Sparse Vector Sensor Array Design Based on Quaternionic Formulations

Matthew B. Hawes and Wei Liu

Communications Research Group
Department of Electronic and Electrical Engineering
University of Sheffield, UK
{elp10mbh, w.liu}@sheffield.ac.uk

***Abstract*—In sparse arrays, the randomness of sensor locations avoids the introduction of grating lobes, while allowing adjacent sensor spacings to be greater than half a wavelength, leading to a larger array size with a relatively small number of sensors. In this paper, for the first time, the design of both robust and non-robust sparse vector sensor arrays is studied, and the proposed method is based on quaternionic formulations. It is a further extension of the recently proposed compressive sensing (CS) based design for traditional sparse arrays and the vector sensors being considered are crossed-dipoles. Design examples are presented to validate the effectiveness of the proposed method.**

***Index Terms*—Sparse array, quaternion beamformer, vector sensor array, compressive sensing, steering vector error.**

I. Introduction

For uniform linear arrays (ULAs) the adjacent antenna spacing has to be less than half a wavelength in order to avoid unwanted grating lobes [1], [2]. This can be problematic when considering arrays with a large aperture size, due to the cost associated with the number of antennas required. As a result, sparse arrays become a desirable alternative [3], which allow separations to be greater than half a wavelength, while still avoiding grating lobes due to the randomness of antenna locations.

However, the tradeoff in using sparse arrays is the unpredictable sidelobe behaviour and it is often necessary to optimise sensor locations in order to achieve a desired performance, (e.g. minimising the peak sidelobe level). Some nonlinear optimisation methods such as genetic algorithms (GAs) [4], [5], [6], [7], [8], and simulated annealing (SA)[9], have been regularly used to achieve the required optimisation. The disadvantage of these methods is the potentially long computation time and the possibility of convergence to a non-optimal solution.

More recently, the area of Compressive Sensing (CS) has been explored [10], and CS-based methods have been proposed in the design of sparse arrays [11], [12], [13], [14], [15] through l_1 norm minimisation. Most previous work tends to assume isotropic array elements, which means signal polarisation is not considered when looking at the array's performance. Instead, a vector sensor array can be considered allowing measurements of both the horizontal and vertical components of the received waveform [16], [17], [18], [19], [20], [21], [22], [23], [24], [25]. However, to our best knowledge, the design of sparse vector sensor arrays has not been addressed yet.

In the past ten years, quaternion-valued signal processing has attracted more and more attention with application areas involving three or four-dimensional signals, such as in wind profile prediction and wireless communications [26], [27], [28]. In particular, a quaternionic signal model was introduced into the field of vector sensor arrays for both adaptive beamforming and direction of arrival (DOA) estimation [19], [20], [22], [23], [24]. In this paper we extend the recently proposed CS-based sparse array design methods to the sparse vector-sensor array case based on such a quaternionic signal model. Similar to the traditional complex-valued l_1 minimisation schemes [29], the design is transformed into a form which can be readily solved using existing convex optimisation toolboxes [30], [31].

Moreover, model perturbations can cause a mismatch between designed and achieved steering vectors, leading to a change in the array's beam response. By assuming a norm-bounded steering vector error, it is possible to find the maximum possible change and add it as a constraint on the CS formulation to ensure it stays below a predetermined acceptable level [14], [15]. We will extend this idea to the vector sensor array case and derive a quaternion-valued formulation for the design of robust sparse vector sensors.

The remainder of this paper is structured as follows: Sec. II gives details of the proposed design method, including some basics about quaternions (II-A), the array model being considered (II-B), and the CS-based design method for sparse quaternionic arrays (II-C). Two design examples are presented in Sec. III, with conclusions drawn in Sec. IV.

