FIR BAND-PASS DIGITAL DIFFERENTIATORS WITH FLAT PASSBAND AND EQUIRIPPLE STOPBAND CHARACTERISTICS

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ABSTRACT

Maximally flat digital differentiators are widely used as narrow-band digital differentiators because of their high accuracy around their center frequency of flat property. To obtain highly accurate differentiation over narrow-band, it is important to avoid the undesirable amplification of noise. In this paper, we introduce a design method of linear phase FIR band-pass differentiators with flat passband and equiripple stopband characteristics. The center frequency at the passband of the designed differentiators can be adjusted arbitrarily. Moreover, the proposed transfer function consists of two functions, i.e. the passband function and the stopband one. The weighting coefficients of the passband function are derived using a closed-form formula based on Jacobi Polynomial. The weighting coefficients of the stopband function are achieved using Remez algorithm.

***Index Terms*—** Digital differentiators, maximally flat, Remez algorithm, closed-form, Jacobi polynomial

1. INTRODUCTION

Digital differentiators (DDs) are widely used, like acceleration estimation in motion controllers [1], edge extraction in image processing [2], and so on. Various FIR approximations of the frequency response of DDs have been reported, which are based on the minimax relative error criterion [3,4] as well as the maximally flat criterion [5–10]. The minimax relative error design methods are highly suitable for wide-band DDs, while the maximally flat designs are particularly adaptable for narrow-band operations centered on a certain frequency. For example, in airborne Doppler navigation systems [11], it is necessary to perform differentiation around the frequency range from $\pi/12$ to $\pi/6$ with extremely high accuracy (relative error $\leq$ -140dB), typically. In such a situation, the minimax relative error designs for narrow-band would be computationally inefficient and uneconomical as compared to the maximally flat designs [5].

To obtain highly accurate differentiation over narrow-band, it is important to avoid undesirable amplification of noise. Several design methods have been proposed to avoid it. One of these methods provides the cascade of a digital filter for band limiting and a DDs. To maximize the stopband attenuation and to obtain the steep cutoff characteristic simultaneously, the digital filter for band limiting would be designed with the minimax relative error criterion. On the contrary, if we cascade the minimax relative error band limiting filter and the maximally flat differentiator (MFDDs), the accuracy of differentiation is reduced because of the ripple in the passband. To avoid the ripple in the passband, the design method of the low-pass MFDDs is proposed [8]. The transfer function of this low-pass MFDDs is provided as a closed form expression. Although the low-pass MFDDs avoid undesirable amplification of noise, the maximally flat designs can not achieve the steep cutoff characteristic. To solve this problem, the design method of low-pass DDs having flat monotonic passband and equiripple stopband have been proposed in [12]. This method provides highly accurate differentiation over narrow-band with the steep cutoff characteristic, however, it can not adjust the center frequency of passband expect the frequency 0.

In this paper, we describe a new design method of TYPE IV linear phase FIR band-pass DDs with flat passband and equiripple stopband (MF-ERDDs). Using a DD with rippleless passband, you can reduce the distortion of the signal caused by the DD. On the other hand, using a DD with an equiripple stopband, you can maximize the stopband attenuation and obtain the steep cutoff characteristic. Furthermore, the proposed method can adjust the center frequency of passband. Hence, the proposed method can perform efficient differentiation according to applications.

2. THE DESIGN ALGORITHM

In this section, we show a design method of the TYPE IV band-pass MF-ERDDs.

