OPTIMUM DISCRETE PHASE-ONLY TRANSMIT BEAMFORMING WITH ANTENNA SELECTION

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ABSTRACT

Phase-only beamforming is used in radar and communication systems due to its certain advantages. Antenna selection becomes an important problem as the number of antennas becomes larger than the number of transmit-receive chains. In this paper, discrete single group multicast transmit phase-only beamformer design with antenna subset selection is considered. The problem is converted into linear form and solved efficiently by using mixed integer linear programming to find the optimum subset of antennas and beamformer coefficients. Several simulations are done and it is shown that the proposed approach is an effective and efficient method of subarray transmit beamformer design.

***Index Terms*—** Transmit beamformer, discrete beamformer, mixed integer linear programming, antenna selection

1. INTRODUCTION

Large antenna arrays become cheaper and more feasible with the advances on fabrication techniques. In modern radar and communication systems, there are more antennas than the transmit-receive chains. As a result, transmit antenna selection for the most appropriate antenna subarray is an important problem [1], [2].

Practical radar [3] and communication systems [4], [5] hardware is composed of discrete phase shifters for transmit-receive beamforming. Phase-only transmission has certain advantages [3], [6], [7], [8], [9]. Current state of the art for discrete-phase transmit beamforming discretizes the known beamforming coefficients or uses iterative schemes for discrete solutions [4], [5], [6], [7]. These discrete beamformers are far from optimum in terms of maximum transmit power to users.

In single group multicast beamforming, a common signal is transmitted to a pool of N users each equipped with a single antenna while the transmitter is composed of an antenna array [10]. The problem is treated in several previous works [10], [11] where suboptimum solutions are found for continuous amplitude and phase beamformers. In this paper, the design of optimum discrete-phase transmit beamformer with antenna selection is considered. Proposed method selects the best L out of M antennas and finds the optimum phase-only transmit beamformer coefficients in a joint manner using branch and cut technique. In some previous papers, optimum discrete phase-only [12] and optimum discrete phase amplitude [13] transmit beamformer designs are considered. This paper extends the work in [12] by including antenna selection into the problem. Max-min fair transmit beamformer design with antenna selection problem is converted to linear form for effective solution using mixed integer linear programming. While the worst case complexity of this method is exponential when all the possible branches are expanded, it is significantly lower than the brute force search in practice due to the fact that solution is usually found before the full tree expansion. This point is further elaborated inside the paper. To our knowledge, this paper is the first work which presents the optimum solution to the aforementioned problem described above.

2. SYSTEM MODEL

In single group multicast beamforming, it is assumed that a base station equipped with M transmit antennas transmits a common signal to N receivers, each having a single antenna. The transmitted signal can be written as,

$$\mathbf{x}(t) = s(t)\mathbf{w} \tag{1}$$

where $s(t)$ is the source signal and $\mathbf{w}$ is the $M \times 1$ complex beamformer weight vector. The received signal at the k^{th} receiver is given as,

$$y_k(t) = \mathbf{h}_k^H \mathbf{x}(t) + n_k(t) \quad k = 1, \dots N \tag{2}$$

where $\mathbf{h}_k$ is the $M \times 1$ complex channel vector for the k^{th} receiver and n_k is additive noise uncorrelated with the source signal and its variance is σ_k^2. Signal-to-noise ratio (SNR) for the k^{th} receiver is,

$$SNR_k = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{h}_k|^2}{\sigma_k^2} \tag{3}$$

where σ_s^2 is the source signal variance. $\sigma_s^2 = 1$ is selected without loss of generality throughout the paper.

In phase-only single group multicast beamforming max-min problem, beamforming vector, $\mathbf{w}$, is chosen to maximize the minimum transmitted power to any user. Suppose that only L out of M antennas can be transmitting simultaneously. In this case, the goal is to select the best L antennas and find the corresponding beamforming vector. Considering P_{an} as the per-antenna power, the problem can be written as follows,

$$\begin{aligned} &\max_{\mathbf{w}\in\mathbb{C}^M} \ t \\ s.t.\quad &\mathbf{w}^H\mathbf{R}_k\mathbf{w} \geq t\gamma_k\sigma_k^2, \quad k=1,...,N \\ &(\mathbf{w}\mathbf{w}^{\mathbf{H}})_{i,i} \in \{0, P_{an}\} \quad i=1,...,M \\ &\mathbf{w}^{\mathbf{H}}\mathbf{w} = LP_{an} \end{aligned} \tag{4}$$