II. Proposed Design Method for Sparse Quaternionic Beamformer

A. Quaternions

A quaternion is a hypercomplex number defined as follows [32]

$$q = R(q) + iI(q) + jJ(q) + kK(q) \tag{1}$$

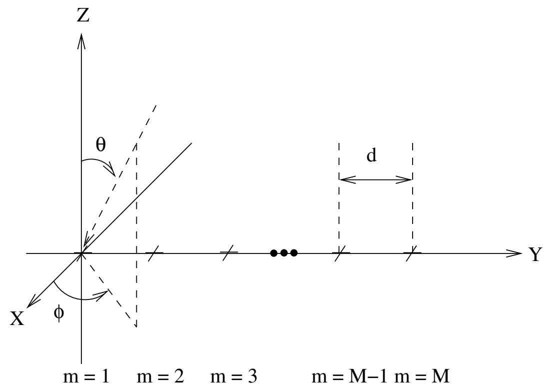

Fig. 1. Array model being considered with crossed-dipole elements.

where $R(q)$ is the real part of the quaternion and $I(q)$, $J(q)$ and $K(q)$ are the three imaginary components. Similarly for vectors and matrices of quaternions we have

$$\begin{aligned} \mathbf{v} &= R(\mathbf{v}) + iI(\mathbf{v}) + jJ(\mathbf{v}) + kK(\mathbf{v}), \\ \mathbf{M} &= R(\mathbf{M}) + iI(\mathbf{M}) + jJ(\mathbf{M}) + kK(\mathbf{M}). \end{aligned} \tag{2}$$

The conjugate and modulus of a quaternion are given by

$$q^* = R(q) - iI(q) - jJ(q) - kK(q), \tag{3}$$

$$|q| = \sqrt{R^2(q) + I^2(q) + J^2(q) + K^2(q)}. \tag{4}$$

The imaginary units i, j and k satisfy the following

$$ii = jj = kk = -1\ , \tag{5}$$

$$ij = -ji = k\ , jk = -kj = i\ , ki = -ik = j\ . \tag{6}$$

Finally $\{.\}^{\triangleleft}$ denotes the conjugate transpose of quaternionic vectors and matrices.

B. Quaternionic Array Model

Fig. 1 shows the array structure that is being considered. There are M potentially active crossed-dipole pairs located along the y-axis, uniformly spaced a distance d apart. At each location one of the dipoles is parallel to the x-axis and the other to the y-axis. Also shown is a signal with its direction of arrival (DOA) defined by the angles θ and ϕ. Without loss of generality, we assume the signals impinge upon the array from the y-z plane, i.e. $\phi = \pi/2$ or $\phi = -\pi/2$. The angle θ is limited as follows $0 \leq \theta \leq \pi/2$. A plane-wave signal model is considered, i.e. the signal impinges upon the array from the far field.

The spatial steering vector of the array is given by

$$\begin{aligned} \mathbf{s_s}(\theta,\phi) &= [1, e^{-j2\pi d \sin\theta \sin\phi/\lambda}, \\ &\quad \ldots, e^{-j2\pi(M-1)d\sin\theta\sin\phi/\lambda}]^T, \end{aligned} \tag{7}$$

where λ is the wavelength of the signal of interest and $\{.\}^T$ denotes the transpose operation. For crossed dipoles the spatial-polarization coherent vector contains information about a signals polarisation and is given by [16], [17], [18], [24]:

$$\mathbf{s_p}(\theta,\phi,\gamma,\eta) = \begin{cases} [-\cos\gamma, \cos\theta\sin\gamma e^{j\eta}] & \text{for } \phi = \pi/2 \\ [\cos\gamma, -\cos\theta\sin\gamma e^{j\eta}] & \text{for } \phi = -\pi/2\ , \end{cases} \tag{8}$$

where $\gamma \in [0, \pi/2]$ is the auxiliary polarization angle and $\eta \in [-\pi, \pi)$ is the polarization phase difference.

Now the array structure can be split into two sub-arrays, i.e. one parallel to the x-axis and one to y-axis. The steering vector of each of these sub-arrays is complex-valued and given by

$$\mathbf{s_x}(\theta,\phi,\gamma,\eta) = \begin{cases} -\cos\gamma\mathbf{s_s}(\theta,\phi) & \text{for } \phi = \pi/2 \\ \cos\gamma\mathbf{s_s}(\theta,\phi) & \text{for } \phi = -\pi/2 \end{cases} \tag{9}$$

and

$$\mathbf{s_y}(\theta,\phi,\gamma,\eta) = \begin{cases} \cos\theta\sin\gamma e^{j\eta}\mathbf{s_s}(\theta,\phi) & \text{for } \phi = \pi/2 \\ -\cos\theta\sin\gamma e^{j\eta}\mathbf{s_s}(\theta,\phi) & \text{for } \phi = -\pi/2. \end{cases} \tag{10}$$

