

# HYBRID BAYESIAN VARIATIONAL SCHEME TO HANDLE PARAMETER SELECTION IN TOTAL VARIATION SIGNAL DENOISING

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## ABSTRACT

Change-point detection problems can be solved either by variational approaches based on total variation or by Bayesian procedures. The former class leads to small computational time but requires the choice of a regularization parameter that significantly impacts the achieved solution and whose automated selection remains a challenging problem. Bayesian strategies avoid this regularization parameter selection, at the price of high computational costs. In this contribution, we propose a hybrid Bayesian variational procedure that relies on the use of a hierarchical Bayesian model while preserving the computational efficiency of total variation optimization procedures. Behavior and performance of the proposed method compare favorably against those of a fully Bayesian approach, both in terms of accuracy and of computational time. Additionally, estimation performance are compared to the Stein unbiased risk estimate, for which the knowledge of the noise variance is needed.

**Index Terms**— Parameter selection, total variation, convex optimization, hierarchical Bayesian model.

## 1. INTRODUCTION

Change-point detection problems are of considerable potential interest in many different applications ranging e.g. from econometrics to signal processing (see [1, 2], for an overview). Formally, a change-point detection problem consists in estimating a piecewise constant signal  $\mathbf{x} \in \mathbb{R}^N$  from noisy observations  $\mathbf{y} = \mathbf{x} + \epsilon$ , where  $\epsilon$  denotes an additive degradation.

To solve this problem, variational methods based on total variation have received considerable interest and research efforts over the past years (see, e.g., [3] for genomic data processing). They aim at providing an estimate for  $\mathbf{x}$  by minimizing the following non-smooth convex criterion by iterative strategies [4–7] or straightforward computations [8, 9]:

$$\mathbf{x}_\lambda^* = \arg \min_{\mathbf{u} \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{u} - \mathbf{y}\|_2^2 + \lambda \sum_{i=1}^{N-1} |u_{i+1} - u_i| \quad (1)$$

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where  $\mathbf{u} = (u_i)_{1 \leq i \leq N}$  and  $\lambda$  is a (positive) regularization parameter that controls the trade-off between the fit to the observations  $\mathbf{y}$  and the amount of regularization. The selection of  $\lambda$  is critical for the performance of (1) because the solution  $\mathbf{x}_\lambda^*$  strongly depends on its value. Indeed, for  $\lambda \rightarrow 0$ , the first term in (1) dominates and  $\mathbf{x}_\lambda^*$  consists of many constant segments. To the contrary, when  $\lambda \rightarrow +\infty$ , the regularization term is dominant and (1) yields a solution with very few constant segments. Currently existing methods for selecting a value for  $\lambda$  rely on the *Stein unbiased risk estimate* (SURE) [10, 11], which minimizes the unbiased estimator of the mean squared error between  $\mathbf{x}$  and  $\mathbf{x}_\lambda^*$ . While this approach is effective, it requires the knowledge of the variance of  $\epsilon$ , which is often unavailable. Another alternative consists in resorting to a Bayesian framework, by formulating (1) as a statistical inference problem. Indeed, in the right-hand side of (1), the first term is related to a likelihood under an additive white Gaussian noise assumption, while the second term refers to a prior distribution parametrized by an hyperparameter  $\lambda$ . By adopting a hierarchical strategy, this hyperparameter could be included within the Bayesian model and estimated jointly with  $\mathbf{x}$ . Unfortunately, in the specific case of (1), the partition function, which depends on  $\lambda$ , cannot be expressed analytically. As a consequence, estimating  $\lambda$  within a hierarchical Bayesian framework would require computationally intensive sampling methods similar to [12].

