DISTRIBUTED CONTROL OF WIRELESS AD-HOC NETWORKS CONNECTIVITY INCORPORATING REALISTIC CHANNEL MODELS

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ABSTRACT

In this paper, we propose a distributed method to control the connectivity of wireless ad hoc networks taking into account the impairments resulting from the propagation through realistic channel models incorporating fading, noise and packet collision. We illustrate a mechanism to estimate the algebraic connectivity of the expected graph and we show that the method is robust against random packet drops. Then, we show how the algebraic connectivity of the expected graph Laplacian depends on the transmit power of each node. The interesting result is that there exists an optimal power that maximizes the algebraic connectivity, as a tradeoff between the degree of each node and the number of collisions. Finally, we propose a distributed algorithm to evaluate the optimal transmission power that maximizes the network connectivity in the presence of realistic MAC protocols.

Index Terms— Ad hoc networks, expected connectivity, collisions, stochastic approximation.

1. INTRODUCTION

The diffusion of information through a network presumes connectivity of the network. In many practical examples, this connectivity can only be assumed to hold in probability because the links among the nodes may be on or off depending on channel conditions. In most applications, channel variability may depend on several factors, such as mobility of the nodes, as in vehicular networks, channel fading due to propagation over multipath channels, packet collisions due to random medium access control (MAC) strategies working on a collision avoidance regime, etc. Furthermore, many distributed processing algorithms running over a graph, such as consensus algorithms or diffusion algorithms for example, have a convergence time strongly dependent on the graph connectivity [1]-[5]. It is then of interest to look at distributed mechanisms to control network connectivity in the presence of realistic channel models. Spectral graph theory [6] has been demonstrated to be a very powerful tool for topology inference. The eigenvalues and/or eigenvectors of the Laplacian matrix of the graph have been exploited, e.g., to estimate the connectivity of the network [7], to find densely connected clusters of nodes [8]-[9], and to search for potential links that would greatly improve the connectivity if they would be established [10]. In all these works, it was argued and demonstrated that the most useful eigenvector for graph partitioning is the one corresponding to the second-smallest eigenvalue of the Laplacian matrix. This eigenvalue is referred to as the *algebraic connectivity* and its eigenvector is often referred to as the *Fiedler vector* [7]. Most of the previous works assumed ideal communications among the network nodes. However, in a realistic scenario, the wireless channel is affected by random fading and additive noise, which induce errors in the received packets. Furthermore, realistic random medium access control (MAC) protocols may determine packet collisions during the exchange of data among the nodes. In such a case, the receiving node could request the retransmission of the erroneous packets, but this would imply random delays in the communication among the nodes and it would be complicated to implement over a totally decentralized system. It is then of interest to analyze networks where the erroneous packets are simply dropped, without requiring a retransmission. In this paper, we investigate on the effect of collisions, induced by a realistic random medium access control protocol, on the connectivity of wireless ad-hoc networks. The contribution of this paper is twofold. First, we show how the presence of realistic medium access control protocols intrinsically limits the connectivity of a wireless ad-hoc network due to the inevitable presence of collisions. It turns out that, if nodes choose a too large transmission power, the network connectivity may be heavily degraded due to an increase in the collision probability. Second, it is provided a distributed stochastic approximation method aimed at finding the maximum of the connectivity in the presence of collisions.

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2. REVIEW OF ALGEBRAIC GRAPH THEORY

We consider a network composed of N nodes interacting according to a communication topology. The interaction among the nodes is modeled as an undirected graph $G = (V, E)$, where $V = 1, 2, ..., N$ denotes the set of nodes and $E \subseteq V \times V$ is the edge set. The structure of the graph is described by a symmetric $N \times N$ adjacency matrix $\boldsymbol{A} := \{a_{ij}\}$, whose entries a_{ij} are either positive or zero, depending on wether there is a link between nodes i and j or not, i.e., if the distance between nodes i and j is less than a coverage radius, which is dictated by nodes' transmit power and the channel between them. The set of neighbors of a node i is $\mathcal{N}_i$, defined as $\mathcal{N}_i = \{j \in V : a_{ij} > 0\}$. Node i communicates with node j if j is a neighbor of i (or $a_{ij} > 0$). Denoting by $d_{ii} = \sum_{j=1}^{N} a_{ij}$ the degree of node i, the degree matrix $\boldsymbol{D}$ is a diagonal matrix with entries d_{ii} that are the row sums of the adjacency matrix $\boldsymbol{A}$. The graph Laplacian $\boldsymbol{L}$ is defined as

