SEMI-DETERMINISTIC TERNARY MATRIX FOR COMPRESSED SENSING

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ABSTRACT

For the random $\{0, \pm 1\}$ ternary matrix, it is interesting to determine the number of nonzero elements required for good compressed sensing performance. By seeking the best RIP, this paper proposes a semi-deterministic ternary matrix, which is of deterministic nonzero positions but random signs. In practice, it presents better performance than common random ternary matrices and Gaussian random matrices.

***Index Terms*—** random matrix, ternary matrix, compressed sensing, RIP, deterministic, semi-deterministic

I. INTRODUCTION

Compressed sensing is an emerging technique that aims to acquire and recover sparse signals through an underdetermined sensing matrix [1]. Its performance is largely determined by the nature of the nonadaptive sensing matrix [2]. It is known that the random matrix with elements i.i.d drawn from some known symmetric distributions, e.g. from Gaussian distribution or Bernoulli distribution [1] [3], can provide good sensing performance. In practice, the sparse random ternary matrix with nonzero elements being ± 1 equiprobably is more attractive due to its low complexity but competitive performance [4]. For this kind of matrix, the optimal number of nonzero elements that provides the best sensing performance is still unknown, although some deterministic ternary matrices have been developed with algebraic codes [5] [6]. In this paper, we aim to address this problem.

Inspired by the most recent study on the optimal sparsity of the zero-one binary sensing matrix [7], this paper proposes a suitable distribution for the nonzero elements of ternary matrix by seeking the best restricted isometry property (RIP). As will be shown latter, the proposed matrix enjoys two obvious advantages: 1) in structure, it can be viewed as 'semi-deterministic', since it holds random signs but fixed nonzero positions; 2) in performance, it outperforms other popular random matrices.

This paper is organized as follows. In the next section, the RIP is first introduced, and then the proposed ternary matrix is derived by searching the best RIP. In section III, the proposed matrix is compared with other popular random matrices by extensive simulations. Finally, the paper is concluded in section IV.

II. MAIN RESULTS

II-A. Fundamentals of RIP

RIP is a predominant tool for evaluating the performance of sensing matrix especially in the presence of interference. Conceptually, it is concerned with a pair of parameters (k, δ_k), where k denotes the number of nonzero elements in the sparse signal, and δ_k is a positive constant called restricted isometry constant (RIC). Let $x \in \mathbb{R}^N$ be a k-sparse signal and $A \in \mathbb{R}^{M \times N}$ be a sensing matrix, $M < N$. The RIP states that x can be perfectly recovered from Ax by solving a ℓ_1-regularized minimization problem, if the RIC is less than some given threshold [8] [9] [10]. The RIC is defined as the minimal positive constant δ_k such that the following inequality

$$(1 - \delta_k)||x_T||^2 \leq ||A_T x_T||^2 \leq (1 + \delta_k)||x_T||^2 \qquad (1)$$

holds for arbitrary submatrix A_T and arbitrary corresponding vectors $x_T \in \mathbb{R}^{|T|}$, where T indicates all possible column index subsets of A with cardinality $|T| = k$. The exact solution to δ_k is NP-hard [11]. In practice, it is generally approximated by exploring the extreme eigenvalues of Gram matrix $A'_T A_T$ [12], where A'_T denotes the transpose of A_T. Note that the value δ_k tends to increase with increasing k. To recover a signal with large k, it is necessary to construct the sensing matrix with small δ_k. So in the following part we attempt to seek a ternary matrix with as small RIC as possible.

II-B. RIP of ternary matrix

For the convenience of analysis, the ternary matrix is denoted by $A(d, s)$, whose two parameters are defined as below:

- the parameter d denotes the number of nonzero elements per column; the nonzero position is selected uniformly at random;
- the parameter s is the maximum number of overlapping nonzero positions between arbitrary two distinct columns; then the correlation values between distinct columns should vary in the integer interval $[-s,\ s]$.

With the definition above, the RIP of ternary matrix is derived in Theorem 1.

