ITERATIVE APPROACH TO ESTIMATE THE PARAMETERS OF A TVAR PROCESS CORRUPTED BY A MA NOISE

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ABSTRACT

A great deal of interest has been paid to the time-varying autoregressive (TVAR) parameter tracking, but few papers deal with this issue when noisy observations are available.
Recently, this problem was addressed for a TVAR process disturbed by an additive zero-mean white noise, by using deterministic regression methods. In this paper, we focus our attention on the case of an additive colored measurement noise modeled by a moving average process. More particularly, we propose to estimate the TVAR parameters by using a variant of the improved least-squares (ILS) methods, initially introduced by Zheng to estimate the AR parameters from a signal embedded in a white noise. Simulation studies illustrate the advantages and the limits of the approach.

***Index Terms*—** Time-varying autoregressive model, unbiased parameter estimation, colored noise, moving average process, deterministic regression approach.

1. INTRODUCTION

Time-varying autoregressive (TVAR) models have been used in a wide range of applications such as [1], [2]. For the last 30 years, two families of methods have been proposed to track the TVAR parameters:

1/ *the so-called "stochastic" family*

It includes recursive methods such as the recursive least squares (RLS) with forgetting factor and Kalman filtering[1].

2/ *the "evolutive method" family, also called deterministic regression based methods*

In that case, the parameter evolutions are assumed to be weighted linear combinations of some basis functions. Power of time, Legendre polynomials and Fourier functions can be considered. If the weights are available or preliminary estimated, the TVAR parameters can be deduced by summing all the weighted basis functions. See for instance [3], [4]. However, when the pth-order TVAR process is disturbed by an additive measurement noise, the above approaches lead to an estimation bias. To avoid this problem when the noise is white, both the TVAR process and the TVAR parameters must be estimated if recursive stochastic methods are used. This leads to a nonlinear estimation issue that can be solved by using an extended Kalman filter (EKF), sigma point Kalman filters (SPKF), quadrature Kalman filter (QKF) or cubature Kalman filter (CKF). Nevertheless, these methods require *a priori* knowledge of the noise variances. To relax this constraint, multiple-model approaches such as the interacting multiple models (IMM) could be considered. In that case, several Kalman filter based estimators would be applied in parallel, but they would differ in the selections of the noise variances. As an alternative to the above recursive approaches, off-line methods based on cumulants could be also used especially in the non-Gaussian case, but this may lead to poor results because their estimates may have a high variance when few data are available. In [5], two authors of the current paper have proposed a recursive errors-in-variables (EIV) based method. It combines a Newton-Raphson algorithm to iteratively estimate the variance of the additive noise and a sliding off-line noise compensated Yule-Walker based method to deduce the TVAR parameters. In [6] and [7], the authors also proposed two kinds of unbiased estimation methods. These deterministic regression methods were based on an EIV algorithm or a subspace approach.

In this paper, our purpose is to track the TVAR parameters when the measurement noise is additive stationary Gaussian and colored. This assumption is probably more realistic than the one corresponding to the additive white noise, because it does not deal with thermal noise alone, but also with other kinds of disturbances.

To our knowledge, when the additive noise is colored, people have rather focused their attention on the estimations of the signal parameters when the signal is "stationary". More particularly, the estimations of the autoregressive (AR) parameters of the signal from noisy colored observations have been addressed. Thus, in [8], the noise is modeled by a first-order AR process. To compensate the estimation bias induced by the additive colored noise, the authors suggest extending the approach initially proposed by Davila in [9].

In the following, we will assume that the noise can be modeled as a qth-order moving average (MA) process.

A first "intuitive" idea would be to use a Kalman filter-based approach with a state vector including the p TVAR parameters, the p last samples of the TVAR process, the q MA parameters and q other quantities describing the dynamics of the MA process. The way to select the MA driving process

[1] In that case, the TVAR parameters to be tracked are stored in the state vector. Random walk is usually used to model the evolution of the TVAR parameters. In addition, the noise appearing in the observation equation in the state space representation corresponds to the AR driving process.

variance and the AR driving process variance could be addressed, as stated above, by using an IMM.