A second important class of methods for the estimation of a piecewise constant signal  $\mathbf{x}$  relies on the estimation of unknown parameters related to  $\mathbf{x}$  such as the number of change points or the value of the signal on each segment, and of a model for the noise  $\epsilon$ . Estimation of the unknown parameters is then performed on the basis of their posterior distribution. By adopting a hierarchical strategy with additional hyperparameters, no tuning of the parameters of the prior and noise distributions is needed, see, e.g., [13, 14]. However, Bayesian methods still suffer from a high computational cost which stems from the fact that the posterior distribution cannot, in most cases, be expressed analytically and needs to be approximated numerically by sampling methods such as *Markov chain Monte Carlo* (MCMC) to construct estimators for the unknown parameters.

In the present contribution, we propose a *hybrid Bayesian variational* (HBV) method that combines the advantages of

both worlds, i.e., no parameter to tune by hand with lower computational cost than usual Bayesian strategies. To achieve this goal, we propose to revisit [14] by restricting the searching space to the change-point locations resulting from (1).

The performance of the proposed hybrid procedure compares favorably against those of a fully Bayesian approach, both in terms of estimation accuracy and computational cost. It is also compared against the benchmark SURE that assumes the a priori knowledge of the noise variance.

The remainder of this contribution is organized as follows. In Section 2, we describe the proposed HBV denoising algorithm. Numerical experiments validating and illustrating the proposed method are conducted in Section 3. Conclusions are drawn in Section 4.

## 2. HYBRID BAYESIAN VARIATIONAL DENOISING

### 2.1. Parametric model

To formalize the change point detection problem, we explicitly model the piecewise constant signal  $\mathbf{x} \in \mathbb{R}^N$  as a signal constituted of  $K$  segments, with corresponding constant values  $\mu_k$ ,  $k = 1, \dots, K$ , referred to as the vector  $\boldsymbol{\mu} = (\mu_k)_{1 \leq k \leq K}$ . Use is also made of the change-point indicator vector  $\mathbf{r} = (r_i)_{1 \leq i \leq N}$ ,

$$r_i = \begin{cases} 1, & \text{if there is a change-point at time instant } i, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Convention  $r_N = 1$  ensures that the number  $K$  of segments is equal to the number of change-points, i.e.,  $K = \sum_{i=1}^N r_i$ . By definition,  $r_i = 1$  indicates that  $x_i$  is the last sample belonging to the current segment, and thus that  $x_{i+1}$  belongs to next segment. Equivalently, we can deduced from  $\mathbf{r}$  the set of time indices  $\mathcal{I}_k \subset \{1, \dots, N\}$  corresponding to the  $k$ -th segment for  $k = 1, \dots, K$ , such that  $\mathcal{I}_k \cap \mathcal{I}_{k'} = \{\emptyset\}$  for  $k \neq k'$  and  $\cup_{k=1}^K \mathcal{I}_k = \{1, \dots, N\}$ . Following [14], we also introduce the point-wise change occurrence probability  $p$ . In addition, the noise  $\epsilon$  is assumed to be zero-mean and with constant variance  $\sigma^2$ . Therefore, observations  $\mathbf{y}$  depend upon the vector parameter  $\boldsymbol{\Theta} = \{\mathbf{r}, \boldsymbol{\mu}, \sigma^2, p\}$ .

### 2.2. Regularization parameter selection

The strategy of the proposed HBV denoising algorithm can now be summarized in the following steps. First, a TV denoising algorithm is used to solve (1) for a large number of candidate values  $\lambda \in \Lambda$ . Then, from each recovered piecewise constant solution  $\mathbf{x}_\lambda^*$ , we derive an estimate  $\hat{\boldsymbol{\Theta}}_\lambda$  of  $\boldsymbol{\Theta}$ . Finally, the optimal value  $\lambda_{\text{HBV}}$  of  $\lambda$  is chosen as the value  $\lambda$  for which  $\hat{\boldsymbol{\Theta}}_\lambda$  maximizes the posterior, i.e.,

$$\lambda_{\text{HBV}} \in \underset{\lambda \in \Lambda}{\text{Argmax}} f(\hat{\boldsymbol{\Theta}}_\lambda | \mathbf{y}) \quad (3)$$

where the notation  $\in \text{Argmax}$  indicates that the solution is not necessarily unique. To model the posterior distribution

$f(\hat{\boldsymbol{\Theta}}_\lambda | \mathbf{y})$ , we have recourse to a hierarchical Bayesian model, detailed in next section.