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}. \tag{1}$$

We denote by $\lambda_i(\boldsymbol{L})$, $i = 1, \ldots, N$, the eigenvalues of $\boldsymbol{L}$, ordered in increasing sense. The $N \times N$ matrix $\boldsymbol{L}$ always has, by construction, a null eigenvalue $\lambda_1(\boldsymbol{L}) = 0$, with associated eigenvector $\mathbf{1}_N$ composed of all ones. For a connected graph, the nullspace of $\boldsymbol{L}$ has dimension 1 and it is spanned by the vector $\mathbf{1}$. The quantity $\lambda_2(\boldsymbol{L})$ is known as the *algebraic connectivity* of the graph. This eigenvalue is greater than 0 if and only if G is a connected graph.

Random link failures : In a realistic communication scenario, the packets exchanged among network nodes may be received with errors, because of collisions, channel fading or noise. The retransmission of erroneous packets can be incorporated into the system, but packet retransmission introduces a nontrivial additional complexity in decentralized implementations and, more importantly, it also introduces an unknown delay and delay jitter. It is then of interest to examine simple protocols where erroneous packets are simply dropped. We take into account random packet dropping by modeling the coefficient a_{ij} describing the network topology as statistically independent random variables. Then, the Laplacian of the graph varies with time as a sequence of i.i.d. matrices $\{\boldsymbol{L}[k]\}$, which can be written, without loss of generality, as

$$\boldsymbol{L}[k] = \bar{\boldsymbol{L}} + \tilde{\boldsymbol{L}}[k] \tag{2}$$

where $\bar{\boldsymbol{L}}$ denotes the mean matrix and $\tilde{\boldsymbol{L}}[k]$ are i.i.d. perturbations around the mean. We do not make any assumptions about the link failure model. Although the link failures and the Laplacians are independent over time, during the same iteration, the link failures can still be spatially correlated.

3. ESTIMATION OF ALGEBRAIC CONNECTIVITY

Since in our setting, the network graph is random due to the presence of random packet drops due to either decoding errors or packet collisions, in the following, we illustrate a method to evaluate the algebraic connectivity in the presence of random packet drops, which is amenable for distributed implementation, as recently proposed in [11]. This method will be instrumental to setup our connectivity control strategy based on power control.

Let us consider a random graph $G[k]$, obtained by using a transmitted power p at each node. We define the transition matrix $\boldsymbol{W}[k]$, at time k, as:

$$\boldsymbol{W}[k] = \boldsymbol{I} - \varepsilon \boldsymbol{L}[k] = \bar{\boldsymbol{W}} + \tilde{\boldsymbol{W}}[k] \tag{3}$$

where $\bar{\boldsymbol{W}} = \boldsymbol{I} - \varepsilon \bar{\boldsymbol{L}}$ is the mean matrix, $\tilde{\boldsymbol{W}}[k] = -\varepsilon \tilde{\boldsymbol{L}}[k]$ are i.i.d. fluctuations around the mean, and $0 < \varepsilon < 2/\lambda_N(\boldsymbol{L})$. The eigenvalues of the expected Laplacian matrix $\bar{\boldsymbol{L}}$ are directly related to those of the expected consensus matrix $\bar{\boldsymbol{W}}$ in (3) through the relation

$$\lambda_i(\bar{\boldsymbol{L}}) = (1 - \lambda_{N+1-i}(\bar{\boldsymbol{W}}))/\varepsilon \tag{4}$$

and, in particular, the algebraic connectivity is given by

$$\lambda_2(\bar{\boldsymbol{L}}) = (1 - \lambda_{N-1}(\bar{\boldsymbol{W}}))/\varepsilon. \tag{5}$$