However, this method is not reliable to get the MA parameters. Indeed, a MA process defined by its MA parameters can be viewed as the output of a filter, the input of which is a specific realization of a zero-mean white Gaussian process. Nevertheless, this driving process can be also the output of any all-pass filter, the input of which would be another realization of this zero-mean white process. Therefore, several filters make it possible to obtain the same correlation function of the MA process. The transfer function of one filter can be deduced from another one by replacing their roots by their inverse conjugates. As a consequence, the estimations of the true MA parameters are not necessarily guaranteed with such a method.

In [11] and [12], variants of Zheng's method [10] were proposed. In [13], we suggested using the prediction error method (PEM). This method is known to be asymptotically unbiased and efficient in the Gaussian case, but its computational cost may be high. In addition, if the state representation is time-varying, the classical PEM is no longer applicable.

For the above reasons, we propose to address the estimations of the TVAR parameters from observations disturbed by a MA noise, by using an evolutive method. More particularly, we aim at extending the method initially proposed by Zheng in [11]. Even if deriving this variant does not really pose theoretical problems, one thing of importance is to analyze its relevance and its limits in terms of signal-to-noise ratio (SNR), number of samples, etc.

The remainder of this paper is organized as follows. The problem statement is given in Section 2. In Section 3, an estimation method for TVAR parameter based on the iterative deterministic regression approach is proposed. In Section 4, simulation results are provided.

2. PROBLEM FORMULATION

Let $x(k)$ be a real signal modeled by a pth-order TVAR process given by:

$$x(k) = -\sum_{n=1}^{p} a_n(k-n)x(k-n) + u(k), \tag{1}$$

where $u(k)$ is assumed to be the driving process which is a zero-mean stationary white noise with variance σ_u^2. In addition, the TVAR parameters $a_n(k)$ are expressed by considering a function basis, as follows:

$$a_n(k) = \sum_{\ell=0}^{m} \beta_{n\ell} f_\ell(k), \tag{2}$$

where $\{f_\ell(\cdot)\}_{\ell=0,1,\cdots,m}$ are the basis functions *a priori* defined and selected by the practitioner and $\{\beta_{n\ell}\}_{n=1,\cdots,p \text{ and } \ell=0,1,\cdots,m}$ are the corresponding weights.

The process $x(k)$ is assumed to be disturbed by an additive zero-mean colored noise $b(k)$ modeled by a MA process:

$$b(k) = \sum_{j=0}^{q} c_j w(k-j), \tag{3}$$

where $w(k)$ is a zero-mean white noise with variance σ_w^2 which is uncorrelated with the driving process $u(k)$. By adjusting the noise variance σ_w^2, c_0 can be set to 1.

The observation data are hence given by:

$$y(k) = x(k) + b(k). \tag{4}$$

However, the TVAR parameters $\{a_n(k-n)\}_{n=1,..,p}$, the MA parameters $\{c_j\}_{j=1,\dots,q}$ and the variances σ_u^2 and σ_w^2 are unknown and hence need to be estimated.

Let us rewrite the above equations in a vector form by means of the following vectors:

$$\underline{\theta} = [\beta_{10} \quad \cdots \quad \beta_{1m} \quad \beta_{20} \quad \cdots \quad \beta_{2m} \quad \cdots \quad \beta_{pm}]^T, \tag{5}$$

$$\underline{f}(k) = [f_0(k) \quad \cdots \quad f_m(k)]^T, \tag{6}$$

and of the $p(m+1) \times 1$ column vector $\underline{X}_p(k)$ defined by:

$$\underline{X}_p(k) = \left[\underline{f}^T(k-1)x(k-1) \quad \dots \quad \underline{f}^T(k-p)x(k-p)\right]^T. \tag{7}$$

