}(\mathbf{A}) = \max_{j\notin S_0} \|\left(\mathbf{A}_{S_0}{}^H \mathbf{A}_{S_0}\right)^{-1} \mathbf{A}_{S_0}{}^H \mathbf{a}_j\|_1$ where S_0 denotes the support of $\mathbf{x}_0$. Also, given a matrix $\mathbf{B}$, let $\lambda^{S_0}_{\min}(\mathbf{B})$ denote the minimum eigenvalue of $\mathbf{B}^H_{S_0}\mathbf{B}_{S_0}$. Then the condition under which OMP yields the true support is given by [21, Theorem 1 and Prop. 1]

Theorem 3: Consider the model (6) where $\|\mathbf{n}\|_2 \leq \epsilon$ and $\mu_1^{\mathbf{A}}(\mid S_0 \mid) + \mu_1^{\mathbf{A}}(\mid S_0 \mid -1) < 1$. Then the OMP algorithm with an appropriate stopping rule recovers exactly the true subset of the correct support if all the non zero coefficients of $\mathbf{x}_0$ satisfy $[\mathbf{x}_0]_i \geq \frac{2\epsilon}{(1-M_{S_0}(\mathbf{A}))\lambda^{S_0}_{min}(\mathbf{A})}$

Hence, two conditions are important to guarantee the success of OMP in the noisy setting:

1) The support S_0 should be such that $\mu_1^{\mathbf{A}}(\mid S_0 \mid) + \mu_1^{\mathbf{A}}(\mid S_0 \mid -1) < 1$. Since $\mu_1^{\mathbf{A}}(s)$ is an increasing function of s, this is ensured if

$$\mu_1^{\mathbf{A}}(\mid S_0 \mid) < \frac{1}{2} \tag{28}$$

2) The signal to noise ratio SNR $= \frac{[\mathbf{x}_0]_i}{\epsilon}$ should be high enough, i.e.,

$$\text{SNR} \geq \frac{2}{(1-M_{S_0}(\mathbf{A}))\lambda^{S_0}_{min}(\mathbf{A})} \tag{29}$$

We would like to know how the above guarantees extend to OMP applied on the vectorized correlation $\mathbf{z}$ given by (5). It is to be noted that here we need to consider the matrix $\mathbf{A}_{KR}$ instead of $\mathbf{A}$. The following lemma relates $\mu_1^{\mathbf{A}_{KR}}(s)$ and $\mu_1^{\mathbf{A}}(s)$

Lemma 2: Given an integer s, The cumulative coherence function of $\mathbf{A}$ and $\mathbf{A}_{KR}$ are related as

$$\mu_1^{\mathbf{A}_{KR}}(s) < \left(\mu_1^{\mathbf{A}}(s)\right)^2 \tag{30}$$

Proof: See [23]. ■

In light of the above results, we can now state the conditions under which OMP applied on $\mathbf{z}$ in (5) can recover the support S_0.

Theorem 4: Consider the representation (5) of the covariance matrix with S_0 denoting the common sparse support. The OMP algorithm with an appropriate stopping rule recovers exactly the true subset of the correct support if

$$\mu_1^{\mathbf{A}}(|S_0|) < \frac{1}{\sqrt{2}} \tag{31}$$

and

$$\frac{p^{min}}{\rho^{max}} \geq \frac{2\sqrt{\mid S_0 \mid^2 - \mid S_0 \mid}\left(\sigma^{max}_{|S_0|}(\mathbf{A})\right)^2\left[1-\left(\mu_1^{\mathbf{A}}(|S_0|-1)\right)^2\right]}{\left[1-\left(\mu_1^{\mathbf{A}}(|S_0|-1)\right)^2-\left(\mu_1^{\mathbf{A}}(|S_0|)\right)^2\right]\lambda^{S_0}_{min}(\mathbf{A}_{KR})} \tag{32}$$

Proof: Follows from the Theorem 3, the inequality (24) and the fact that $M_{S_0}(\mathbf{A}) \leq \frac{\mu_1^{\mathbf{A}}(|S_0|)}{1-\mu_1^{\mathbf{A}}(|S_0|-1)}$ ■

The above theorem also shows that it is possible to recover any support S_0 provided its size is such that (31) holds and the correlation terms are small compared to the power of non zero elements so that (32) holds. Since $\mu_1^{\mathbf{A}}(s)$ is an increasing function of s, it is clear that (31) indicates that a larger support can be recovered when compared with (28).