### 2.3. Hierarchical Bayesian model

We assume that the  $\epsilon_i$ , for  $i = \{1, \dots, N\}$ , are i.i.d. zero mean Gaussian variables with constant (but unknown) variance  $\sigma^2$ :  $\mathcal{N}(0, \sigma^2)$ . The joint likelihood function of the observations  $\mathbf{y}$ , depending on the piecewise constant model and noise parameters  $\{\mathbf{r}, \boldsymbol{\mu}, \sigma^2\}$ , then reads:

$$f(\mathbf{y} | \mathbf{r}, \boldsymbol{\mu}, \sigma^2) = \prod_{k=1}^K \prod_{i \in \mathcal{I}_k} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \mu_k)^2}{2\sigma^2}\right). \quad (4)$$

To model the posterior distribution, further assumptions are needed. Following [13], it is first assumed that  $\mathbf{r}$  are a priori independent and distributed according to a Bernoulli distribution with parameter  $p$ , quantifying the prior probability of having a change-point at a given location:

$$f(\mathbf{r} | p) = \prod_{i=1}^N p^{r_i} (1-p)^{1-r_i}. \quad (5)$$

Furthermore, a Beta( $\alpha_0, \alpha_1$ ) distribution is assigned to  $p$ :

$$f(p | \alpha_0, \alpha_1) = \frac{\Gamma(\alpha_0, \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} p^{\alpha_0-1} (1-p)^{\alpha_1-1}. \quad (6)$$

Segment values  $\mu_k$ , for  $k = \{1, \dots, K\}$ , are also assumed to be i.i.d., distributed according to a common Gaussian distribution  $\mathcal{N}(\mu_0, \sigma_0^2)$ :

$$f(\boldsymbol{\mu} | \mu_0, \sigma_0^2) = \prod_{k=1}^K \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(\mu_k - \mu_0)^2}{2\sigma_0^2}\right). \quad (7)$$

Finally, a non-informative Jeffreys prior is assigned to the noise variance  $\sigma^2$ :

$$f(\sigma^2) \propto \frac{1}{\sigma^2}. \quad (8)$$

With these assumptions, the posterior distribution reads:

$$f(\boldsymbol{\Theta} | \mathbf{y}) = \frac{1}{f(\mathbf{y})} f(\mathbf{y} | \mathbf{r}, \boldsymbol{\mu}, \sigma^2) f(\mathbf{r} | p) f(p | \alpha_0, \alpha_1) \times f(\boldsymbol{\mu} | \mu_0, \sigma_0^2) f(\sigma^2) \quad (9)$$

where  $f(\mathbf{y} | \mathbf{r}, \boldsymbol{\mu}, \sigma^2)$ ,  $f(\mathbf{r} | p)$ ,  $f(p | \alpha_0, \alpha_1)$ ,  $f(\boldsymbol{\mu} | \mu_0, \sigma_0^2)$ , and  $f(\sigma^2)$  have been defined in (4), (5), (6), (7), and (8), respectively.

### 2.4. Proposed algorithm

The proposed HBV algorithm consists in repeating for a large number of candidates  $\lambda \in \Lambda$  the following procedure.

Step 1 consists in computing  $\mathbf{x}_\lambda^* \in \mathbb{R}^N$ , the solution of (1), using a variational strategy, based on our own implementation of Condat 1D-TV algorithm [8].