Hence, given (1), (2), (5) and (6), the TVAR process can be rewritten as follows:

$$x(k) = -\underline{X}_p^T(k)\underline{\theta} + u(k). \tag{8}$$

By pre-multiplying both sides in (8) by $\underline{X}_p(k)$ and taking the expectation, one has:

$$R_X\underline{\theta} = E\left[\underline{X}_p(k)\underline{X}_p^T(k)\right]\underline{\theta} = -r_X = -E\left[\underline{X}_p(k)x(k)\right]. \tag{9}$$

By introducing the square matrices of size $(m+1)$ $F_{i,j} = \underline{f}(k-i)\underline{f}^T(k-j)$ and the $(m+1)p \times 1$ vectors:

$$\underline{Y}_p(k) = \left[\underline{f}^T(k-1)y(k-1) \quad \dots \quad \underline{f}^T(k-p)y(k-p)\right]^T,$$

$$\underline{B}_p(k) = \left[\underline{f}^T(k-1)b(k-1) \quad \dots \quad \underline{f}^T(k-p)b(k-p)\right]^T,$$

the corresponding correlation matrices and vectors satisfy:

$$R_Y = E\left[\underline{Y}_p(k)\underline{Y}_p^T(k)\right], \ \underline{r}_Y = E\left[\underline{Y}_p(k)y(k)\right], \tag{10}$$

$$\begin{aligned} R_B &= E\left[\underline{B}_p(k)\underline{B}_p^T(k)\right] \\ &= \begin{bmatrix} F_{1,1}r_b(0) & \cdots & F_{1,p}r_b(p-1) \\ \vdots & \ddots & \vdots \\ F_{p,1}r_b(p-1) & \cdots & F_{p,p}r_b(0) \end{bmatrix}, \end{aligned} \tag{11}$$

$$\begin{aligned} \underline{r}_B &= E\left[\underline{B}_p(k)b(k)\right] \\ &= \left[\underline{f}^T(k-1)r_b(1) \cdots \underline{f}^T(k-p)r_b(p)\right]^T \end{aligned} \tag{12}$$

where $r_b(\tau)$ is the correlation function of the noise.

At that stage, one can easily show that $\underline{r}_Y = \underline{r}_X + \underline{r}_B$ and $R_Y = R_X + R_B$. Therefore, (9) can be rewritten as follows:

$$(R_Y - R_B)\underline{\theta} = -(\underline{r}_Y - \underline{r}_B). \tag{13}$$

If the influence of the additive noise is not compensated in the above equation, the weight estimations would satisfy:

$$(R_X + R_B)\underline{\theta}_{LS} = R_Y\underline{\theta}_{LS} = -\underline{r}_Y = -\underline{r}_X - \underline{r}_B, \tag{14}$$

or

$$\underline{\theta}_{LS} = -R_Y^{-1}\underline{r}_Y\,. \tag{15}$$

It is the least squares estimation of $\underline{\theta}$ from the noisy observations. Given (13) and (15), one has:

$$\underline{\theta} = \underline{\theta}_{LS} + R_Y^{-1}\left(R_B\underline{\theta} + \underline{r}_B\right), \tag{16}$$

or equivalently:

$$\underline{\theta}_{LS} = (I - R_Y^{-1}R_B)\underline{\theta} - R_Y^{-1}\underline{r}_B. \tag{17}$$

From (16), it is easy to deduce that the LS estimate bias is equal to – $(R_Y^{-1}R_B\underline{\theta} + R_Y^{-1}\underline{r}_B)$.

In the next section, our purpose is to estimate the weight vector $\underline{\theta}$ by compensating the influence of the MA noise.

Remark 1: let us recall that the correlation function $r_b(\tau)$ of the MA process $b(k)$ is given by:

$$r_b(\tau) = \begin{cases} \sigma_w^2 \left(\sum_{j=\tau}^{q} c_j c_{j-\tau} \right) \text{ with } c_0 = 1, \text{for } 0 \le \tau \le q \\ 0 \text{ for } \tau > q \,. \end{cases}$$