Theorem 1. *For a ternary matrix $A(d, s)$, the RIC can be approximately derived as*

$$\delta_k \approx (k-1)s/d \tag{2}$$

Proof: We customarily denote by $\lambda_1(X) \geq \ldots \geq \lambda_k(X)$ the eigenvalues of matrix X. Two extreme eigenvalues are derived as follows.

1) Let $B = A'_T A_T - (d+s) \times I$, then $B_{ii} = -s$ and $B_{ij,i\neq j} \in [-s, s]$. Recall that A_T is the submatrix of A with columns indexed by $T \subset \{1, 2, 3..., n\}$, $|T| = k$. Let the normalized $x = (x_1, \ldots, x_k)'$ be the eigenvector corresponding to $\lambda_k(B)$. Then the minimal eigenvalue can be formulated as

$$\lambda_k(B) = x'Bx = \mathbb{1}'[B \circ (xx')]\mathbb{1}$$

where $\circ$ denotes the Hadamard product and $\mathbb{1} = (1, \ldots, 1)' \in \mathbb{R}^{k\times 1}$. Since B is symmetric, by simultaneous permutations of the rows and columns of B, we can suppose $x_i \geq 0$ for $i = 1, \ldots, n$ and $x_i < 0$ for $i = n+1, \ldots, k$, and then xx' is divided into four parts:

$$xx' = \begin{bmatrix} X_{n\times n} & X_{n\times(k-n)} \\ X_{(k-n)\times n} & X_{(k-n)\times(k-n)} \end{bmatrix}$$

where the entries in $X_{n\times n}$ and $X_{(k-n)\times(k-n)}$ are nonnegative, while the entries in $X_{n\times(k-n)}$ and $X_{(k-n)\times n}$ are nonpositive. Further, define a novel matrix $\tilde{B}$ of same size with B

$$\tilde{B} = \begin{bmatrix} -s \times \mathbb{1}_{n\times n} & s \times \mathbb{1}_{n\times(k-n)} \\ s \times \mathbb{1}_{(k-n)\times n} & -s \times \mathbb{1}_{(k-n)\times(k-n)} \end{bmatrix}$$

where $\mathbb{1}_{a\times b}$ is an $a \times b$ matrix with all entries equal to 1. It is easy to deduce that

$$\begin{aligned}\lambda_k(\tilde{B}) &= \min\{y'\tilde{B}y : \|y\| = 1\} \leq x'\tilde{B}x \leq x'Bx \\ &= \lambda_k(B).\end{aligned}$$

Since the rank of $\tilde{B}$ is at most 2, it has at most two nonzero eigenvalues. Considering the trace and the Frobenius norm, we have

$$\lambda_k(\tilde{B}) = -ks, \; 0 \leq n \leq k.$$

Then with $\lambda_k(B) \geq \lambda_k(\tilde{B})$, we have the minimum eigenvalue

$$\lambda_k(A'_T A_T) = \lambda_k(B) + (d+s) \geq d - (k-1)s$$

2) Let $C = A'_T A_T - (d-s) \times I$, then $C_{ii} = s$ and $C_{ij,i\neq j} \in [-s, s]$.
Let normalized $x = (x_1, \ldots, x_k)'$ be the eigenvector corresponding to $\lambda_1(C)$. By simultaneous permutations of C and x, we can suppose $x_i \geq 0$ for $i = 1, \ldots, n$ and $x_i < 0$ for $i = n+1, \ldots, k$, and the maximal eigenvalue is formulated as

$$\lambda_1(C) = x'Cx = \mathbb{1}'[C \circ (xx')]\mathbb{1}.$$

Further define

$$\tilde{C} = \begin{bmatrix} s \times \mathbb{1}_{n\times n} & -s \times \mathbb{1}_{n\times(k-n)} \\ -s \times \mathbb{1}_{(k-n)\times n} & s \times \mathbb{1}_{(k-n)\times(k-n)} \end{bmatrix},$$

then

$$\begin{aligned}\lambda_1(\tilde{C}) &= \max\{y'\tilde{C}y : \|y\| = 1\} \geq x'\tilde{C}x \geq x'Cx \\ &= \lambda_1(C).\end{aligned}$$