where γ_k is the power proportion for the k^{th} receiver and t is the max-min parameter corresponding to the $min\{\frac{SNR_k}{\gamma_k}\}$. $\mathbf{R}_k = \mathbb{E}\{\mathbf{h}_k\mathbf{h}_k^H\}$ is the correlation matrix of the channel vector. Solving the above problem requires a combinatorial search over all $\binom{M}{L}$ NP hard problems. In the following part, we consider the discrete version of this problem and find the optimum solution using mixed integer linear programming with moderate complexity with branch and cut strategy.

3. DISCRETE PROBLEM

In [12], an optimum solution for the max-min style phase-only single group multicast problem without antenna selection is obtained when the phase angles of the beamformer are selected from a discrete set. In this paper, optimum solution is found when the antenna selection is added to the problem. The problem in (4) can be written for the discrete phase as,

$$\begin{aligned} &\max_{\psi_i,\alpha_i} \ t \\ s.t.\quad &\mathbf{w}^H\mathbf{R}_k\mathbf{w} \geq t\gamma_k\sigma_k^2, \quad k=1,...,N \\ &\alpha_i \in \{0,1\} \quad i=1,...,M \\ &\sum_{i=1}^{M}\alpha_i = L \\ &\psi_i \in \{0, \Delta\theta, 2\Delta\theta, ..., (2^n-1)\Delta\theta\},\ \Delta\theta = \frac{360^\circ}{2^n} \end{aligned} \tag{5}$$

where i^{th} element of the beamformer vector $\mathbf{w}$ is $w_i = \alpha_i\sqrt{P_{an}}e^{j\psi_i}$ and ψ_i is the discrete phase with n bits. α_i is the antenna selection coefficient. $\Delta\theta$ is the discrete step size for phase. Since $\mathbf{R}_k$ is a Hermitian symmetric matrix, the inequality in (5) can be expressed as,

$$\begin{aligned} &\sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2\alpha_i\alpha_p|R_k(i,p)|cos(\angle R_k(i,p)+\psi_p-\psi_i)+ \\ &\sum_{i=1}^{M}\alpha_i^2R_k(i,i) \geq \frac{t\gamma_k\sigma_k^2}{P_{an}}, \quad k=1,...,N \end{aligned} \tag{6}$$

The inequality in (6) can be simplified further if additional variables $\beta_{i,p}$ and $\mu_{i,p}$ are defined as,

$$\begin{aligned} &\beta_{i,p} = -\psi_i + \psi_p \quad \mu_{i,p} = \alpha_i\alpha_p, \\ &i=1,2,...,M-1,\ p=i+1,...,M \end{aligned}$$

Using the trigonometric identity, the optimization problem can be written as,

$$\max_{\psi_i,\alpha_i,\beta_{i,p},\mu_{i,p}} \ t$$

$$\begin{aligned} s.t.\ &\sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2\mu_{i,p}|R_k(i,p)|[cos(\angle R_k(i,p))cos\beta_{i,p} \\ &-sin(\angle R_k(i,p))sin\beta_{i,p}] + \sum_{i=1}^{M}\alpha_i^2R_k(i,i) \\ &\geq \frac{t\gamma_k\sigma_k^2}{P_{an}}, \quad k=1,...,N \end{aligned} \tag{7}$$

$$\psi_i \in \{0, \Delta\theta, 2\Delta\theta, ...(2^n-1)\Delta\theta\},\ \Delta\theta = \frac{360^\circ}{2^n}$$

$$\alpha_i \in \{\,0, 1\,\},$$

$$\beta_{i,p} = -\psi_i + \psi_p \tag{8}$$

$$\mu_{i,p} = \alpha_i\alpha_p, \tag{9}$$

$$i=1,2,...,M-1,\ p=i+1,...,M$$

$$\sum_{i=1}^{M}\alpha_i = L \tag{10}$$

The above problem setting is not linear. In the following section, the above problem is converted into linear form.