3. ESTIMATING TVAR PARAMETERS FROM NOISY OBSERVATIONS

Even if another case can be considered, we assume that $q< p$ for convenience of illustration. The same kind of development could be done for $q \ge p$. In the following, let us introduce the $2p(m+1) \times 1$ weight vector $\underline{\tilde{\theta}}$ as follows:

$$\underline{\tilde{\theta}} = \left[\underline{\theta}^T \quad 0_{1\times p(m+1)}\right]^T. \tag{18}$$

Then, let us define two other data and noise vectors:

$$\underline{Y}_{+p}(k) = \left[\underline{f}^T(k-p-1)y(k-p-1) \cdots \underline{f}^T(k-2p)y(k-2p)\right]^T,$$

$$\underline{B}_{+p}(k) = \left[\underline{f}^T(k-p-1)b(k-p-1) \cdots \underline{f}^T(k-2p)b(k-2p)\right]^T,$$

and also introduce the corresponding correlation matrices:

$$\tilde{R}_Y = E\left[\begin{bmatrix} \underline{Y}_p(k) \\ \underline{Y}_{+p}(k) \end{bmatrix} \begin{bmatrix} \underline{Y}_p^T(k) & \underline{Y}_{+p}^T(k) \end{bmatrix}\right] = \begin{bmatrix} R_Y & \bar{R}_Y \\ \bar{R}_Y^+ & R_Y^+ \end{bmatrix},$$

$$\tilde{R}_B = E\left[\begin{bmatrix} \underline{B}_p(k) \\ \underline{B}_{+p}(k) \end{bmatrix} \begin{bmatrix} \underline{B}_p^T(k) & \underline{B}_{+p}^T(k) \end{bmatrix}\right] = \begin{bmatrix} R_B & \bar{R}_B \\ \bar{R}_B^+ & R_B^+ \end{bmatrix},$$

$$\underline{\tilde{r}}_B = E\left[\begin{bmatrix} \underline{B}_p(k) \\ \underline{B}_{+p}(k) \end{bmatrix} b(k)\right] = \begin{bmatrix} \underline{r}_B \\ \underline{\bar{r}}_B \end{bmatrix}.$$

Here, as $q < p$, $\underline{\bar{r}}_B$ is necessarily a null vector.

Similarly, let us consider the following correlation vector:

$$\underline{\tilde{r}}_Y = E\left[\begin{bmatrix} \underline{Y}_p(k) \\ \underline{Y}_{+p}(k) \end{bmatrix} y(k)\right] = \begin{bmatrix} \underline{r}_Y \\ \underline{\bar{r}}_Y \end{bmatrix}.$$

With the above vectors and covariance matrices, (13) can be extended as follows:

$$\left(\tilde{R}_Y - \tilde{R}_B\right)\underline{\tilde{\theta}} = -\left(\underline{\tilde{r}}_Y - \underline{\tilde{r}}_B\right). \tag{19}$$

By taking into account the structure of $\tilde{R}_Y$, $\tilde{R}_B$, $\underline{\tilde{r}}_B$, and $\underline{\tilde{r}}_Y$, (19) leads to:

$$\begin{cases} \underline{r}_Y = -R_Y\underline{\theta} + R_B\underline{\theta} + \underline{r}_B \\ \underline{\bar{r}}_Y = -\bar{R}_Y^+\underline{\theta} + \bar{R}_B^+\underline{\theta} \end{cases} \tag{20}$$

The above equations are the key expressions to estimate the weight vector $\underline{\theta}$. However, the correlation function of the noise that appears in $\tilde{R}_B$ and $\underline{\tilde{r}}_B$ must be estimated.

By pre-multiplying both sides of the first equality in (20) by $\bar{R}_Y^+ R_Y^{-1}$, we have

$$\begin{cases} \bar{R}_Y^+ R_Y^{-1}\underline{r}_Y = -\bar{R}_Y^+\underline{\theta} + \bar{R}_Y^+ R_Y^{-1} R_B\underline{\theta} + \bar{R}_Y^+ R_Y^{-1} r_B \\ \underline{\bar{r}}_Y = -\bar{R}_Y^+\underline{\theta} + \bar{R}_B^+\underline{\theta} \end{cases} \tag{21}$$