Now, under the assumption that every instance of the random matrix $\boldsymbol{W}[k]$ in (3) is doubly stochastic, we deflate the original matrix $\boldsymbol{W}[k]$, obtaining the matrix $\boldsymbol{B}[k]$ given by:

$$\begin{aligned} \boldsymbol{B}[k] &= \boldsymbol{W}[k] - \frac{1}{N}\mathbf{1}\mathbf{1}^T \\ &= \bar{\boldsymbol{W}} - \frac{1}{N}\mathbf{1}\mathbf{1}^T + \tilde{\boldsymbol{W}}[k] = \bar{\boldsymbol{B}} + \tilde{\boldsymbol{B}}[k] \end{aligned} \tag{6}$$

where $\bar{\boldsymbol{B}} = \bar{\boldsymbol{W}} - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ and $\tilde{\boldsymbol{B}}[k] = \tilde{\boldsymbol{W}}[k] = -\varepsilon \tilde{\boldsymbol{L}}[k]$. In this way, the maximum eigenvalue of the deflated matrix $\bar{\boldsymbol{B}}$ coincides with the second largest eigenvalue of $\bar{\boldsymbol{W}}$. The main steps of the algorithm are listed in the following.

Stochastic power iteration method: Initialize $\boldsymbol{x}[0]$, $y[0]$, and $z[0]$ randomly. Then, perform the following steps for $k \geq 0$:

1. Build the deflated matrix $\boldsymbol{B}[k] = \boldsymbol{W}[k] - \frac{1}{N}\mathbf{1}\mathbf{1}^T$;
2. Evaluate the estimate $y[k+1]$ of $\lambda_{N-1}(\bar{\boldsymbol{W}})$ at time $k+1$ as:

$$\bar{y}[k] = \frac{\boldsymbol{x}^T[k]\boldsymbol{B}[k]\boldsymbol{x}[k]}{\boldsymbol{x}^T[k]\boldsymbol{x}[k]} \tag{7}$$

$$y[k+1] = y[k] + \alpha[k]\left(\bar{y}[k] - y[k]\right) \tag{8}$$

   where $\alpha[k]$ is a step-size sequence satisfying (11);

3. Perform the following power iteration

$$\boldsymbol{x}[k+1] = \frac{\boldsymbol{B}[k]\boldsymbol{x}[k]}{\|\boldsymbol{B}[k]\boldsymbol{x}[k]\|}; \tag{9}$$

4. Compute the estimate $z[k+1]$ of $\lambda_2(\bar{\boldsymbol{L}})$ at time k as:

$$z[k+1] = (1 - y[k+1])/\varepsilon; \tag{10}$$

5. Go to step 1 and repeat until convergence.

The stochastic power iteration method in (7)-(9) computes an estimate for the largest eigenvalue of the expected matrix $\bar{\boldsymbol{B}}$, which is directly related to the second eigenvalue $\lambda_2(\bar{\boldsymbol{L}})$ of the expected Laplacian through (10). To obtain convergence of the stochastic power iteration method, the step-size sequence $\alpha[k]$ in (8) must satisfy the conditions:

$$\alpha[k] > 0, \quad \sum_{k=0}^{\infty} \alpha[k] = \infty, \quad \sum_{k=0}^{\infty} \alpha^2[k] < \infty. \tag{11}$$

Conditions (11) are standard in stochastic approximation [12]; the effect of the step-size in (11) is to drive to zero the variance of the additive disturbance due to the presence of link failures. Then, the convergence of the iterative procedure is determined only by the expected graph of the network. In [11], we proved that the sequence $z[k]$ generated in (10) by the stochastic power iteration algorithm converges almost surely to the second smallest eigenvalue of the expected Laplacian matrix of the graph, i.e.,

$$\lim_{k\to\infty} z[k] = \lambda_2(\bar{\boldsymbol{L}}), \quad \text{almost surely (w.p.1)}. \tag{12}$$

The stochastic power iteration method has been described up to now in a centralized fashion. In [11], we showed how to implement such a method using a decentralized approach based on average consensus [3].