Since the rank of $\tilde{C}$ is at most 2, it has at most two nonzero eigenvalues. Considering the trace and the Frobenius norm, we have

$$\lambda_1(\tilde{C}) = ks.$$

Then $\lambda_1(C) \leq \lambda_1(\tilde{C}) \leq ks$. Finally, we obtain

$$\lambda_1(A'_T A_T) = \lambda_1(C) + (d-s) \leq d + (k-1)s$$

To sum up, we have the RIC:

$$\delta_k = \frac{\lambda_1(A'_T A_T) - \lambda_k(A'_T A_T)}{\lambda_1(A'_T A_T) + \lambda_k(A'_T A_T)} \approx (k-1)s/d$$

■

II-C. Semi-deterministic ternary matrix

From Theorem 1, it can be observed that the RIC will decrease as s decreases or d increases. To reduce the value of RIC as much as possible, it is natural to minimize s, i.e. let $s = 1$, while maximizing d [7]. In practice, the desired matrix can be constructed with progressive edge-growth (PEG) algorithm [7] [13]. This allows us to define a semi-deterministic ternary (SDT) matrix, which is constructed in two steps:

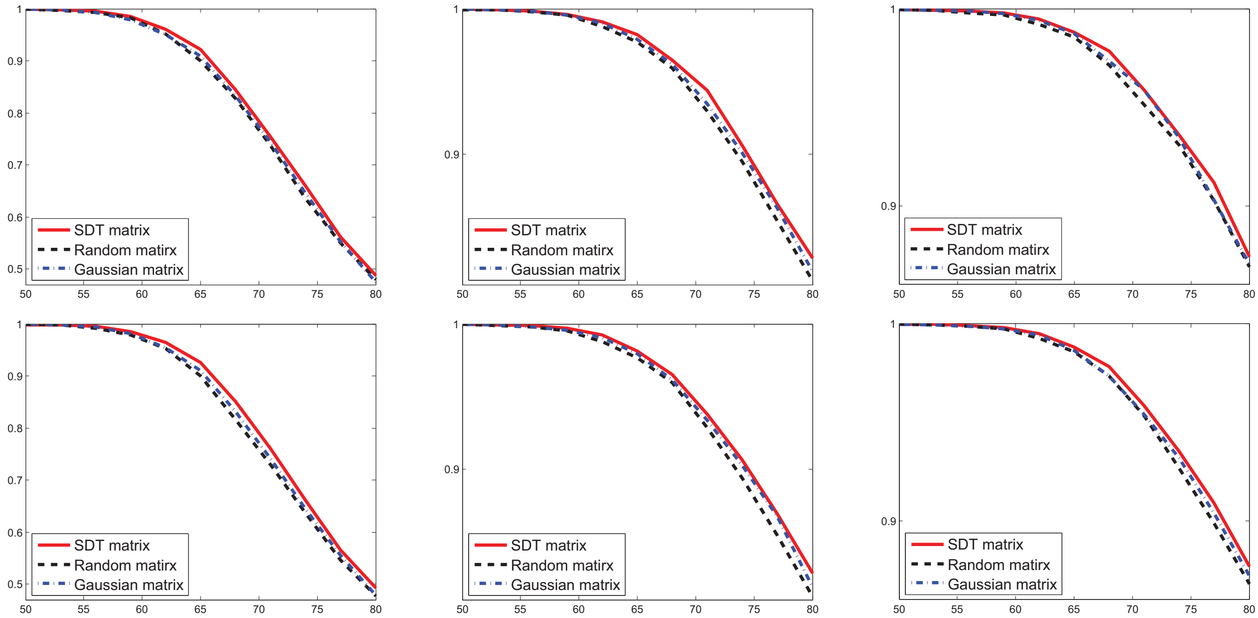
1) the nonzero position is determined by PEG algorithm based on the principle of maximizing d while setting $s = 1$;
2) the nonzero position is randomly signed at each compressed sensing process.