Substituting $\bar{R}_Y^+\underline{\theta}$ in the first equation by its expression given in the second one, one has:

$$\bar{R}_B^+\underline{\theta} - \bar{R}_Y^+ R_Y^{-1} R_B\underline{\theta} - \bar{R}_Y^+ R_Y^{-1}\underline{r}_B = \underline{\bar{r}}_Y - \bar{R}_Y^+ R_Y^{-1}\underline{r}_Y\,. \tag{22}$$

The above equation can be rewritten in the following way:

$$Q\left(\underline{\theta}\right)\rho_b = \underline{\bar{r}}_Y - \bar{R}_Y^+ R_Y^{-1}\underline{r}_Y, \tag{23}$$

where $\rho_b$is a $p \times 1$ vector storing the correlation values of the MA process for the first lags, namely:

$$\rho_b = [r_b(0) \quad \cdots \quad r_b(p-1)]^T, \tag{24}$$

and $Q\left(\underline{\theta}\right)$ is $p(m+1) \times p$ matrix defined as follows:

$$Q\left(\underline{\theta}\right) = T_1\left(\underline{\theta}\right) - \bar{R}_Y^+ R_Y^{-1} T_2\left(\underline{\theta}\right) - \bar{R}_Y^+ R_Y^{-1} T_3, \tag{25}$$

where:

$$T_1\left(\underline{\theta}\right) = \begin{bmatrix} 0_{p(m+1)\times 1} & \Sigma_{p+1,p}^1\underline{\theta} & \Sigma_{p+1,p-1}^2\underline{\theta} & \cdots & \Sigma_{p+1,2}^{p-1}\underline{\theta} \end{bmatrix}$$

$$T_2\left(\underline{\theta}\right) = \begin{bmatrix} \Sigma_{1,1}^p\underline{\theta} & \left(\Sigma_{1,2}^{p-1} + {\Sigma_{1,2}^{p-1}}^T\right)\underline{\theta} & \cdots & \left(\Sigma_{1,p}^1 + {\Sigma_{1,p}^1}^T\right)\underline{\theta} \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0_{(p-1)(m+1)\times 1} & D_f \\ 0_{(m+1)\times 1} & 0_{1\times(p-1)(m+1)} \end{bmatrix},$$

$$\Sigma_{i,j}^{\ell} = \begin{bmatrix} 0_{(\ell-1)(m+1)\times(p-\ell+1)(m+1)} & D_{F,i,j}^{\ell} \\ 0_{(p-\ell+1)(m+1)\times(p-\ell+1)(m+1)} & 0_{(p-\ell+1)(m+1)\times(\ell-1)(m+1)} \end{bmatrix},$$

and, D_f and $D_{F,i,j}^{\ell}$ are the following block diagonal matrices:

$$\begin{bmatrix} F_{i,j} & & 0 \\ & \ddots & \\ 0 & & F_{i+\ell-1,j+\ell-1} \end{bmatrix}, \text{ and } \begin{bmatrix} \underline{f}(k-1) & & 0 \\ & \ddots & \\ 0 & & \underline{f}(k-p-1) \end{bmatrix}$$

So, the unknown correlation vector ρ_b can be estimated as follows:

$$\rho_b = Q^{\dagger}\left(\underline{\hat{\theta}}\right)\left(\underline{\bar{r}}_Y - \bar{R}_Y^+ R_Y^{-1}\underline{r}_Y\right), \tag{26}$$

where $Q^{\dagger}(\cdot)$ is the Moore Penrose pseudo-inverse matrix of $Q(\cdot)$ and $\underline{\hat{\theta}}$ is an estimate of $\underline{\theta}$.