4. CONNECTIVITY OF WIRELESS AD-HOC NETWORKS WITH REALISTIC MAC

The method illustrated above converges to algebraic connectivity of the expected Laplacian. It is then of interest to establish the relation between such a value and the value obtained under ideal channel propagation conditions. In a realistic communication scenario, nodes communicate with each other by accessing to a shared channel according to a specified MAC protocol. Let us assume that, in the considered wireless ad-hoc scenario, each node has M wireless channels that are dedicated to the exchange of data with its own neighbors. To establish a communication, a node then randomly selects one of these channels independently of the choices of its neighbors. Let us further assume that the nodes are deployed according to a random geometric graph (RGG) model [13]. It is well known that asymptotically, as the number of nodes goes to infinity, RGG networks tend to satisfy a regularity condition, i.e., each node tends to have, asymptotically as the number of nodes tends to infinity, the same number d of neighbors, on average. The average number d of neighbors depends on the coverage radius of each node, which is dictated by the transmitted power and the channel conditions. Let us assume a simple free-space propagation model so that the power received by a node is related to the transmitted power as $P_R = P_T/r^2$, where r is the covered distance.

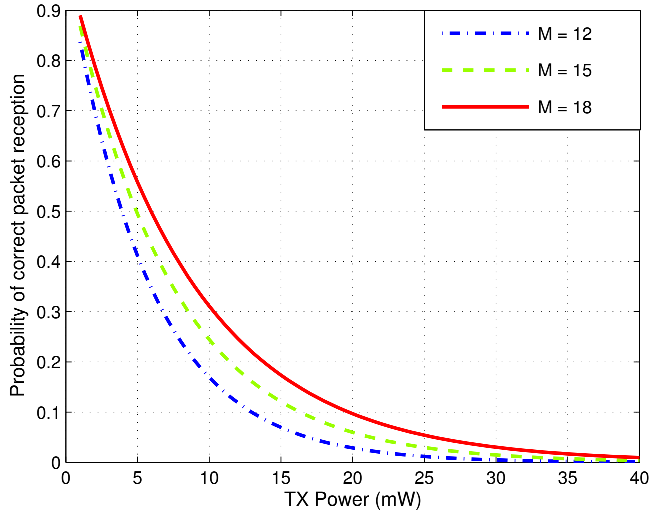

Fig. 1: Probability of correct packet reception versus the power transmitted by each node, for different number M of available channels.

Now, setting a minimum threshold value P_{th} for the power at the receiver node, the coverage radius is simply obtained by inverting the previous expression as $r^2 = P_T/P_{th}$. The average number d of neighbors is then related to the covering radius and, consequently, to the transmitted power P_T, as

$$d = \pi r^2 \varrho = \pi \frac{P_T}{P_{th}} \varrho \tag{13}$$

where ϱ is the spatial density of nodes inside a circle of area πr^2. In this setting, it is clear that the number M of channels used to establish a communication must be designed with respect to the average number d of neighbors, in order to keep the probability to have a collision among the communications of two nodes sufficiently small. Assuming independence among the channel selections of different nodes and exploiting (13), the probability that a packet is correctly exchanged over the selected channel is given by

$$p_c(M, P_T) = \left(\frac{M-1}{M}\right)^d = \left(1 - \frac{1}{M}\right)^{\mu P_T} \tag{14}$$

where $\mu = \pi\varrho/P_{th}$. A numerical example is shown in Fig. 1, where we illustrate the behavior of the probability in (14) versus the power transmitted by each node, for different values of the number of channels M. The simulation considers a network composed of $N = 400$ nodes randomly deployed over a geographic area of 10^4 m^2. The threshold power value at the receiver node is given by $P_{th} = 0.01$ mW. As expected, from Fig. 1 we can notice how the probability to establish correctly a communication link gets worse by increasing the transmitted power P_T, because it translates in having more neighbors which to communicate with, whereas, for a fixed transmitted power, it of course improves by taking a larger number of channels M.