DDs are designed as one of linear phase digital FIR filters. Generally, linear phase digital FIR filters are classified into four types [13]. These four types differ in order (even or odd) and the symmetry of the impulse response (symmetry or antisymmetry). Due to purely imaginary frequency response, DDs can only be designed by antisymmetric impulse response sequences, i.e. $h(n) = -h(N-1-n)$, where $h(n)$ are the filter coefficients and N is the filter order. Consequently, the frequency response of DDs is given by

$$H(e^{j\omega}) = e^{j(\pi/2-\omega N/2)} H_0(\omega), \tag{1}$$

where $H_0(\omega)$ is a real function of ω given by

$$H_0(\omega) = \sum_{n=0}^{(N-1)/2} \tilde{h}(n) \sin\left\{\left(n+\frac{1}{2}\right)\omega\right\},\ N \text{ odd}, \tag{2}$$

and

$$\tilde{h}(n) = 2h\left(\frac{N-1}{2} - n\right), \quad 0 \leq n \leq \tfrac{N-1}{2}. \tag{3}$$

The frequency response of the ideal (full-band) DD is described as

$$H_F\left(e^{j\omega}\right) = e^{j\pi/2}\omega, \ \mid \omega \mid < \pi, \tag{4}$$

while the frequency response of the ideal band-pass MF-ERDD is described as

$$H_{BP}\left(e^{j\omega}\right) = \begin{cases} 0, & \mid \omega \mid < \omega_L \\ j\omega, & \omega_L < \mid \omega \mid < \omega_H \\ 0, & \omega_H < \mid \omega \mid < \pi. \end{cases} \tag{5}$$

Here ω_L and ω_H are lower and higher cutoff frequency, respectively. From (5), we have the constraints for the passband of the proposed band-pass MF-ERDD given as

$$H_0(\omega)|_{\omega=\omega_0} = \omega_0 \tag{6a}$$

$$\left.\frac{dH_0(\omega)}{d\omega}\right|_{\omega=\omega_0} = 1 \tag{6b}$$

$$\left.\frac{d^n H_0(\omega)}{d\omega^n}\right|_{\omega=\omega_0} = 0 \quad n = 2, 3, \cdots, L, \tag{6c}$$

where L is the degree of flatness for ω_0 and an integer. In the proposed method, we give the function of TYPE IV MF-ERDDs as

$$\begin{aligned} H_0(\omega) &= \arccos(-x) - (1+x)^{1/2}(x-x_0)^{L+1} H_s(x) \\ &= (1+x)^{1/2}\left\{\frac{\arccos(-x)}{(1+x)^{1/2}} - (x-x_0)^{L+1} H_s(x)\right\}, \end{aligned} \tag{7}$$

where

$$\begin{cases} x = -\cos\omega, & 0 \leq \omega \leq \pi \\ x_0 = -\cos\omega_0, & 0 \leq \omega_0 < \pi, \end{cases} \tag{8}$$

and $H_s(x)$ is the correction function to make $H_0(e^{j\omega})$ having equiripple stopband.

To design $H_0(e^{j\omega})$ satisfying (6), $\arccos(-x)/(1+x)^{1/2}$ must have the degree of flatness at least L. Then, we consider the following power series expansion:

$$\frac{\arccos(-x)}{(1+x)^{1/2}} = \sum_{n=0}^{\infty} t_n(x_0)(x-x_0)^n, \tag{9}$$

where

$$\begin{aligned} & t_n(x_0) \\ &= \left.\frac{1}{n!}\frac{d^n}{dx^n}\frac{\arccos(-x)}{(1+x)^{1/2}}\right|_{x=x_0} \\ &= \left.\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{d^i}{dx^i}\arccos(-x)\frac{d^{n-i}}{dx^{n-i}}(1+x)^{-1/2}\right|_{x=x_0} \\ &= \left.\arccos(-x)\frac{1}{n!}\frac{d^n}{dx^n}(1+x)^{-1/2}\right|_{x=x_0} \\ &\quad + \frac{1}{n!}\sum_{i=1}^{n}\binom{n}{i}\left\{\frac{d^{i-1}}{dx^{i-1}}(1-x)^{-1/2}(1+x)^{-1/2}\right. \\ &\quad \left.\left.\cdot\ \frac{d^{n-i}}{dx^{n-i}}(1+x)^{-1/2}\right\}\right|_{x=x_0}. \end{aligned} \tag{10}$$

