A LOWER BOUND FOR PASSIVE SENSOR-NETWORK AUTO-LOCALIZATION

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ABSTRACT

In this paper a lower-bound for estimation of inter-sensor propagation delay using sources of opportunity is presented. This approach is referred to as passive identification. It relies on Ward identity, which is extended to the case of non white sources. Performances are studied in the case of an homogeneous non dispersive linear and time invariant wave propagation medium, under the assumption that many independent sources impige on the sensor array.

***Index Terms*—** Passive sensor network autolocalization, variance lower bound, Ward identity.

1. PASSIVE TRAVEL-TIME ESTIMATION

Estimating the geometry of a sensor network can be approached by inversing intersensor-distance matrices. Those distances can be assessed from the local speed of sound and the intersensor propagation delays in a non dispersive medium [7, 20].

We aim at providing a study of the performances of passive estimation of those propagation delays. The framework will be restrained to a single pair of sensors embedded in the propagation medium. The term passive here means that sources in the medium do exist but are uncontrolled. The combination of ergodicity with such sources allows the computation of *noise correlations* [5], which in turn allows to derive the Green function between the two sensors using generalized Ward identities. The propagation delay is eventually estimated as the argument that maximizes the retrieved Green function. Domains of applications are broad and include seismology [4, 6, 11, 12], acoustic [5, 10], structural analysis [13] and electromagnetism [6].

The approach raises challenging signal processing issues. In particular, given that the sources are uncontrolled, they can be assumed to be random processes with unknown temporal and spatial spectral properties. Performances of estimation are then to be investigated. Equivalent areas of the concept are *output-only* approaches in mechanical structure analysis or *seismic interferometry* in acoustics and seismology.

One may therefore recall that identification of linear systems is achievable through the knowledge of second order statistics of the inputs and those of corresponding outputs. This reasoning is appliable to our framework, where information on the medium is indeed contained in the statistics of the sensed fields. 2^{nd} order statistics theoretically suffice to achieve identification.

This approach leads to combine 2^{nd} order statistics of the sensed fields with the dispersion relation of the medium, which is referred to as the Ward identity (9). Such combination provides a Green's function estimate from the measures [21]. It is important to point out that most of the descriptions of this problem were derived with *ambient "noise"* as sources: white noise sources in the sense that they are spatially and temporally uncorrelated. In practice, this assumption is of course never met due to the non existence of such perfect natural sources or even for the band-limited measurement devices. However, feasability of the approach is unarguable as the protocol has repeatedly proven efficient and this is why we believe it is of importance to study the passive estimation performances given a relaxed assumption on the sources distribution, thus introducing non white source distributions.

The closest investigations to our purpose are [15], studying the impact of instrumentation time shifts on travel-time retrieval and [16, 17], studying the impact of the spatial distribution of sources on the Green function retrieval for a free acoustic propagation medium. However, up to our knowledge, no lower bound for passive propagation delay estimation has never been proposed.

First, an extension of Ward identity to non white sources is presented in section 2. A lower bound for the variance of the presented passive propagation delay estimator is then presented in section 3, which takes the same form as the active delay estimation Cramer-Rao Lower Bound. Results are illustrated with a simulation.

2. ON PASSIVE AUTO-LOCALIZATION

In this section, an overview of passive propagation delay estimation protocol using Ward identity is presented. Ward identity is recalled and derived for non white sources. The section ends with the passive propagation delay estimator expression.

2.1. From the propagation equation to Green's function

Classical physics allows to describe interactions between a source and a resulting field with propagation equations:

$$\mathcal{A}f(t,\vec{x}) = \left[\frac{\partial^2}{\partial t^2} + \mathcal{D}\frac{\partial}{\partial t} + \mathcal{L}\right] f(t,\vec{x}) = s(t,\vec{x}) \quad (1)$$

Equation (1) is a general instance of wave propagation equation in a medium that is or can be approximated as homogeneous, isotropic and non dispersive. $s(t,\vec{x})$ is the source and $f(t,\vec{x})$ is the resulting field. $\mathcal{L}$ is a propagation operator and $\mathcal{D}$ is a dissipation operator. Both of them are spatial operators and in the case of a visco-acoustic medium, they are proportional to the 3D-Laplacian operator [19]. Equation (1) can be used to describe mechanical structures with finite number of degrees of freedom, vibrating strings and membranes, beams and plates, acoustic and electromagnetic fields in fluids and elastic propagation media.