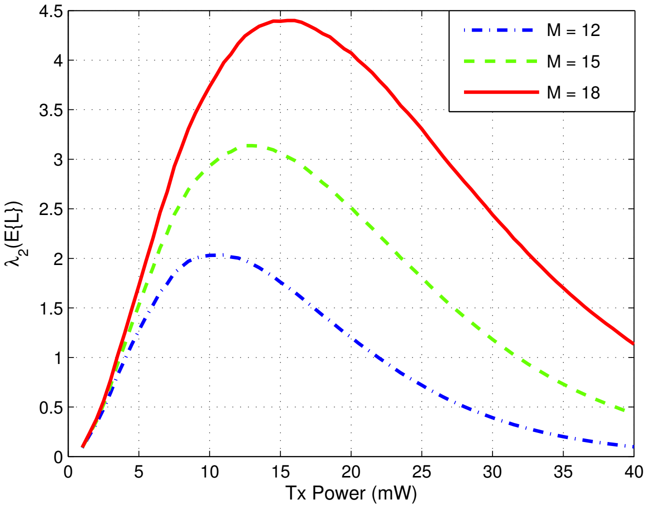

Fig. 2: Algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus the power transmitted by each node, for different number M of available channels.

In an ideal communication case where no collisions occur, the increment of the power transmitted by each node leads to a monotonic increment of the network connectivity. Thus, in an ideal case, it is always convenient to increase the power in order to increase the connectivity of the network, until full connectivity is reached. However, in a real communication scenario, the presence of collisions due to the adoption of a random medium access protocol, e.g. the one we have introduced before, makes the graph describing the network topology a random graph, where each link is on with a probability given by (14). It is then of interest to check the effect of collisions on the connectivity of the expected graph, which is actually the effective connectivity of the network. Thus, considering RGG networks and assuming the number of nodes is sufficiently high to approach a regularity condition of the graph, we have

$$\lambda_2(\bar{\boldsymbol{L}}(P_T)) \simeq p_c(M, P_T) \cdot \lambda_2(\boldsymbol{L}(P_T)). \tag{15}$$

Intuitively, this happens because all the coefficients of the expected adjacency matrix $\bar{\boldsymbol{A}}$ can be approximated as $\bar{a}_{ij} \simeq p_c(M, P_T)$, i.e., each communication link has almost the same probability to be established, if the number of nodes is sufficiently high to approach a regularity condition of the graph. An example is given in Fig. 2, where we show the behavior of the algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus the power transmitted by each node, for different number M of available channels. The simulation settings are the same as before. As we can notice from Fig. 2, $\lambda_2(\bar{\boldsymbol{L}})$ shows approximately a quasi-concave behavior with respect to the transmitted power P_T. In fact, at low power values, the algebraic connectivity of the expected graph increases because the number of neighbors of each node increases. However, as power goes above a certain threshold, the number of neighbors becomes too high and the probability of collision increases, thus leading to a reduction of the overall connectivity of the network. From Fig. 2, as expected, we also notice how, increasing the number of available channels M for a fixed transmitted power, the connectivity of the expected graph improves. The behavior of $\lambda_2(\bar{\boldsymbol{L}})$ determines that there is an optimal transmitted power that nodes should use to maximize the connectivity of the expected graph. An increment of the power with respect to this threshold value would lead to a waste of energy due to the effect of collisions, which becomes the dominant effect that drives the effective connectivity to zero. In summary, while in an ideal communication scenario nodes would always improve the network connectivity by increasing their transmitted power, considering a realistic random MAC, a too large transmission power may degrade the connectivity due to an increase in the collision probability.

5. DISTRIBUTED MAXIMIZATION OF EXPECTED GRAPH'S CONNECTIVITY

In the previous section, we have shown that, for a sufficiently large number of nodes composing the network, the behavior of the algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus the power p transmitted by each node is approximately continuous and unimodal, thus leading to the presence of a unique maximum point (Fig. 2). The goal of this section is to find the optimal power value p^* that maximizes the connectivity of the expected graph, without assuming knowledge of the analytical relation between $\lambda_2(\bar{\boldsymbol{L}})$ and p. The method is based on a stochastic algorithm that approximates the derivative of the function on the basis of noisy measurements of $\lambda_2(\bar{\boldsymbol{L}})$. For a given power value p, using the stochastic power iteration method in (7)-(9), it is possible to achieve an estimate $\hat{z}(p)$ of the second smallest eigenvalue of the expected Laplacian matrix in a totally distributed fashion. In practice, stopping the iterative method in (7)-(9) at a finite number of iterations, induces an inevitable estimation error, so that we can write

$$\hat{z}(p) = \lambda_2(\bar{\boldsymbol{L}}(p)) + \xi \tag{16}$$

where ξ is a realization of a zero-mean random variable with bounded variance σ_ξ^2.