Step 2 relies on the fact that, by nature, the solution  $\mathbf{x}_\lambda^*$  is piecewise constant with  $\widehat{K}_\lambda$  segments. Therefore, from each solution  $\mathbf{x}_\lambda^*$ , an estimate, denoted  $\widehat{\Theta}_\lambda = \{\widehat{\mathbf{r}}_\lambda, \widehat{\boldsymbol{\mu}}_\lambda, \widehat{\sigma}_\lambda^2, \widehat{p}_\lambda\}$ , of the parameter vector  $\Theta$  involved in the hierarchical Bayesian model, can be obtained: Estimates  $\widehat{\mathbf{r}}_\lambda$  of the change-point locations (or, equivalently, of the sets  $(\widehat{\mathcal{I}}_{\lambda,k})_{1 \leq k \leq \widehat{K}_\lambda}$ ) can be easily computed and empirical estimates of the remaining parameters can be derived as follows:

$$\widehat{\boldsymbol{\mu}}_\lambda = (\widehat{\mu}_{\lambda,k})_{1 \leq k \leq \widehat{K}_\lambda} \text{ where } \widehat{\mu}_{\lambda,k} = \frac{1}{|\widehat{\mathcal{I}}_{\lambda,k}|} \sum_{i \in \widehat{\mathcal{I}}_{\lambda,k}} y_i, \quad (10)$$

$$\widehat{\sigma}_\lambda^2 = \frac{1}{N} \sum_{k=1}^{\widehat{K}_\lambda} \sum_{i \in \widehat{\mathcal{I}}_{\lambda,k}} (y_i - \widehat{\mu}_{\lambda,k})^2, \quad (11)$$

$$\widehat{p}_\lambda = \widehat{K}_\lambda / N. \quad (12)$$

We denote  $\widehat{\mathbf{x}}_\lambda = (\widehat{x}_{\lambda,i})_{1 \leq i \leq N}$ , the estimate of  $\mathbf{x}$  such that:

$$(\forall k \in \{1, \dots, \widehat{K}_\lambda\})(\forall i \in \widehat{\mathcal{I}}_{\lambda,k}), \quad \widehat{x}_{\lambda,i} = \widehat{\mu}_{\lambda,k}. \quad (13)$$

Note that  $\widehat{\mathbf{x}}_\lambda$  is different from  $\mathbf{x}_\lambda^*$  but both estimates share the same change-point locations.

Step 3 computes the posterior distribution  $f(\widehat{\Theta}_\lambda | \mathbf{y})$  based on the hierarchical Bayesian model (9).

Finally, the optimal regularization parameter  $\lambda_{\text{HBV}}$  is selected as the one that maximizes the posterior distribution (9) according to (3). Consequently we denote  $\widehat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  the estimate  $\widehat{\mathbf{x}}_\lambda$  when  $\lambda = \lambda_{\text{HBV}}$ . The HBV procedure is sketched in Algorithm 1.

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#### Algorithm 1 HBV algorithm

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**Input:** Observed signal  $\mathbf{y} \in \mathbb{R}^N$ .  
 Predefined set of regularization parameters  $\Lambda$ .  
 Prior parameters  $\Phi = \{\alpha_0, \alpha_1, \mu_0, \sigma_0^2\}$ .

**Iterations:**

- 1: **for**  $\lambda \in \Lambda$  **do**
- 2: Estimate  $\mathbf{x}_\lambda^*$  with Condat 1D-TV algorithm.
- 3: Estimate  $\widehat{\Theta}_\lambda = \{\widehat{\mathbf{r}}_\lambda, \widehat{\boldsymbol{\mu}}_\lambda, \widehat{\sigma}_\lambda^2, \widehat{p}_\lambda\}$  from  $\mathbf{x}_\lambda^*$ .
- 4: Compute  $f(\widehat{\Theta}_\lambda | \mathbf{y})$  with (9) and compute  $\widehat{\mathbf{x}}_\lambda$ .
- 5: **end for**