V. Numerical Example

In this section we conduct a short numerical experiment to study the performance of OMP applied on $\mathbf{z}$ in (5) with varying levels of cross correlation ρ_{ij}. The main idea is to see how the matrices $\mathbf{A}$ and $\mathbf{A}_{KR}$ compare in terms of recovery performance of OMP. Let $\mathbf{A}$ be a random Gaussian matrix with $M = 20, N = 150$. We consider a sparsity level of $\mid S_0 \mid= 15$. We first apply OMP directly to the measurement $\mathbf{y} = \mathbf{Ax}$ where $\mathbf{x}$ has a sparsity of $\mid S_0 \mid= 15$. It is to be noted that for the given values of M, N and the sparsity $\mid S_0 \mid$, OMP performs poorly in terms of recovering the original support (see, for example, Fig. 1 in [20]). Fig. 2 further demonstrates this by plotting the true support and that recovered by the OMP algorithm. We also define the percentage of successful recovery as the percentage of correct atoms identified by OMP averaged over 500 random realizations of $\mathbf{A}$. The percentage of successful recovery turns out to be 26.67% in this case.

We now demonstrate that OMP applied to the covariance matrix represented by (5) performs significantly better than the previous approach. Fig. 1 shows a sample run for three different levels of cross correlation ρ^{max} given by 0, 0.05 and 0.5 respectively. The plot shows the correct indices as well as the indices recovered by the OMP algorithm in the first 15 iterations of the algorithm. We also tabulate the percentage of successful recovery in Table I by averaging over 500 Monte Carlo runs for the three different levels of ρ^{max}. It can be seen that OMP applied on proposed framework is significantly better than that applied directly on the measurement model for the given level of sparsity. Also, as expected, the performance of OMP improves as the cross correlation terms become smaller.

TABLE I
Percentage of indices recovered by OMP applied on $\mathbf{R_{yy}}$

	$\rho^{max}=0$	$\rho^{max}=0.05$	$\rho^{max}=0.5$
% Recovered	94.1	89.3	66.67

VI. Conclusion

We studied the problem of sparse support recovery from the covariance matrix of measurements and what advantages it can offer over recovering the support from the measurement itself. It was previously shown that when the non zero elements of the unknown sparse vector are statistically uncorrelated, recovery algorithms based on the covariance matrix can lead to an order of magnitude increase in the size of the recoverable support. In this paper, we investigated if this continues to hold when there is correlation among the non zero elements. We showed that the relative magnitude of the correlation terms and the power of the non zero elements play a crucial role in this regard. We derived a fundamental guarantee on the

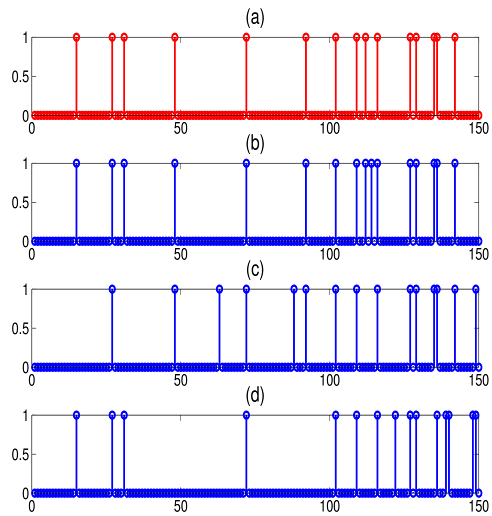

Fig. 1. Support Recovered by OMP applied to the covariance matrix $\mathbf{R_{yy}}$ of the measurement $\mathbf{y}$ plotted for different values of ρ^{max}: (a) True Support, (b)$\rho^{max} = 0$, (c) $\rho^{max} = 0.05$ (d) $\rho^{max} = 0.5$. Here $M = 20, N = 150, \mid S_0 \mid = 15$ and $p^{min} = 1$.

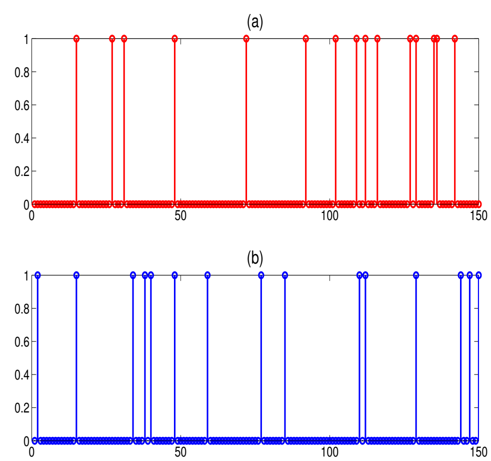

Fig. 2. (a) True Support (b) Support Recovered by OMP applied directly to the measurement $\mathbf{y}$. Here $M = 20, N = 150, \mid S_0 \mid = 15$.

maximum size of the sparse support which ensures that the sparse representation of the covariance matrix is unique even when there is correlation among the non zero elements. This guarantee is independent of any specific algorithm used. We also developed guarantees for the OMP algorithm for sparse support recovery from the covariance matrix. The uniqueness condition as well as the recovery guarantees of OMP indicate that it is possible to recover larger sparse support from the covariance matrix even in the presence of correlation, provided the correlation terms are small enough compared to the power of the non zero elements.