The proposed improved least squares (ILS) algorithm for TVAR estimation can be summarized as follows:

1. Compute $\tilde{R}_Y$ and $\underline{\tilde{r}}_Y$, and extract R_Y, $\bar{R}_Y^+$, $\underline{r}_Y$, and $\underline{\bar{r}}_Y$.
2. Calculate $\underline{\theta}_{LS}$ using (15), and set the initial values to the biased estimates $\underline{\hat{\theta}}^1 = \underline{\theta}_{LS}$.
3. Estimate the i-step correlation vector ρ_b^i as follows:
$$\rho_b^i = Q^{\dagger}\left(\underline{\hat{\theta}}^{i-1}\right)\left(\underline{\bar{r}}_Y - \bar{R}_Y^+ R_Y^{-1}\underline{r}_Y\right). \tag{27}$$
and construct R_B^i and $\underline{r}_B^i$ from estimates ρ_b^i.
4. Estimate the weights $\underline{\hat{\theta}}^i$ from R_B^i, $\underline{r}_B^i$ and the previous estimated weights $\underline{\hat{\theta}}^{i-1}$: $\underline{\hat{\theta}}^i = \underline{\theta}_{LS} + R_Y^{-1}\left(R_B^i\underline{\hat{\theta}}^{i-1} + \underline{r}_B^i\right)$.
5. Repeat steps 3 and 4 until $\frac{\|\underline{\hat{\theta}}^{i+1} - \underline{\hat{\theta}}^i\|}{\underline{\hat{\theta}}^i} < \delta$, where δ is a small positive number.

4. SIMULATION RESULTS

In this section, our purpose is to test the performance of the proposed method in order to analyze its limits.

Since the signal $x(k)$ is non-stationary, we suggest defining the signal-to-ratio $SNR(k)$ by $10\log_{10}\left(\frac{\sigma_x^2(k)}{r_b(0)}\right)$ where $\sigma_x^2(k) = E[x^2(k)]$ is the variance of the TVAR process $x(k)$, which can be easily expressed by using (1), under the assumptions of local stationarity i.e., $E[x(k)x(k-i)] = E[x(k-1)x(k-i-1)] = \cdots = E[x(k-p)x(k-i-p)]$, for any k and $i = 0, \cdots, p-1$.

In the following, two TVAR processes, of order $p = 2$ and $p = 4$ are considered where $\sigma_u^2 = 1$ and the basis functions are $f_i(k) = \begin{cases} \cos\left((i+1)\lambda\pi k/2\right) & i \text{ is odd} \\ \sin(i\lambda\pi k/2) & i \text{ is even} \end{cases}$ $(i = 1, \cdots, m)$, where λ is tuned by the practitioner and $f_0(k) = 1$.

Two 1st-order MA noises are studied, which are characterized by the coefficients $c_1 = -1.5$ and $c_1 = 1.5$.

For all simulations, the number of data is 3000 and the number δ is set to 0.001.

1/ *Results for* $p = 2$

Here, $m = 1$ and $\lambda = 3.3 \times 10^{-3}$. The true values of the weights to be estimated are set to $\beta_{10} = -0.5$, $\beta_{11} = -0.9$,

Table 1. True and estimated values of weights and correlation function of the MA noise, with M =100 runs ($p = 2$, c_1 =−1.5 and then 1.5).

	β_{10}	β_{11}	β_{20}	β_{21}	NRMSE
True values	-0.5	-0.9	0.8	-0.1	
ILS ($r_b(1)$ = -1.5)	-0.55±0.048	-0.65±0.067	0.79±0.109	-0.15±0.053	0.22
LS (15) ($r_b(1)$ = -1.5)	0.12±0.022	-0.40±0.033	0.30±0.021	-0.44±0.031	0.76
ILS ($r_b(1)$ = 1.5)	-0.59±0.068	-0.60±0.043	0.87±0.083	-0.05±0.049	0.30
LS (15) ($r_b(1)$ = 1.5)	-0.55±0.022	-0.41±0.033	0.50±0.021	-0.07±0.031	0.44

Table 2. True and estimated values of weights and correlation function of the MA noise, with M =100 runs ($p = 4$, c_1 = 1.5).