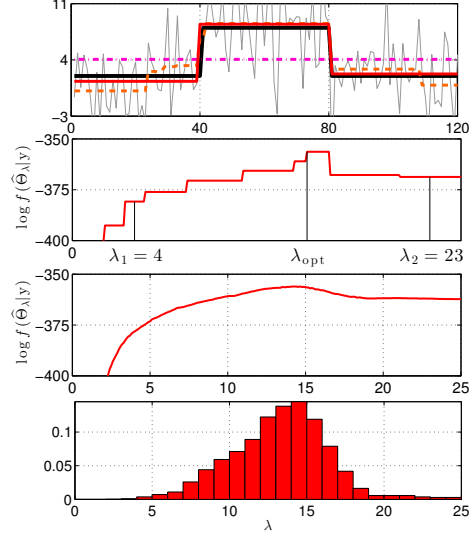
**Output:**  $\lambda_{\text{HBV}} \in \text{Argmax}_{\lambda \in \Lambda} f(\widehat{\Theta}_\lambda | \mathbf{y})$ ;  
 Solution  $\widehat{\mathbf{x}}_{\lambda_{\text{HBV}}}$ .

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### 3. PERFORMANCE ASSESSMENT

#### 3.1. Experimental settings

Non informative prior parameters are used for the Bayesian hierarchical model ( $\alpha_0 = \alpha_1 = 1$ , from which it follows that  $p$  has an a priori uniform distribution on  $[0, 1]$ ;  $\mu_0 = \frac{1}{N} \sum_{i=1}^N y_i$  and  $\sigma_0^2 = \frac{1}{10} \widehat{\text{Var}}(\mathbf{y})$ , where  $\widehat{\text{Var}}$  stands for the



**Fig. 1: Illustration of HBV procedure.** Top row: true signal  $\mathbf{x}$  (solid black), single realization of observations  $\mathbf{y}$  (gray, SNR = 1.8 dB), solution  $\widehat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  (solid red), non-optimal solutions  $\widehat{\mathbf{x}}_\lambda$ , with  $\lambda_1 = 4$  (dashed orange) and  $\lambda_2 = 23$  (mixed magenta);  $\log f(\widehat{\Theta}_\lambda | \mathbf{y})$  evaluated for the set  $\Lambda$  (second row). Ensemble average of  $\log f(\widehat{\Theta}_\lambda | \mathbf{y})$  (third row) and empirical distribution of  $\lambda_{\text{HBV}}$  (bottom row).

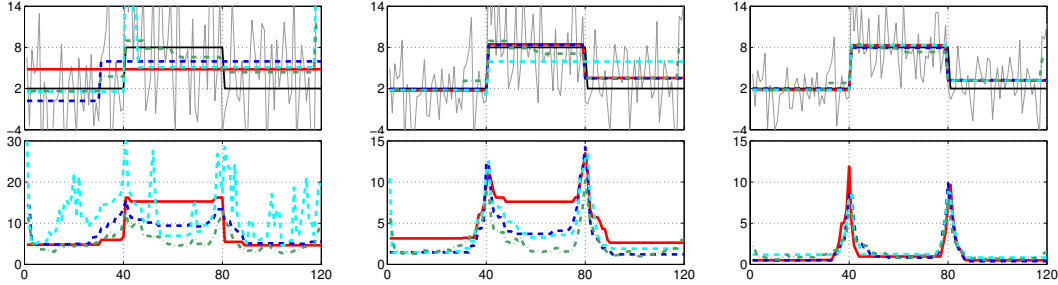
variance).

**First example** – To illustrate and quantify the performance of the proposed HBV denoising procedure, we consider the piecewise constant signal  $\mathbf{x} \in \mathbb{R}^N$  with  $N = 120$  samples plotted in black in Fig.1 (top) together with the resulting data  $\mathbf{y}$  (gray) for one realization of an additive Gaussian noise with signal-to-noise ratio (SNR) of 1.8 dB.

**Second example** – Complementary results arise from the study of another example, where  $\mathbf{x}$  is made of  $N = 240$  samples whose segments are 40 samples long with different amplitudes  $\mu_k$ , and whose interest will be shown in the following.