The Green function $G(t,\vec{x}|t_0,\vec{y})$ is the impulse response for a wave propagation medium $\mathcal{X}$. It stands for the response sensed at $\vec{x}$ at time t, when a pulse is triggered from $\vec{y}$ at time t_0. When (1) is invertible, its kernel turns out to be Green's function. Green's' function indeed allows to fully characterize the medium $\mathcal{X}$, and expresses the link between the source and the field as a filtering process:

$$f(t,\vec{p}) = \int G(t,\vec{p}|t',\vec{x})s(t',\vec{x})dt'd\vec{x} = (G \underset{T,S}{\otimes} s)(t,\vec{p}) \quad (2)$$

where $\otimes_T$ denotes the classical time convolution and $\otimes_S$ denotes the generalized spatial convolution.

2.2. Green's correlation and Ward identity

In the passive context, sources are unknown stochastic processes in time and space. While propagating, their wavefronts are shaped by the medium; filtering induces information on the medium which remains present in the field. Techniques have therefore been developped to study n^{th} order statistics of signals; cyclostationarity or high order statistics are examples of tools used *e.g.* on telegraph waves, or in 'blind channel equalization' [22]. Here however the approach relies on Ward identity for which the framework focuses on 2^{nd} order statistics only. Let $\gamma_f(\tau,\vec{x},\vec{y})$ be the cross-correlation of a stochastic and stationary field f measured at two points $\vec{x}$ and $\vec{y}$:

$$\gamma_f(\tau,\vec{x},\vec{y}) = \mathbb{E}\left[f(u,\vec{x})f(u+\tau,\vec{y})\right] \quad (3)$$

By combining Equation (2) and (3), a generalized interference formula is derived to relate the field f generated by a source in $\vec{s}$, to the signals sensed at two locations $\vec{x}$ and $\vec{y}$:

$$\gamma_f(\tau) = \left(G(t,\vec{x}|0,\vec{s}) \underset{T,S}{\otimes} \gamma_s(t,\vec{s},\vec{s}) \underset{T,S}{\otimes} G(0,\vec{s}|t,\vec{y})\right)(\tau) \quad (4)$$

To assess the effect of non white sources on the Ward identity, we restrict the scope of the study to the case where only one wave vector impiges on the 'array' formed by the sensors. This is case for *e.g.* a point source placed at large distances from the sensors. In the frequency domain, for any function z, let $\tilde{z}(\omega,\vec{k}) = \int_{\mathbb{R},\mathcal{X}} z(t,\vec{s})e^{-j(\omega t-\vec{k}\cdot\vec{s})}dtd\vec{s}$ be its Fourier transform in both time and space. Note that $\widetilde{G}(\omega,\vec{k})$ is the spatio-temporal Fourier transform of the solution of (1) when $s(t,\vec{x}) = \delta(t-t_0,\vec{x}-\vec{x_0})$.

The correlation γ_G of two measures of the field is a function of the source auto-correlation function, that comes as a general interference formula:

$$|\widetilde{A}(\omega,\vec{k})|^2\widetilde{\gamma}_G(\omega,\vec{k}) = \widetilde{\gamma}_s(\omega,\vec{k}) \quad (5)$$

which can also write:

$$\widetilde{\mathcal{A}}(\omega,\vec{k})^*\widetilde{\gamma}_G(\omega,\vec{k}) = \widetilde{G}(\omega,\vec{k})\widetilde{\gamma}_s(\omega,\vec{k}) \quad (6)$$

Green's correlation emerges from (6) under the assumption of white sources, which was conducted in [12, 18, 21]. The key to access the Ward identity from (6) is the dispersion relation extracted from (1):

$$\omega^2 + j\eta\omega^\alpha - v^2k^2 = 0 \quad (7)$$

where α,η account for the dissipation model. In geophysics, the signals propagating over hectokilometric distances are often spectrally contained in $\Delta f < 100$ Hz and a constant dissipation model can be assessed. On the contrary in more general viscoelastic propagation (including short distances), $\mathcal{D} \propto \mathcal{L}$. Both ways, this leads to the Ward identity (9), relating more explicitly Green's correlation function to Green's function. The Ward identity was shown in [18, 21]. Here we present it for non white sources. From (6) and (7) and knowing that $\widetilde{\gamma}_s(\omega,\vec{k})$ is real, the imaginary part of (6) writes:

$$\eta\omega^\alpha\widetilde{\gamma}_G(\omega,\vec{k}) = \widetilde{\gamma}_s(\omega,\vec{k}) \times \mathrm{Im}\left[\widetilde{G}(\omega,\vec{k})\right] \quad (8)$$

Equation (8) allows to derive a Ward identity in a boundary free medium for non white sources:

$$\frac{\partial^\alpha}{\partial t^\alpha}\gamma_G(t,\vec{x},\vec{y}) = \frac{1}{\eta}\gamma_s(t,\vec{s}) \underset{T,S}{\otimes} \mathrm{Odd}\left[G(t,\vec{x},\vec{y})\right] \quad (9)$$

In this expression, it was noted in [12, 18] that the dissipation operator appears necessary for passive Green function retrieval. If $\mathcal{D}$ is null, η equals zero, which prevents to retrieve the Green function from (9). Note the odd part of the Green function suffices to reconstruct the function.