4. DISCRETE OPTIMIZATION IN LINEAR FORM

Let the first part of the left hand side of the inequality in (7) be represented as A. A can be expressed in linear form by using some known vectors $\mathbf{c}$ and $\mathbf{s}$. $\mathbf{c}$ and $\mathbf{s}$ are composed of all possible $cos\beta_{i,p}$ and $sin\beta_{i,p}$ terms and "0" corresponds to a term related to the antenna selection which nullifies the corresponding element in the beamformer weight vector $\mathbf{w}$, i.e.,

$$\mathbf{c} = [\,0\,, cos(0\cdot\Delta\theta)\,, cos(1\cdot\Delta\theta),\ ...\,, cos((2^n-1)\cdot\Delta\theta)]^T$$
$$\mathbf{s} = [\,0\,, sin(0\cdot\Delta\theta)\,, sin(1\cdot\Delta\theta),\ ...\,, sin((2^n-1)\cdot\Delta\theta)]^T$$

In order to access each term in A, indicator vectors $\mathbf{u}_{i,p}$ whose elements are all zero except a single element are defined. $\mathbf{u}_{i,p}$'s are the new variables of the optimization.

A in (7) can be expressed in terms of $\mathbf{u}_{i,p}$ as,

$$\begin{aligned} A = &\sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2|R_k(i,p)|[cos(\angle R_k(i,p))\mathbf{c}^T \\ &-sin(\angle R_k(i,p))\mathbf{s}^T]\cdot\mathbf{u}_{i,p} \end{aligned} \tag{11}$$

Note that $u_{i,p}(1)$ and $\mu_{i,p}$ are complements of each other as binary variables described as in Table 1. The relationship between $\mathbf{u}_{i,p}$ vectors should be established and used during the optimization. Such relationships can be established over binary indicator vectors, $\mathbf{v}_i$. $(2^n+1)\times 1$ vector, $\mathbf{v}_i$, carries the phase and amplitude information of the beamformer vector. The first element of $\mathbf{u}_{i,p}$ and $\mathbf{v}_i$, namely $u_{i,p}(1)$ and $v_i(1)$ are the antenna selection parameters. $v_i(1)$ is the complement of α_i as given in Table 1.

Once $\mathbf{u}_{i,p}$ and $\mathbf{v}_i$ indicator vectors are defined, it is possible to write (8) in terms of these vectors. First (8) is normalized by $\Delta\theta$, then

$$\frac{\beta_{i,p}}{\Delta\theta} = \frac{-\psi_i}{\Delta\theta} + \frac{\psi_p}{\Delta\theta} \tag{12}$$

The above expression can be written as,

$$\mathbf{d}^T \cdot \mathbf{u}_{i,p} = \mathbf{d}^T \cdot (-\mathbf{v}_i + \mathbf{v}_p) \pmod{2^n} \tag{13}$$

where $\mathbf{d} = [\ 0\ ,0\ ,1\ ,2\ ,\ldots,(2^n-1)\ ]^T$. Note that the first zero in $\mathbf{d}$ is for the antenna selection and the second one is for zero phase ($0 \cdot \Delta\theta$). $\mathbf{d}^T \cdot \mathbf{u}_{i,p}$ is always nonnegative and modulo operation on the right hand side ensures the equality. Modulo operation can be removed if an additional binary variable $a_{i,p}$ is used, i.e.,

$$\mathbf{d}^T \cdot \mathbf{u}_{i,p} = \mathbf{d}^T \cdot (-\mathbf{v}_i + \mathbf{v}_p) + a_{i,p}2^n \tag{14}$$

Note that (14) is defined when i^{th} and p^{th} antennas are used simultaneously. For the other cases of antenna existences, (ex: i^{th} antenna does not exists), equation (14) can be still made valid if an additional variable $b_{i,p} \geq 0$ is added to the left hand side of equation (14). In this case, the right hand side of (14) is always positive and $\mathbf{d}^T \cdot \mathbf{u}_{i,p} = 0$ requiring a positive $b_{i,p}$ to satisfy the equality, i.e.,

$$\mathbf{d}^T \cdot \mathbf{u}_{i,p} + b_{i,p} = \mathbf{d}^T \cdot (-\mathbf{v}_i + \mathbf{v}_p) + a_{i,p}2^n \tag{15}$$

Note that (15) is more comprehensive form of linear expression in (8).