Jacobi polynomial [14] is given as

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= (-2)^n(1-x)^{-\alpha}(1+x)^{-\beta}\frac{1}{n!}\frac{d^n}{dx^n}(1-x)^{\alpha+n}(1+x)^{\beta+n} \\ &= 2^{-n}\sum_{v=0}^{n}\binom{n+\alpha}{v}\binom{n+\beta}{n-v}(x-1)^{n-v}(x+1)^v. \end{aligned} \tag{11}$$

By using (11), we define $q_n^{\kappa,\mu}(x_0)$ as

$$\begin{aligned} q_n^{\kappa,\mu}(x_0) &= (-2)^n(1-x_0)^{-\kappa-n}(1+x_0)^{-\mu-n} \\ &\quad \cdot P_n^{-\kappa-n,-\mu-n}(x_0). \end{aligned} \tag{12}$$

Then, (9) can be denoted as

$$\frac{\arccos(-x)}{(1+x)^{1/2}} = \sum_{n=0}^{\infty} t_n(x_0)(x-x_0)^n \tag{13}$$

$$t_n(x_0) = \sum_{i=0}^{n} r_i(x_0) q_{n-i}^{0,1/2}(x_0) \tag{14}$$

$$r_i(x_0) = \begin{cases} \arccos(-x_0), & i = 0 \\ 1/i q_{i-1}^{1/2,1/2}(x_0), & \text{other}. \end{cases} \tag{15}$$

Because (13) is decomposed as

$$\frac{\arccos(-x)}{(1+x)^{1/2}}=\sum_{n=0}^{L} t_n(x_0)(x-x_0)^n + \sum_{n=L+1}^{\infty} t_n(x_0)(x-x_0)^n = \sum_{n=0}^{L} t_n(x_0)(x-x_0)^n + (x-x_0)^{L+1} \cdot \sum_{n=0}^{\infty} t_{n+L+1}(x_0)(x-x_0)^n, \quad (16)$$

the minimal degree of $\arccos(-x)/(1+x)^{1/2}$ to satisfy (6) is L. Hence, (7) is rewritten as

$$H_0(\omega)=(1+x)^{1/2}\left\{T_L(x)-(x-x_0)^{L+1}H_s(x)\right\} \quad (17)$$

$$T_L(x)=\sum_{n=0}^{L} t_n(x_0)(x-x_0)^n \quad (18)$$

$$H_s(x)=\sum_{n=0}^{N_s} h_s(n)x^n, \quad (19)$$

where $h_s(n)$ is the coefficient of $H_s(x)$. The filter order N of the proposed MF-ERDDs is given as

$$N = 2\left(L + \frac{3}{2} + N_s\right), \quad (20)$$

where N_s is the order of $H_s(x)$ and L is an odd integer since $H_0(e^{j\omega})$ with the band-pass property should be convex at ω_0.

Now, we introduce the weighted error function $E(x)$ defined as

$$E(x) = W(x) \mid D(x) - H_0(x) \mid . \quad (21)$$

In the proposed method, the estimation of $E(x)$ is done only for stopband so that $D(x)$ for stopband is set to 0. By substituting (17) into (21), $E(x)$ can be rearranged as

$$\tilde{E}(x) = \tilde{W}(x) \mid \tilde{D}(x) - H_s(x) \mid, \quad (22)$$

where

$$\tilde{W}(x) = (1+x)^{1/2}(x-x_0)^{L+1}W(x) \quad (23)$$

$$\tilde{D}(x) = \frac{T_L(x)}{(x-x_0)^{L+1}}. \quad (24)$$

With

$$\boldsymbol{h}_s = [h_s(0)\ h_s(1)\ \cdots\ h_s(N_s)]^{\mathrm{T}},$$

and X_s which is a set of x in the stopband, then we have the following min-max design problem of $\boldsymbol{h}_s$

$$\underset{\boldsymbol{h}_s}{\text{minimize}}\ [\underset{x\in X_s}{\text{maximize}}\ \tilde{E}(x)]. \quad (25)$$

By using Remez algorithm, this design problem is solved. Then, the equiripple stopband is achieved.