From (8) and (9), we can observe that the spectral context of the signals of interest depend on both the frequency spectrum of the source and the transfer function of the propagation medium. Furthermore, the physical nature of this latter may lead to quite significant modifications of the relative importance of $\mathcal{D}$ and $\mathcal{L}$.

2.3. Parameter extraction from the Green function

When the Green function satisfies (1) (*e.g.* electromagnetic waves in air, acoustic waves in fluids, P and S seismic waves in homogeneous media), the estimated travel-time $\hat{\tau}(\vec{x},\vec{y})$ between the sensors located at $\vec{x}$ and $\vec{y}$ is the argmax of the amplitude of the *estimated* Green function.

For example, for an homogeneous and isotropic boundary free and dissipative acoustic propagation medium, the studied field is the pressure field. In general, for a low dissipation case and unbounded medium, the Green function is shown to express as [20]:

$$G(t,\vec{y}|t_0,\vec{x}) \approx \frac{\exp(-a(t-t_0))}{4\pi d(\vec{y},\vec{x})}\delta(t-t_0-\tau(\vec{x},\vec{y})) \quad (10)$$

where a is the damping factor, d the intersensor distance.

Note that a bounded medium will exhibit a series of attenuated delays. Again here, we must emphasize that the time delay is nothing bu the distance $||\vec{x}-\vec{y}||$ divided by the velocity of the wave (assumed constant). Given that attenuation gets stronger with time, it makes sense to design the passive travel-time estimator noted $\widehat{\tau}$ as:

$$\widehat{\tau}(\vec{x},\vec{y}) = \arg\max_t \ \widehat{G}(\vec{y},t|\vec{x}) \quad (11)$$

where $\widehat{G}$ is an estimate of the Green function.

3. A VARIANCE LOWER BOUND

Feasability of passive identification using Ward identity is unarguable and yet performances are seldom studied. In this section, a lower bound for the variance of the passive travel-time estimator $\widehat{\tau}$ is presented. To that purpose, we consider the simple case of a set of independent plane waves imping-ing on the sensors set. Note that this assumption corresponds to the case where the distance source-sensors is much larger than the inter-sensor distance and to a set of independent point sources. Discussing the consequences of a departure from this model is deferred to a further communication.

To account for spatial diversity in an accurate and easy manner, let us work under the assumption of plane wave sources that will be described by their temporal auto-correlation function $\gamma_{s_i}(t)$ and incidence angle θ_i. Spatial diversity is attained by the existence of a distribution of incidence angles whereas the temporal auto-correlation function $\gamma_s(t)$ of the source informs on the temporal statistical properties.

Suppose that a single source with incidence angle noted θ_i be selected. A classical estimator for the propagation delay for this special case is:

$$\widehat{\tau}(\vec{x},\vec{y}) = \frac{1}{\cos(\theta_i)} \times \arg\max_t |\gamma_{x^i,y^i}(t)| \times \Pi(t \geq 0) \quad (12)$$

where $x^i(t) = y^i(t-\Delta_i/v)$ and $\Pi(t \geq 0)$ is the gate function.

In an active context, travel time estimation relies on a controlled transmitter and receiver. Received signal is comonly designed as a delayed attenuated version of the emitted signal, corrupted by an additive measurement noise. Estimation theory allows to derive a Cramer-Rao lower bound $\sigma^2_{CRLB}(\tau)$ on the active travel time estimator [8], of the form:

$$\sigma^2_{CRLB}(\tau) = \left[\ \mathrm{SNR} \times \bar{F}^2\ \right]^{-1} \quad (13)$$

where SNR is the signal-to-noise ratio and $\bar{F}$ measures the bandwidth of the signal. The problem is fundamentally identical to ours, as it consists in the detection of a maximum amplitude of a correlation function computed from noisy data.