	β_{10}	β_{11}	β_{12}	β_{20}	β_{21}	β_{22}
True values	0	-1.6	3.14	3.11	-1.15	-1.59
ILS	-0.48±0.71	-1.05±0.69	2.98±0.20	1.05±0.76	-0.81±0.73	-0.99±0.24
LS (15)	-0.53±0.88	-0.97±0.86	2.80±0.22	0.08±1.43	1.70±1.39	-0.82±0.41

	β_{30}	β_{31}	β_{32}	β_{40}	β_{41}	β_{42}	NRMSE
True values	0	-1.57	3.08	0.96	0	0	
ILS	-0.26±0.78	-1.22±0.75	2.84±0.25	-0.26±0.68	1.19±0.66	0.27±0.18	0.65
LS (15)	-0.45±1.26	-0.96±1.21	2.58±0.38	-0.58±1.45	1.44±1.42	0.26±0.39	0.98

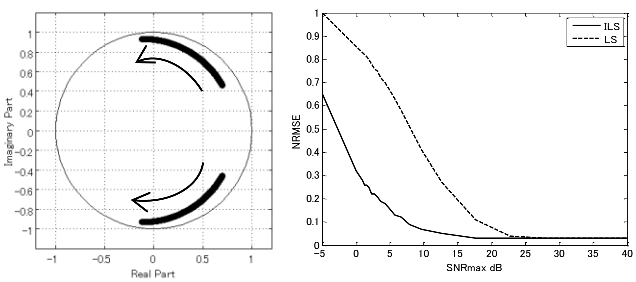

Fig. 1. Evolution of the poles of the AR process in the z-plane (left) and NRMSE for various $SNRmax$ (right).

$\beta_{20} = 0.8$, and $\beta_{21} = -0.1$. The poles of the corresponding transfer functions $H(z,k) = \frac{1}{1+\sum_{n=1}^{p} a_n(k-n)z^{-n}}$ are given in the left hand side of Fig. 1. As σ_w^2 is equal to 1, $r_b(0)$ =3.25 and $r_b(1) = -1.5$ or 1.5 and the SNR depends on k and varies between $SNRmax = \max_k SNR(k) = 2.7$dB and $SNRmin = \min_k SNR(k) = -0.38$dB for both MA noises in this simulation.

According to Table 1, the estimations based on our method are more reliable than the standard LS estimate deduced from (15). Indeed, the NRMSE[2] is considerably reduced.

Then, we have investigated the limits of the proposed method by analyzing its performance for various $SNRmin$ and $SNRmax$. We provide the results for $c_1 = -1.5$, for 50 independent trials. The noise variance varies by modifying the variance σ_w^2 of the MA driving process whereas the other model parameters are still set to the same values. This hence leads to a $SNRmin$ and $SNRmax$ varying from -8.1dB to 37dB and from -5dB to 40dB respectively.

According to the right hand side of Fig. 1, the results obtained with the proposed approach are still reliable even for low $SNRmax$ (i.e. 5dB).

[2] NRMSE $= \frac{1}{\|\theta\|}\sqrt{\frac{1}{M}\sum_{i=1}^{M}\|\hat{\theta}_i - \theta\|^2}$ where $\hat{\theta}_i$ denotes the estimates obtained during the i-th trial of the Monte Carlo simulation and M denotes the number of Monte Carlo runs.

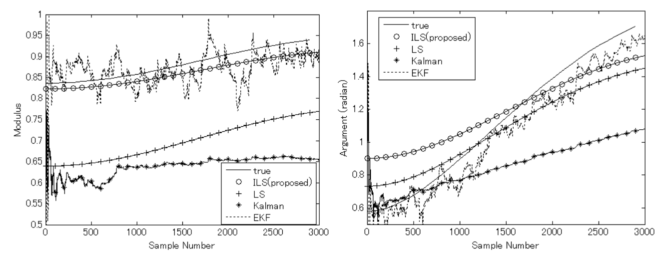

Fig. 2. Estimation of the pole modulus (left) and argument (right) for a 2nd-order TVAR process

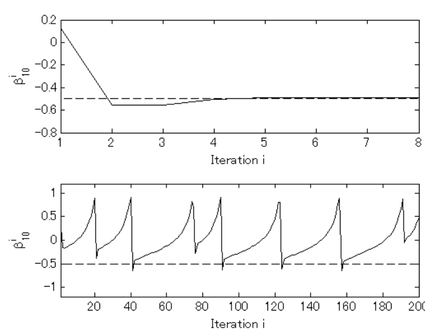

Fig. 3. Convergence features of the weight estimation β_{10} for $SNRmax$ =2.7 (top) and -5.0 dB (bottom), (the break line indicates true value $\beta_{10} = -0.5$).