#### 3.2. Illustration of regularization parameter selection

In Fig. 1 (top),  $\widehat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  (solid red) is compared against two non-optimal solutions  $\widehat{\mathbf{x}}_\lambda$  obtained with  $\lambda_1 = 4$  (dashed orange) and  $\lambda_2 = 23$  (mixed magenta), respectively, for one single realization. While the non-optimal solutions clearly fail to reproduce the true signal  $\mathbf{x}$ ,  $\widehat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  obtained with the proposed procedure provides a visually good estimate for  $\mathbf{x}$ . In Fig.1 (second row), the corresponding log posterior distribution  $\log f(\widehat{\Theta}_\lambda | \mathbf{y})$  is plotted as a function of  $\lambda$ . Its piecewise behavior stems from the fact that the  $\mathbf{x}_\lambda^*$  have the same discontinuities and are identical for ranges of  $\lambda$ . We denote by  $\Lambda_{\text{HBV}}$  the range of values of  $\lambda$  for which the posterior is maximal and define, by convention,  $\lambda_{\text{HBV}}$  as the smallest value in  $\Lambda_{\text{HBV}}$ . We note that the log posterior distribution for a single realization provides a relevant approximation for the ensemble average of the posterior distribution, plotted in Fig. 1 (third row) for 100 realizations. The empirical distribution



**Fig. 2: Estimation for single realizations (Top) and MSE over 50 realizations (Bottom) for different SNR.** From left to right SNR = -1.5dB, 0.4dB and 1.8dB. True signal  $\mathbf{x}$  (solid black), observations  $\mathbf{y}$  (solid gray),  $\hat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  (solid red),  $\hat{\mathbf{x}}_{\text{MAP}}$  (dashed blue),  $\hat{\mathbf{x}}_{\text{MMSE}}$  (dashed cyan) and  $\mathbf{x}_{\lambda_{\text{SURE}}}^*$  (mixed green).

| SNR (dB) | -1.5       |      |      |              | 0.4        |      |      |              | 1.8   |            |      |              |
|----------|------------|------|------|--------------|------------|------|------|--------------|-------|------------|------|--------------|
|          | GSURE      | MAP  | MMSE | HBV          | GSURE      | MAP  | MMSE | HBV          | GSURE | MAP        | MMSE | HBV          |
| MAD      | <b>1.8</b> | 2.4  | 2.4  | 2.6          | <b>1.2</b> | 1.2  | 1.3  | 1.8          | 0.9   | <b>0.6</b> | 0.8  | <b>0.6</b>   |
| MSE      | <b>5.6</b> | 7.8  | 12.1 | 8.4          | <b>2.5</b> | 3.0  | 3.4  | 5.0          | 1.4   | <b>1.2</b> | 1.6  | <b>1.2</b>   |
| time (s) | 37.2       | 31.6 | 31.6 | <b>0.015</b> | 37.5       | 29.6 | 29.6 | <b>0.016</b> | 37.4  | 29.5       | 29.5 | <b>0.015</b> |

**Table 1: Estimation performance vs. SNR (first example).**

of  $\lambda \in \Lambda_{\text{HBV}}$  is reported in Fig. 1 (bottom). Its mean (respectively median) equals 13.2 (respectively 13.4), which is consistent with the position of the maximum of the ensemble average of the log posterior distribution at  $\lambda \simeq 13$ .

### 3.3. Estimation performance

**Comparisons with state-of-the-art methods** – We proceed with comparing against a *fully Bayesian* (FB) procedure in which one may naturally compute the Bayesian estimators associated with the posterior distribution  $f(\Theta|\mathbf{y})$  in (9). Deriving the closed-form expression of the *maximum a posteriori* (MAP) or *minimum mean squared error* (MMSE) estimators associated with  $f(\Theta|\mathbf{y})$  is not straightforward. Alternatively, these estimators can be approximated by using MCMC procedures that essentially rely on a partially collapsed Gibbs sampler [15] similar to the algorithm derived in [14]. It consists in iteratively drawing samples (denoted  $\cdot^{(t)}$ ) according to conditional posterior distributions that are associated with the joint posterior (9). The resulting procedure, detailed in Algorithm 2, provides a set of samples  $\vartheta = \{\mathbf{r}^{(t)}, \boldsymbol{\mu}^{(t)}, \sigma^{2(t)}, p^{(t)}\}_{t=1}^T$  that are asymptotically distributed according to (9). These samples can be used to approximate the MAP and MMSE estimators of the parameters of interest [16]. The corresponding solutions are referred to as  $\hat{\mathbf{x}}_{\text{MAP}}$  and  $\hat{\mathbf{x}}_{\text{MMSE}}$ .