4.1. Final Form of the Discrete Problem

The final variable of optimization, t, corresponds to the $min\{\frac{SNR_k}{\gamma_k}\}$ and hence it is real and positive, i.e., $t \in \mathbb{R}^+$. Furthermore all the expressions in (16-22) are linear. Therefore the problem can be solved using mixed integer linear programming [14]. The new optimization problem can be written as,

$$\max_{\mathbf{v}_i,\mathbf{u}_{i,p},a_{i,p},b_{i,p}} t$$

$$s.t. \sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2|R_k(i,p)|[cos(\angle R_k(i,p))\mathbf{c}^T$$

$$-sin(\angle R_k(i,p))\mathbf{s}^T]\cdot\mathbf{u}_{i,p} + \sum_{i=1}^{M}(1-v_i(1))R_k(i,i)$$

$$\geq \frac{t\gamma_k\sigma_k^2}{P_{an}} \quad k=1,\ldots,N \tag{16}$$

$$\mathbf{d}^T \cdot \mathbf{u}_{i,p} + b_{i,p} = \mathbf{d}^T \cdot (-\mathbf{v}_i + \mathbf{v}_p) + a_{i,p}2^n \tag{17}$$

$$a_{i,p} \in \{0,1\} \tag{18}$$

$$b_{i,p} \geq 0 \tag{19}$$

$$b_{i,p} + 2^n(1-u_{i,p}(1)) \leq 2^n \tag{20}$$

$$-1 \leq v_i(1) + v_p(1) - 2u_{i,p}(1) \leq 0 \tag{21}$$

Table 1.

α_i	α_p	$\mu_{i,p}$	$v_i(1)$	$v_p(1)$	$u_{i,p}(1)$	$b_{i,p}$
0	0	0	1	1	1	≥ 0
0	1	0	1	0	1	≥ 0
1	0	0	0	1	1	≥ 0
1	1	1	0	0	0	0

$$\sum_{i=1}^{M} v_i(1) = M - L \tag{22}$$

$$i=1,2,\ldots,M-1,\ p=i+1,\ldots,M$$

$$v_i(k),u_{i,p}(k) \in \{0,1\},\ \sum_{k=1}^{2^n+1} v_i(k) = 1,\ \sum_{k=1}^{2^n+1} u_{i,p}(k) = 1.$$

The expression in (16) stands for (7) in the previous section. The expressions (17), (18), (19) and (20) are for (8) where $a_{i,p}$ and $b_{i,p}$ are the additional variables to make (17) valid for all cases including the case when any one of the antennas is not selected. (19) and (20) guarantees that $b_{i,p}$ is 0 when i^{th} and p^{th} antennas are selected in accordance with Table 1. (21) is used for the antenna selection as in (9) and it realizes the Table 1. Note that (21) is an inequality over binary variables to realize (9). (22) stands for (10) and it shows the number of selected antennas.

The problem in (16-22) is always feasible and the global optimum can be found using mixed integer linear programming [15], [16], [17] with branch and cut technique. Once the solution for $\mathbf{v}_i$'s are found, the phase angles and the antenna selection coefficients of the beamformer vector are obtained as,

$$\psi_i = \mathbf{f}_\psi^T\mathbf{v}_i, \quad \alpha_i = 1 - v_i(1) \quad i=1,\ldots,M \tag{23}$$

where $\mathbf{f}_\psi = [\ 0\ ,0\ ,\Delta\theta\ ,\ldots,(2^n-1)\Delta\theta\ ]^T = \Delta\theta\mathbf{d}$.

The following example shows the structure of $\mathbf{u}_{i,p}$ and $\mathbf{v}_i$ vectors as well as their interrelations. The construction of (17) is also elaborated in this example.

Example: Let $M = 3$, $n = 2$ ($\Delta\theta = 90°$) and the beamformer weight vector be $\mathbf{w} = [\ 0\ e^{j180°}\ e^{j90°}\ ]^T$, i.e., the first antenna is not used. In this case, α_i's are given as $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = 1$ respectively. $\psi_2 = 180°$ and $\psi_3 = 90°$. Then $\mathbf{v}_i$ vectors are;

$$\mathbf{v}_1 = [\ \underbrace{1}_{=1-\alpha_1}\ \underbrace{0}_{0°}\ \underbrace{0}_{90°}\ \underbrace{0}_{180°}\ \underbrace{0}_{270°}\ ]^T$$

$$\mathbf{v}_2 = [\ \underbrace{0}_{=1-\alpha_2}\ \underbrace{0}_{0°}\ \underbrace{0}_{90°}\ \underbrace{1}_{\psi_2=180°}\ \underbrace{0}_{270°}\ ]^T$$

$$\mathbf{v}_3 = [\ \underbrace{0}_{=1-\alpha_3}\ \underbrace{0}_{0°}\ \underbrace{1}_{\psi_3=90°}\ \underbrace{0}_{180°}\ \underbrace{0}_{270°}\ ]^T$$