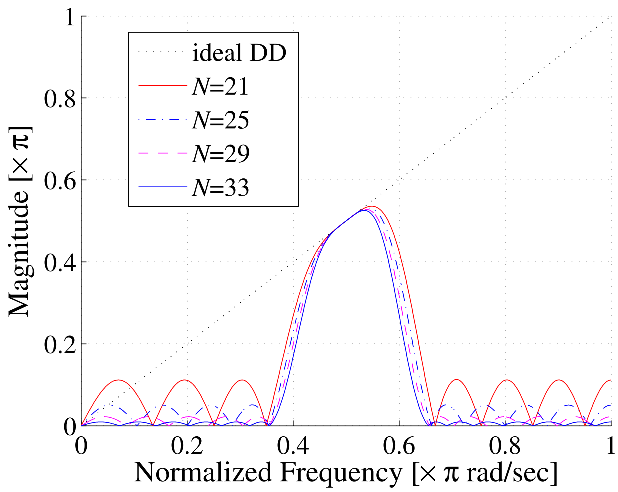

Fig. 1. The design examples with different N

Table 1. The minimum stopband attenuation of Fig. 1

N	21	25	29	33
Atten.[dB]	-9.00	-15.88	-23.10	-30.43

3. THE DESIGN EXAMPLES

In this section, we will illustrate some magnitude responses of the TYPE IV band-pass MF-ERDDs designed by the proposed method. The parameters of the proposed method are the filter order N, the normalized center angular frequency ω_0, the degree of flatness L, the stopband normalized edge angular frequencies $\boldsymbol{\omega}_s = (\omega_{s1}, \omega_{s2})$, and the weight function $W(\omega)$. Note that in this section, $W(\omega)$ is set as $W(\omega) = 1$.

Example 1: In this example, we illustrate the relation between N and the stopband attenuation. Fig. 1 shows the magnitude response for a family of the band-pass MF-ERDDs with $\omega_0 = 0.5\pi$, $L = 3$ and $\boldsymbol{\omega}_s = (0.35\pi, 0.65\pi)$, where N is varied from 21 to 33 in increments of 4. The minimum stopband attenuation of each MF-ERDDs are shown in table 1. From Fig. 1, we confirm that the proposed method can design DDs having maximally flat passband and equiripple stopband. It is also seem from Fig. 1 and table 1 that the larger stopband attenuation is achieved by increasing the filter order N with fixed L. Moreover, it is seen from Fig. 1 that the transition bands become steeper by increasing N.

Example 2: In this example, we illustrate to realize the band-pass MF-ERDDs with an arbitrary center frequency. Fig. 2 shows the magnitude responses for a family of the band-pass MF-ERDDs with $N = 41$ and $L = 3$, where ω_0 is varied from 0.2π to 0.8π in increments of 0.2π. According to ω_0, we set $\boldsymbol{\omega}_s$ as $(0.05\pi, 0.35\pi)$, $(0.25\pi, 0.55\pi)$, $(0.45\pi, 0.75\pi)$, $(0.65\pi, 0.95\pi)$. Note that the bandwidth between ω_{s1} and ω_{s2} is not changed. From Fig. 2, we con-