In addition,  $\hat{\mathbf{x}}_{\lambda_{\text{HBV}}}$  is also compared against  $\mathbf{x}_{\lambda}^*$  for  $\lambda = \lambda_{\text{SURE}}$  that minimizes the *Stein unbiased risk estimate* (SURE). It is used as a benchmark solution since it requires the a priori knowledge of the noise variance  $\sigma^2$ . To do so, we have replaced steps 3 and 4 in Algorithm 1 with the computation of the Generalized SURE estimator proposed in [11], which requires to deal with an expectation that we approximate by an empirical mean computed over  $10^4$  realizations. Moreover, the output  $\mathbf{x}_{\lambda}^*$ , which

#### Algorithm 2 FB Algorithm

**Input:** Observed signal  $\mathbf{y} \in \mathbb{R}^N$ .

Prior parameters  $\Phi = \{\alpha_0, \alpha_1, \mu_0, \sigma_0^2\}$ .

**Iterations:**

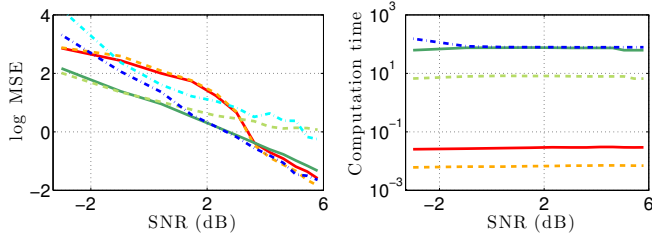
- 1: **for**  $t = 1, \dots, T$  **do**
- 2:   **for**  $i = 1, \dots, N - 1$  **do**
- 3:     Draw  $r_i^{(t)} \sim \text{P}[r_i|\mathbf{y}, \mathbf{r}_{\setminus i}, p, \sigma^2, \mu_0, \sigma_0^2]$
- 4:   **end for**
- 5:   **for**  $k = 1, \dots, \sum_{i=1}^N r_i^{(t)}$  **do**
- 6:     Draw  $\mu_k^{(t)} \sim f(\mu_k|\mathbf{y}, \mathbf{r}, \sigma^2, \mu_0, \sigma_0^2)$
- 7:   **end for**
- 8:   Draw  $\sigma^{2(t)} \sim f(\sigma^2|\mathbf{y}, \mathbf{r}, \boldsymbol{\mu})$
- 9:   Draw  $p^{(t)} \sim f(p|\mathbf{r}, \alpha_0, \alpha_1)$
- 10: **end for**

**Output:**  $\vartheta = \{\mathbf{r}^{(t)}, \boldsymbol{\mu}^{(t)}, \sigma^{2(t)}, p^{(t)}\}_{t=1}^T$ .

minimizes the risk between  $\mathbf{x}_{\lambda}^*$  and  $\mathbf{x}$ , is obtained with  $\lambda = \lambda_{\text{SURE}} \in \text{Argmin}_{\lambda \in \Lambda} \text{GSURE}(\mathbf{y}, \mathbf{x}_{\lambda}^*, \sigma^2)$ .