In our case, the estimator relies on unknown multiple incidence angles, each of which being associated to a multiplicative bias in the estimation. Furthermore, in our passive context, the use of SNR deserves clarificaton. 'Ambient noise' being the signal of interest, it is not to be mistaken with any part of the signal that does not carry information useful to the inter-sensor distance estimation. A physical discrimination between those two noises would be that ambient noise is sensed by both sensors whereas the other one is an additive contribution visible by either one of them, *e.g.* measurement noise. Whilst normally presented in the case of active identification, this concept is here extended to the passive case.

3.1. Information carried by one source

From (11), the finite precision of the measure is embodied by the deterioration of the path delay difference estimator $\widehat{\Delta}_i$ by an additive zero-mean Gaussian noise with variance σ^2_{CRLB} suggested from (13). The inter-sensor travel-time estimator, for a single source, along the optical path is biased by a multiplicative as expressed in Equation (12). Assuming normal distribution for the estimation error, we obtain from Equations (12) and (13):

$$\widehat{\tau}(\vec{x},\vec{y}) \sim \mathcal{N}\Big(\frac{\widehat{\Delta}_i}{v\cos(\theta_i)}, \frac{\sigma^2_{CRLB}}{\cos^2(\theta_i)}\Big) \quad (14)$$

Note that in this formula, the bias has been taken into account, although in a passive context θ_i is usually unknown. From (14), the Fisher information associated to that single source can be expressed as:

$$\mathcal{I}(\theta_i) = \frac{\cos^2(\theta_i)}{\sigma^2_{CRLB}} \quad (15)$$

Note that (15) allows to identify symetries and invariances as $\mathcal{I}(\theta_i) = \mathcal{I}(\pi-\theta_i) = \mathcal{I}(-\theta_i)$ which enables to pick θ_i in $[0;\pi/2]$ only. The information carried by a source i whose incidence vector $\vec{k}$ statisfies $\langle \vec{x}-\vec{y}, \vec{k}(\theta_i)\rangle = 0$, appears to be zero. On the other hand, the variance tends to its minimum σ^2_{CRLB} as the source and the sensor array are aligned.

3.2. Extension to a set of sources

The passive approach leads to have no prior on θ, leaving an unsolvable bias to the travel time estimate, from (14). However, the passive identification protocol consists in accounting for the contributions of a set of sources with random incidence angles (or multipath propagation) providing a wide angle of arrival diversity. Consequently, let $\Theta = \{\theta_i, i = 1 \dots N\}$ be a set of N incidence angles. It is a random sequence with values in $\mathcal{S} \subset [0; \pi/2]$ for reasons explaned above. Take a set of i.i.d. sources with incidence angles Θ and temporal spectral bandwidth $\bar{F}^2$. Furthermore, their contribution to identification will be assessed of identical weight through identical SNRs at receivers. Spatial information brought by each source consequently adds up to form the total Fisher information $\mathcal{I}(\Theta)$:

$$\mathcal{I}(\Theta) = \frac{1}{\sigma^2_{CRLB}} \sum_{\theta \in \Theta} \cos^2(\theta) \tag{16}$$

The variance due to the spatial distribution of N sources with incidence angles in $\mathcal{S}$ is here exhibited with continuous sums, using the law of large numbers on the set Θ with a Monte-Carlo process for the computation of $\int_{\mathcal{S}} \cos^2(\theta) d\theta$:

$$\sigma^2_{\mathcal{S}} = \mathcal{I}^{-1}_{\mathcal{S}} = \sigma^2_{CRLB} \frac{\mu(\mathcal{S})}{N \int_{\mathcal{S}} \cos^2(\theta) d\theta} \tag{17}$$

where $\mu(\mathcal{S})$ measures the volume of the support $\mathcal{S}$. The angular bandwidth as it is presented in [8] can then be extended to the spatial dimension. $\sigma^2_{\mathcal{S}}$ can then be written as a function of the spatial angular bandwidth $\bar{\beta}^2$:

$$\bar{\beta}^2 = \frac{-1}{\mu(\mathcal{S})} \frac{\partial^2}{\partial \zeta^2} \Big[\int_{\mathcal{S}} \cos(\theta) \cos(\theta - \zeta) d\theta \Big]_{\zeta = 0} \tag{18}$$

Eventually, by combining Equations (17) and (18), a lower bound on the variance of passive propagation delay estimation writes:

$$\sigma^2_{passive}(\hat{\tau}) = \frac{1}{N} \frac{1}{\text{SNR}} \frac{1}{\bar{F}^2} \frac{1}{\bar{\beta}^2} \tag{19}$$

where β and F respectively relate to spatial and temporal bandwidths. N is the number of sources. As a consequence, (19) writes as an extension the active CRLB (13).