Since the first antenna is not used $\mu_{i,p}$'s are given as $\mu_{1,2} = \mu_{1,3} = 0$ and $\mu_{2,3} = 1$. $\beta_{2,3}$ is $\beta_{2,3} = -\psi_2 + \psi_3 = -90° =$

270°. Then $\mathbf{u}_{i,p}$ vectors are;

$$\mathbf{u}_{1,2} = \mathbf{u}_{1,3} = [\ \underbrace{1}_{=1-\mu_{i,p}}\ \underbrace{0}_{0°}\ \underbrace{0}_{90°}\ \underbrace{0}_{180°}\ \underbrace{0}_{270°}\ ]^T$$

$$\mathbf{u}_{2,3} = [\ \underbrace{0}_{=1-\mu_{2,3}}\ \underbrace{0}_{0°}\ \underbrace{0}_{90°}\ \underbrace{0}_{180°}\ \underbrace{1}_{\beta_{2,3}=270°}\ ]^T$$

$\mathbf{d}$ is given as $\mathbf{d} = [\,0\,,0\,,1\,,2\,,3\,]^T$. Then (17) for $i = 2, p = 3$ is,

$$\mathbf{d}^T \cdot \mathbf{u}_{2,3} + b_{2,3} = \mathbf{d}^T \cdot (-\mathbf{v}_2 + \mathbf{v}_3) + a_{2,3}2^2 \qquad (24)$$

In the above expression $\mathbf{d}^T \cdot (-\mathbf{v}_2 + \mathbf{v}_3)$ is negative and $a_{2,3}$ should be 1 in order to have the equality with the positive right hand side. Note that $b_{2,3} = 0$ in accordance with Table 1.

(17) for $i = 1, p = 2$ is given as,

$$\mathbf{d}^T \cdot \mathbf{u}_{1,2} + b_{1,2} = \mathbf{d}^T \cdot (-\mathbf{v}_1 + \mathbf{v}_2) + a_{1,2}2^2 \qquad (25)$$

$a_{1,2}$ and $b_{1,2}$ are selected appropriately by the optimization program in order to satisfy the equality.

5. SIMULATIONS

In this paper, "Gurobi" [14] which is an efficient mixed integer linear programming solver is used by employing branch and cut strategy. The evaluation of the proposed method is done for a uniform linear array (ULA) with M antennas where L elements are chosen at each case. For simplicity, $P_{an} = 1W$ is selected.

In the first experiment, $M = 8$ and $L = 4$ antennas are selected as in LTE standard [18]. Line of sight condition is assumed and there are $N = 4$ users at azimuth angles 50°, 90°, 120° and 150° respectively. The power proportions for each user are selected as $\gamma_1 = 1$, $\gamma_2 = 4$, $\gamma_3 = 1$, and $\gamma_4 = 2$ respectively. $n = 5$ bits are used for the discrete phase-only beamformer. Fig. 1 shows the beampatterns for the proposed discrete phase-only beamformer (DPOB) and the fixed ULA with 4 elements. Both of the DPOB's in Fig. 1 are optimum [12] but DPOB with antenna selection chooses the best 4-element subarray out of $M = 8$. The transmit power for each user is significantly improved as seen Fig. 1 for the antenna selection case.

In the second experiment, the azimuth angles for $N = 4$ users are selected randomly at each trial. In this case, $L = 4$ but M is changed as 8, 12 and 16 respectively. Brute force result is shown only for $M = 8$ due to computational complexity. As it is seen from the Fig. 2, proposed method and brute force are exactly the same since the method is optimum. It is also seen that $L = 4$ out of $M = 16$ antenna case gives the best result for $t = \min\{\frac{|\mathbf{w^H h_k}|^2}{\gamma_k}\} \le \frac{L^2}{\gamma_{max}} = \frac{16}{4} = 4$ and it is almost equal to the upper bound $t_{max} = 4$.