Obviously, the proposed SDT matrix should have better RIP than other random ternary matrices with no specific constraint on both s and d. The following simulation also verifies this conjecture.

III. SIMULATION

The proposed SDT matrix is compared with other two typical random matrices: random ternary matrices and Gaussian random matrices. Recall that the recovery performance is sensitive to the distribution and sign of sparse signal [14] [15] [16]. So the *signed* and *unsigned* (absolute value) sparse signals are both tested, which have nonzero entries drawn from three typical distributions:

- Bernoulli distribution $\{\pm 1\}$;
- uniform distribution $U(-1, 1)$;
- standard normal distribution $N(0, 1)$.



(a) (b) (c)

Fig. 1. The BP recovery rates (the vertical axis) over sparse signals of varying sparsity $k \in [50, 80]$. Each column corresponds to one class of sparse signals: $\{\pm 1\}$ in (a), $U(-1, 1)$ in (b) and $N(0, 1)$ in (c). The sparse signals are *unsigned* in the upper row, and *signed* in the lower.

To obtain convincing results, we exploit three popular recovery algorithms: BP [17] [18], SP [19] and OMP [20]. The matrix is constructed with size (200, 400). Here the SDT matrix with d=7 is derived with PEG algorithm. The column degree of random ternary matrix is empirically set to 8 for achieving its best performance. The recovery performance is measured with the correct recovery rates over the sparse signals of varying sparsity $k \in [50, 80]$. Each simulation point is derived after 10000 simulation runs.

The correct recovery rates of three recovery algorithms are shown in Figures 1-3, respectively. It is clear that for all kinds of sparse signals, the proposed SDT matrix always performs better than the random ternary matrix and Gaussian matrix, while the latter two matrices show comparable performance. It confirms the RIP advantage of the proposed SDT matrix.

IV. CONCLUSION

This paper has proposed a kind of random ternary matrix with good RIP. The proposed matrix can be viewed as semi-deterministic due to its deterministic nonzero positions. In practice, as demonstrated by experiments, the proposed matrix presents better performance than other two popular random matrices: random ternary matrices and Gaussian random matrices. Intuitively, it probably performs better than all other possible random or deterministic ternary matrices.