References

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation", *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276-280, Mar 1986.

[2] D. L. Donoho, and Michael Elad."Optimally sparse representation in general (nonorthogonal) dictionaries via l_1 minimization", *Proceedings of the National Academy of Sciences*, vol. 100, no. 5 pp. 2197-2202, 2003.

[3] D. Donoho, M. Elad, and V.N. Temlyakov, "Stable recovery of sparse overcomplete representations in the presence of noise", *IEEE Transactions on Information Theory*, vol. 52, no. 1, pp. 6-18, 2006.

[4] E. Candes,"The restricted isometry property and its implications for compressed sensing", *Comptes Rendus Mathematique*, 2008.

[5] I. F. Gorodnitsky, B. D. Rao, "Sparse signal reconstruction from limited data using FOCUSS: a re-weighted minimum norm algorithm", *IEEE Transactions on Signal Processing*, vol. 45, no. 3, pp. 600-616, Mar 1997.

[6] E. Axell, G. Leus, E. G. Larsson, H. V. Poor, "Spectrum Sensing for Cognitive Radio : State-of-the-Art and Recent Advances", *IEEE Signal Processing Magazine*, vol. 29, no. 3, pp. 101-116, May 2012.

[7] H. Krim and M. Viberg, "Two decades of array signal processing research: the parametric approach," *IEEE Signal Process. Mag.,* vol. 13, pp. 67-94, Jul. 1996.

[8] J.B. Kruskal, "Three-way arrays: Rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics", *Linear Algebra Applications*, vol. 18, 1977.

[9] D. Malioutov, M. Cetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays",*IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3010-3022, Aug. 2005.

[10] Piya Pal and P. P. Vaidyanathan, "Nested arrays: a novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. SP*, pp. 4167-4181, August 2010.

[11] P. P. Vaidyanathan and Piya Pal, "Sparse Sensing With Co-Prime Samplers and Arrays'", *IEEE Trans. on Signal Processing* , vol. 59, pp. 573-586, Feb. 2011.

[12] Y. Bresler, "Spectrum-blind sampling and compressive sensing for continuous-index signals", *in Proc. Inf. Theory Appl. Workshop (ITA 2008)*, pp. 547554, Jan. 2008.

[13] Kiryung Lee, Y. Bresler, and M. Junge, "Subspace Methods for Joint Sparse Recovery", *IEEE Transactions on Information Theory*, vol. 58, no. 6, pp. 3613-3641, June 2012.

[14] Piya Pal and P. P. Vaidyanathan, "Correlation-Aware Techniques for Sparse Support Recovery", *IEEE Statistical Signal Processing Workshop, 2012*, Ann Arbor, MI, Aug. 5-8, 2012.

[15] Piya Pal and P. P. Vaidyanathan, "On Application of LASSO for Sparse Support Recovery With Imperfect Correlation Awareness", *Proc. of 46th Asilomar Conference on Signals, Systems, and Computers,* Pacific Grove, CA, Nov. 4-7, 2012.

[16] Zhi Tian, and Georgios B. Giannakis, "Compressed sensing for wideband cognitive radios" ,*IEEE International Conference on Acoustics, Speech and Signal Processing, 2007 (ICASSP)* Vol. 4., 2007.

[17] D. D. Ariananda and G. Leus, "Direction of arrival estimation of correlated signals using a dynamic linear array", *Proc. of the Asilomar Conference on Signals, systems, and Computers* (Asilomar 2012), Pacific Grove, California, USA, November 2012.

[18] D. D. Ariananda, G. Leus, "Compressive Wideband Power Spectrum Estimation", *IEEE Transactions on Signal Processing*, vol. 60, no. 9, pp. 4775-4789, Sept. 2012.

[19] J. A. Tropp, "Greed is good: Algorithmic results for sparse approximation", *IEEE Transactions on Information Theory*, vol. 50, no. 10, pp. 2231-2242, 2004.

[20] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit ", *IEEE Transactions on Information Theory*, vol. 53, no. 12, pp. 4655-4666, 2007.

[21] T. T. Cai, Lie Wang, "Orthogonal Matching Pursuit for Sparse Signal Recovery With Noise," *IEEE Transactions on Information Theory*, vol. 57, no. 7, pp 4680-4688, July 2011.

[22] R. A. Horn, and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, New York: 1991.

[23] P. Pal and P. P. Vaidyanathan, "A Correlation Aware Framework for Sparse Support Recovery", *in preparation*.