Now, exploiting the noisy measurements in (16), we can build a stochastic approximation Kiefer-Wolfowitz (KW) method [12] that can be used to attempt to find the maximum of the function $\lambda_2(\bar{\boldsymbol{L}}(p))$. The algorithm is run in parallel by each node in the network, which updates its own transmitted power according to the recursive rule:

$$p[t+1] = p[t] + \alpha[t]\frac{\hat{z}(p[t]+c[t]) - \hat{z}(p[t]-c[t])}{2c[t]} \tag{17}$$

where $p[0]$ is chosen at random, and $\alpha[t]$ and $c[t]$ are two pos-

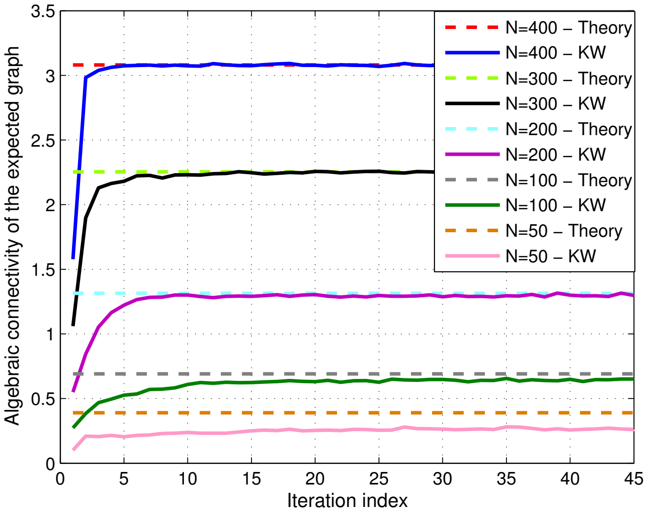

Fig. 3: Algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus iteration index, for different number N of nodes.

itive sequences that satisfy (11) and the further conditions

$$c[t] \to 0, \qquad \sum_{t=0}^{\infty} \frac{\alpha^2[t]}{c^2[t]} < \infty. \tag{18}$$

For any t, the stochastic power iteration method in (7)-(9) must be run twice in order to get $\hat{z}(p[t] + c[t])$ and $\hat{z}(p[t] - c[t])$. The procedure in (17) is then repeated until a convergence criterion is satisfied.

A numerical example is shown in Fig. 3, where we illustrate the behavior of the algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus iteration index, for different values of the number of network nodes N, obtained by using the KW method in (17). The theoretical values of the maximum connectivity are also reported for comparison purposes. The simulation considers a network composed of N nodes randomly deployed over a geographic area of 10^4 m^2. The threshold power value at the receiver node is given by $P_{th} = 0.01$ mW. The number of channels is set to $M = 15$. As we can notice from Fig. 3, when the number of nodes is sufficiently large, the KW method in (17) is able to find the maximum of $\lambda_2(\bar{\boldsymbol{L}}(p))$ in a few iterations. At the same time, reducing the number of nodes in the network, the behavior of $\lambda_2(\bar{\boldsymbol{L}}(p))$ becomes less continuous and unimodal. This implies that the algorithm in (17) can get stuck in some local maximum, thus explaining the gap between the theoretical maximum value and the KW method in Fig. 3, at low number of nodes N.

6. CONCLUSIONS

In this paper we have studied the effect of a realistic random medium access control protocol on the connectivity of wireless ad-hoc networks. It has been shown how the presence of collisions limits the connectivity of a wireless ad-hoc network and, contrarily to what is typically believed, by increasing the covering radius of each node, the network connectivity may be heavily degraded due to an increase in the collision probability. In particular, numerical results show that, for sufficiently large number of nodes, the behavior of the algebraic connectivity of the expected graph $\lambda_2(\bar{\boldsymbol{L}})$ versus the power transmitted by each node is approximately continuous and unimodal. Building on such a result, we have proposed a distributed Kiefer-Wolfowitz stochastic approximation algorithm to find the transmit power that maximizes the expected connectivity in the presence of collisions.

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