The results for a single realization are plotted in Fig. 2 (top) for different SNR values. The usual measure of performance for an estimator of a piecewise constant signal is the mean squared error  $\text{MSE}(\hat{\mathbf{x}}) = \widehat{\mathbb{E}}[\frac{1}{N} \|\hat{\mathbf{x}} - \mathbf{x}\|_2^2]$  (where  $\widehat{\mathbb{E}}$  stands for the empirical mean estimator computed over 50 realizations) reported in Table 1 along with the mean absolute deviations  $\text{MAD}(\hat{\mathbf{x}}) = \widehat{\mathbb{E}}[\frac{1}{N} \|\hat{\mathbf{x}} - \mathbf{x}\|_1]$  and average execution times. Fig. 2 (bottom) provides a performance analysis in terms of MSE at each location  $i \in \{1, \dots, N\}$ .

Overall, the proposed HBV method yields estimates that are comparable with FB (i.e., MAP and MMSE) results in terms of MSE and MAD, while its computation time is reduced by a factor of 2000 with respect to the FB approach. For high SNR values, HBV clearly outperforms FB, the latter having wider MSE peaks at the change point locations, indi-



**Fig. 3: Estimation performance vs. SNR (second example).** MSE (left) and average computation time (right) associated to  $\hat{x}_{\lambda_{\text{HBV}}}$  ( $|\Lambda| = 10$  dashed orange,  $|\Lambda| = 100$  solid red),  $\hat{x}_{\text{MAP}}$  (mixed blue),  $\hat{x}_{\text{MMSE}}$  (mixed cyan) and  $x_{\lambda_{\text{SURE}}}^*$  ( $|\Lambda| = 10$  dashed light green,  $|\Lambda| = 100$  solid dark green).

ating less precision. Performance are consistent with those obtained with the second example and reported in Fig. 3 (left: MSE, right: average computation time) for different SNR values and two different search grids  $\Lambda$ . It shows that HBV is less sensitive to  $|\Lambda|$  than GSURE. In addition, the computational time associated with HBV is reduced by a larger factor as the sample size  $N$  grows.

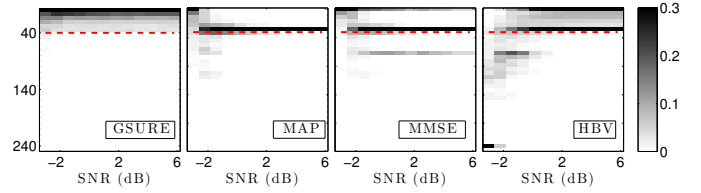
The comparisons with GSURE illustrate that, at high SNR, the estimation performance of the proposed approach are close to those of the GSURE oracle (which requires the a priori knowledge of the noise variance) while the computation cost for the proposed approach is 3 orders of magnitude smaller. Note that  $\hat{x}_{\lambda_{\text{HBV}}}$  is estimated a posteriori, cf., (13), while  $x_{\lambda_{\text{SURE}}}^*$  is a direct solution of (1), resulting in a slightly lower MSE for high SNR values.

### 3.4. Behavior comparisons

In addition, Fig. 2 shows that when increasing noise variance HBV tends to detect less change points and eventually no change point, while MAP, MMSE and GSURE yield a larger number of change point detection, with highly variable locations. Visually, no clear preference can be given to any of the methods, yet HBV has a lower MSE than FB at very low SNR. These pronouncedly different behaviors at low SNR are further illustrated in Fig. 4 where segment size empirical distributions are plotted as functions of SNR.

## 4. CONCLUSIONS

We developed a hybrid Bayesian variational method for the change point detection problem. The originality and advantage of the proposed procedure reside in combining the computational efficiency of variational methods with the statistical flexibility of a hierarchical Bayesian model, that thus permit to efficiently handle the automated regularization parameter selection. The proposed procedure compares favorably against a fully Bayesian approach in terms of estimation performance, while reducing computational cost by more than 2 orders of magnitude, and achieves, at large SNR, performance that are similar to those obtained with the GSURE “oracle”



**Fig. 4: Size empirical distributions of estimated segment vs. SNR.** Second example where  $x$  consists of six segments which are 40 samples long (dashed red).

procedure, which requires the a priori knowledge of the noise variance. Extensions to image denoising and segmentation are under current investigation.

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