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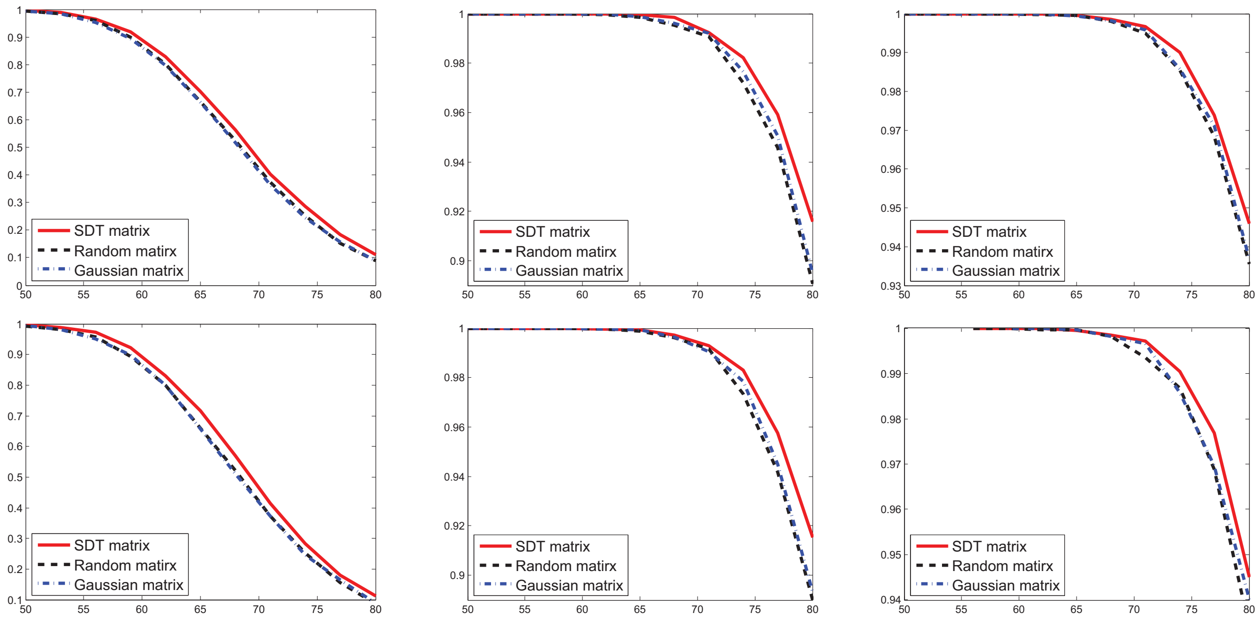
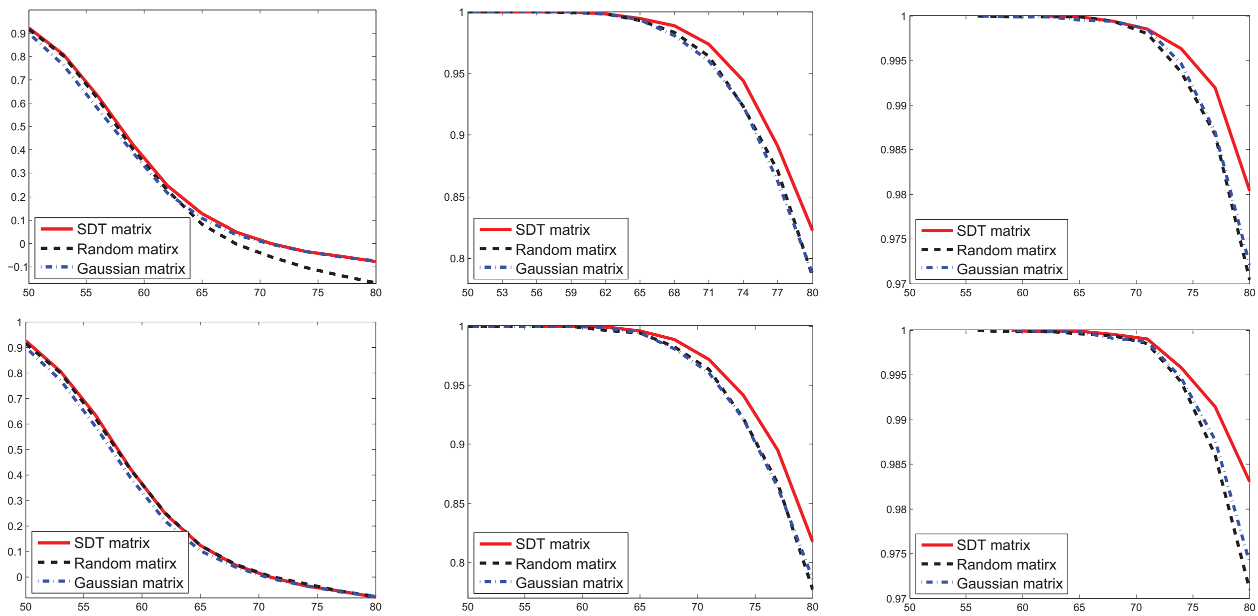


(a) (b) (c)

Fig. 2. The SP recovery rates (the vertical axis) over sparse signals of varying sparsity $k \in [50, 80]$. Each column corresponds to one class of sparse signals: $\{\pm 1\}$ in (a), $U(-1, 1)$ in (b) and $N(0, 1)$ in (c). The sparse signals are *unsigned* in the upper row, and *signed* in the lower.



(a) (b) (c)

Fig. 3. The OMP recovery rates (the vertical axis) over sparse signals of varying sparsity $k \in [50, 80]$. Each column corresponds to one class of sparse signals: $\{\pm 1\}$ in (a), $U(-1, 1)$ in (b) and $N(0, 1)$ in (c). The sparse signals are *unsigned* in the upper row, and *signed* in the lower.

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