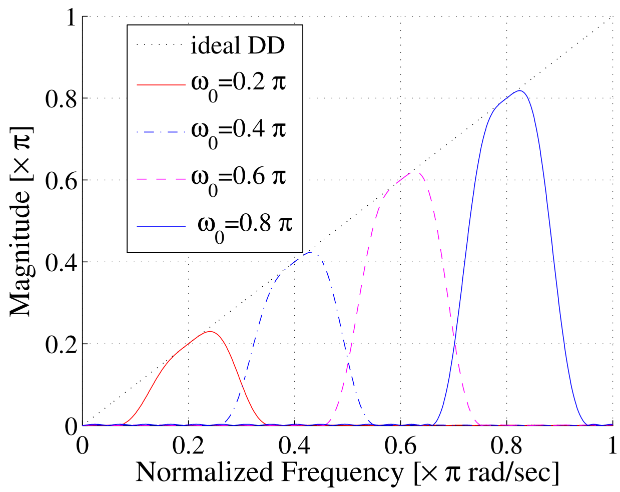

Fig. 2. The design examples with different ω_0

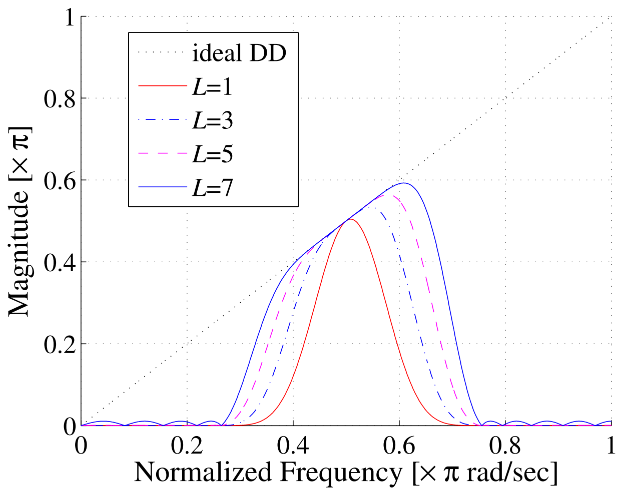

Fig. 3. The design examples with different L

Table 2. The minimum stopband attenuation of Fig. 2

ω_0	0.2π	0.4π	0.6π	0.8π
Atten.[dB]	-48.12	-46.10	-45.45	-39.62

Table 3. The minimum stopband attenuation of Fig. 3

L	1	3	5	7
Atten.[dB]	-108.61	-76.20	-50.12	-29.20

firm that the proposed transfer function can adjust the center frequency arbitrarily. The minimum stopband attenuation of each MF-ERDDs are shown in table 2. From table 2, the minimum stopband attenuation becomes a little smaller with setting ω_0 as a high frequency. Hence, when you set ω_0 as a high frequency, N should be large compared with MF-ERDDs whose ω_0 is set as a low frequency.

Example 3: In this example, we illustrate the relation between L and the bandwidth having maximally flat property (flat bandwidth). Fig. 3 shows the magnitude response for a family of the band-pass MF-ERDDs with $N = 33$, $\omega_0 = 0.5\pi$ and $\boldsymbol{\omega}_s = (0.25\pi, 0.75\pi)$, where L is varied from 1 to 7 in increments of 2. From Fig. 3, the flat bandwidth of the band-pass MF-ERDDs is widened with increasing value of L. The minimum stopband attenuations of each MF-ERDDs are shown in table 3. As shown in table 3, the minimum stopband attenuation becomes smaller with increasing value of L because of fixed N and $\boldsymbol{\omega}_s$.

4. CONCLUSION

In this paper, we presented a design method of TYPE I V linear phase FIR band-pass digital differentiators having maximally flat passband and equiripple stopband (MF-ERDDs). For passband, MF-ERDDs can reduce the distortion of the signal caused by DDs. On the other hand, for stop band, MF-ERDDs can maximize the stopband attenuation and achieve the steep cutoff characteristic. It is most important that the proposed method can adjust the arbitrary center frequency of passband. Hence, the proposed method can perform efficient differentiation according to applications. To design the band-pass MF-ERDDs, we indicated the function of MF-ERDDs which satisfy the constraint of maximally flat passband at an arbitrary frequency. Then, using Remez algorithm, the coefficient of the correction function is achieved, which realizes the equiripple stopband. The parameters of the proposed method are the filter order, the normalized center frequency of passband, the stopband edge frequencies and the weight function. Finally, through some design examples, we confirm the relation between the magnitude response of MF-ERDDs and the parameters.

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