In the third experiment, Rayleigh channel model is assumed and the channels are selected randomly at each trial. Results of both DPOB with fixed array ($M = L = 4$) and antenna selection ($M = 8$, $L = 4$) are shown for different number of bit values, $n = 3, 4, 5$. Fig. 3 shows the minimum SNR for 100 different channel realizations. Minimum SNR obtained with antenna selection is higher than that of fixed array even with small number of bits.

Table 2 shows the computational complexity of the brute force and the proposed method where the average of 10 trials are reported. As it is seen from this table, the proposed optimum method has significantly lower complexity thanks to the efficiency of the mixed integer linear programming with branch and cut technique [14].

Table 2. Computational time of DPOB and brute force search

$L = 4$	$M = 8$	$M = 12$	$M = 16$
DPOB	7.96 s	12.18 s	46.91 s
BRUTE FORCE	88 s	430 s	1250 s

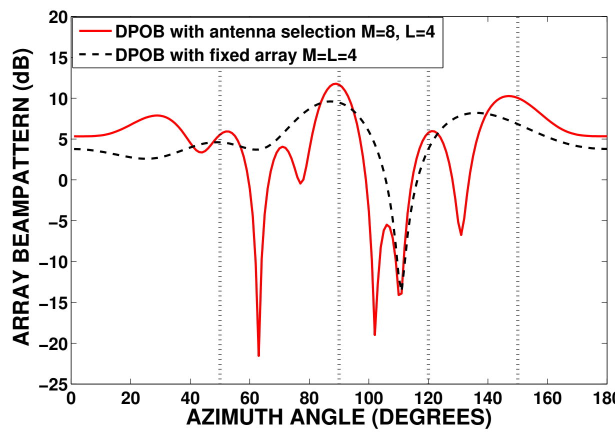

Fig. 1. Beampatterns of DPOB with antenna selection and fixed array (The dotted lines show the azimuth angles of the users).

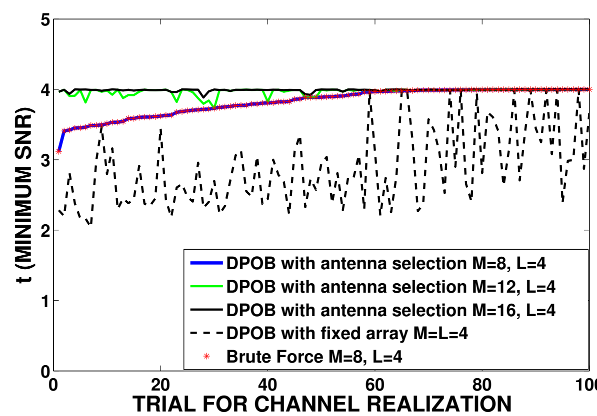

Fig. 2. Minimum SNR for 100 channel realizations for DPOB with antenna selection for different number of antennas and fixed array in line of sight condition with ULA.

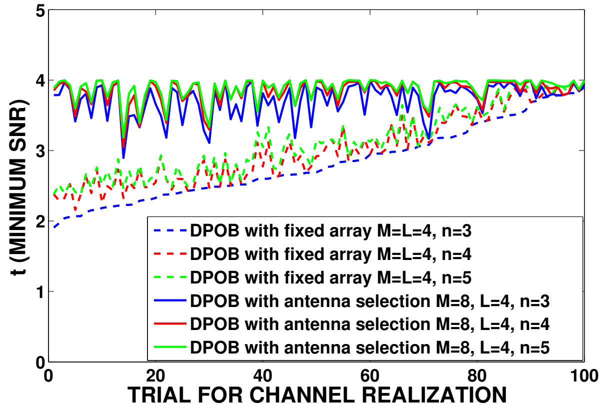

Fig. 3. Minimum SNR for 100 realizations of Rayleigh fading channels for DPOB with antenna selection and fixed array for different number of bits.

6. CONCLUSION

Single group multicast transmit beamformer design with antenna subarray selection problem is considered. A method for optimum discrete phase-only beamformer with antenna selection is presented. It is shown that the proposed method performs significantly better compared to optimum fixed array. Computational complexity is much better than the exhaustive search thanks to the efficiency of the selected algorithm for optimization.

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