3.3. Illustration : narrow band signals

In this simulation, passive distance estimation is performed for a set of temporally narrow band sources with uniform spatial distribution. In this configuration, Ward identity (9) applies. The simulated medium is 3-dimensional, boundary free and the Green function satisfies (10):

$$G(t, \vec{y}|0, \vec{x}) = \frac{\exp(-at)}{4\pi d(\vec{x}, \vec{y})} \text{sinc}\big(2\pi f_s (t - \tau(\vec{x}, \vec{y}))\big)$$

where $f_s = 1Hz$. The damping is here assumed constant in the sense that $\mathcal{D} = a = 10$.

Sensors are $200u$ distant[1] from each other while the waves propagates with a velocity of $34u/sec$. The sampling frequency is $100Hz$. The fields recorded at each sensor are used to compute the Green correlation ; records of length 100 times longer than the travel time between the sensors are used for the estimation. The Green correlation is averaged on all 2048 sources and then numerically time-differenciated.

Firstly, sources are white Gaussian i.i.d. random processes with unit variance. They are sequentially lowpass filtered with cut-off frequencies f_c ranging from f_s to $0.01 f_s$. SNR was set to $50dB$; it was chosen considering that only low electronic noise is spoiling the measurement. Furthermore, sources are uniformely spatially distributed on a sphere around the sensors with radius $1000u$: their contributions to passive identification is thus approximatively equivalent in the sense that have similar energy at the sensors. TOA estimation was averaged over 250 iterations.

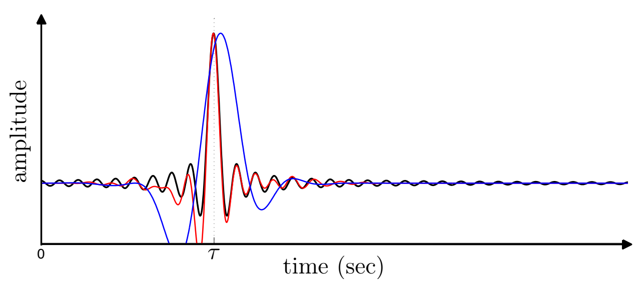

Fig. 1. Passive Green functions estimates with narrowed time spectral sources psd. The black line shows the theoretical Green function G_{th}. The red line shows the result of passive identification when $f_c = f_s$. The blue line shows the result when $f_c = 0.5 f_s$.

Results are displayed in fig. 1 and 2. Although estimation is bandlimited and thus less accurate, the argmax of the amplitude still holds as a TOA estimator. The lower curvature however induces a higher variance in the estimation. The existence of a bias will be investigated in further work.

4. CONCLUSIONS

In this paper, we presented an extension of the Ward identity for non white sources. The ward identity leads to reconstruct a Green function on a bandwidth that is the bandwidth of the sources. A lower bound for the variance of the passive travel-time estimator was then derived in a plane wave sources context. This expression allows to propose an analytical expression of the minimal attainable variance when the sources bandwidth shrinks. The signal to noise ratio that

[1] u is an arbitrary unit of distance.

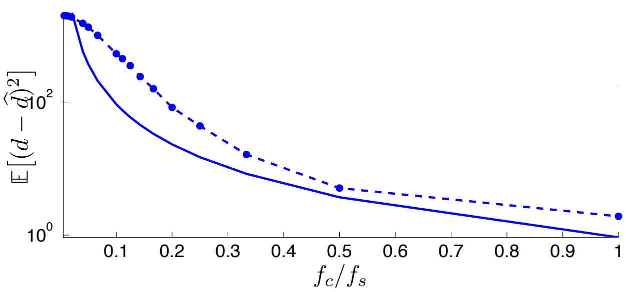

Fig. 2. (dotted line) Mean squared error of the passive distance estimation as a function of the bandwidth of the sensors. The normalized bandwidth of the sensors is represented on the x-axis as the ratio f_c/f_s. The mean squared error increases as the spectral bandwidth decreases. (continuous line) Theoretical variance of the estimator for the corresponding setup. $\mu(\mathcal{S}) = \pi/2$ so $\bar{\beta}^2 = 1/2$ from Equation (18). Eventually, $\bar{F}^2$ is equal to $3/fc^2$.

appears in the bound accounts for the transformation of the Ward identity in a numerical filtering scheme.

Upcoming work will focus on a geometrical formulation of the information carried by a set of sources in any Franhauffer propagation regime. The results shown in this paper offer a possible framework for studying the reliability of autolocalization of large sensor networks from the estimation of some distances between sensor pairs. Such extension can now be tackled on both theoretical and practical (real or simulated) experiments.

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