According to Fig. 2, the proposed approach and the EKF provide estimates that almost vary along the true curves, unlike Kalman filtering directly used with noisy observations.

Remark 2: When the MA-process variance becomes too large, i.e. when the $SNRmax$ becomes low, the iterative estimation does not necessarily converge on most trials. Thus, Fig. 3 illustrates the convergence feature of the weight estimation β_{10} for a small and a large variance of the MA noise, respectively. The successful-convergence rate is equal to 100%, 90%, and 22% for a maximum SNR equal to 10dB, 0dB, and -5dB respectively. Therefore, the risks of the iteration failures increase as the SNR becomes small.

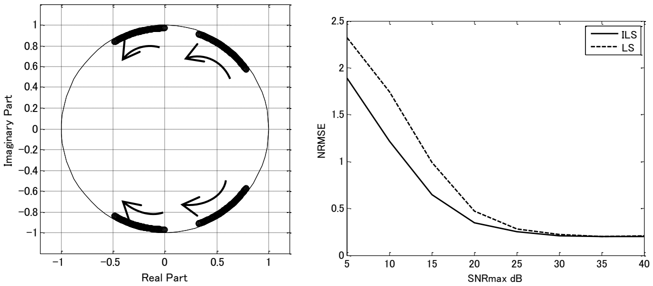

Fig. 4. Evolution of the poles of the AR process in the z-plane for $p = 4$ (left) and NRMSE for various SNR (right).

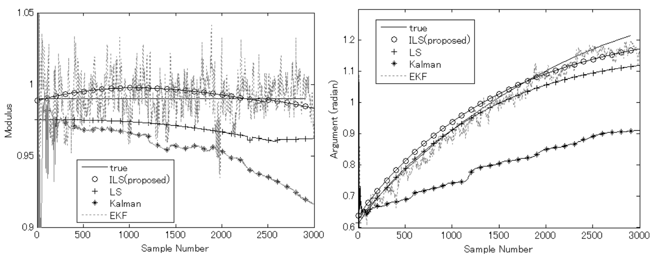

Fig. 5. Estimation of a pole modulus (left) and argument (right) for a 4th order TVAR order

2/ *Results for* $p = 4$

Here, $m = 2$ and $\lambda = 6.0 \times 10^{-5}$. σ_w^2 is equal to 0.31 and $c_1 = 1.5$. Simulation results are summarized in Table 2, where the SNR varies between $SNRmin = 13.5$ dB and $SNRmax$ =15.0dB.

The right hand side of Fig. 4 reports the NRMSE for various variance σ_w^2 and SNR such that $SNRmin$ and $SNRmax$ vary from 3.5dB to 38.5dB and from 5dB to 40dB respectively. The evolution in time of one estimated pole modulus and its argument are reported in Fig. 5.

5. CONCLUSIONS

This paper provides a method for estimating the TVAR parameters when the AR process is disturbed by a MA noise by using a variant of the improved least-squares (ILS) method, initially introduced by Zheng. According to the comparative study, in any case, the approach outperforms the standard LS approach, standard Kalman filter, and the extended Kalman filter. The convergence of the proposed iterative algorithm mainly depends on the SNR and the positions of the poles of the TVAR process

Concerning the order of the TVAR model p, according to other simulations, we have confirmed that the ILS method is feasible for large p ($p = 12$). However, the NRMSE becomes large as the order p becomes large for the same SNR. Furthermore, the risks of the iteration failures increase as the size of the function basis m becomes large.

Concerning the positions of the poles, we have confirmed that the ILS approach has high performance when the modulus of all poles vary between 0.8 and 1.0.

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