These are then combined to give an overall quaternionic steering vector as follows:

$$\hat{\mathbf{s}}(\theta,\phi,\gamma,\eta) = \mathbf{s_x}(\theta,\phi,\gamma,\eta) + i\mathbf{s_y}(\theta,\phi,\gamma,\eta). \tag{11}$$

The response of the array is given by

$$p(\theta,\phi,\gamma,\eta) = \mathbf{w}^{\triangleleft}\hat{\mathbf{s}}(\theta,\phi,\gamma,\eta), \tag{12}$$

where $\mathbf{w}$ is the quaternionic weight coefficient vector defined as

$$\mathbf{w} = [w_1\ \ w_2\ \ \ldots\ \ w_M]^T\ , \tag{13}$$

and w_m is a quaternionic value for $m = 1, 2, \cdots, M$.

However, it is possible that there can be a mismatch between designed and actual steering vectors due to all kinds of model errors. Suppose that the actual steering vector is related to the assumed steering vector by $\mathbf{s} = \hat{\mathbf{s}} + \tilde{\mathbf{s}}$, where $\hat{\mathbf{s}}$ is the designed steering vector (given by (11)) and $\tilde{\mathbf{s}}$ is the error introduced by model perturbations. If this error is norm-bounded, i.e.

$$||\tilde{\mathbf{s}}||_2 \leq \varepsilon, \tag{14}$$

where ε is the upper bound, then the possible difference between designed and actually achieved array responses satisfies

$$|\mathbf{w}^{\triangleleft}\mathbf{s} - \mathbf{w}^{\triangleleft}\hat{\mathbf{s}}| = |\mathbf{w}^{\triangleleft}\tilde{\mathbf{s}}| \leq \varepsilon||\mathbf{w}||_2\ . \tag{15}$$

Later this can be added as an extra constraint on the formulation to ensure the change stays below a predetermined acceptable level so that a robust design can be achieved.

C. Quaternionic Compressive Sensing

Suppose the desired/reference response is given by $P_r(\theta,\phi,\gamma,\eta)$. First, consider Fig. 1 as being a grid of potentially active crossed dipole locations. In this instance, $(M-1)d$ is the aperture of the array and M is a large number. Sparseness is then introduced by selecting the weight coefficients to give as few active crossed-dipoles as possible, while still giving a designed response that is close to the desired one.

This problem is formulated as

$$\begin{array}{ll} \min & ||\mathbf{w}||_0 \\ \text{subject to} & ||\mathbf{p}_r - \mathbf{w}^{\triangleleft}\mathbf{S}||_2 \leq \alpha \end{array} \tag{16}$$

where $||\mathbf{w}||_0$ is the number of nonzero weight coefficients in $\mathbf{w}$, $\mathbf{p}_r$ is the vector holding the desired beam response at the sampled angular and polarisation points of interest, $\mathbf{S}$ is the matrix composed of the corresponding steering vectors and α places a limit on the allowed difference between the desired and the designed responses. In this constraint $||\cdot||_2$ denotes the l_2 norm.

In detail, $\mathbf{p}_r$ and $\mathbf{S}$ are respectively given by

$$\begin{aligned} \mathbf{p}_r &= [P_r(\theta_1,\phi_1,\gamma_1,\eta_1),\cdots,P_r(\theta_L,\phi_L,\gamma_L,\eta_L)] \\ \mathbf{S} &= [\mathbf{s}(\theta_1,\phi_1,\gamma_1,\eta_1),\cdots,\mathbf{s}(\theta_L,\phi_L,\gamma_L,\eta_L)], \end{aligned}$$

where L is number of points sampled at each dimension of the desired beam response. In this work the desired response is that of the ideal response, i.e. a value of one for the mainlobe and zeros for the other entries.

However, (16) is computationally expensive and the problem can be more efficiently expressed as a minimisation of the l_1 norm of the weight coefficients [10], i.e.

$$\begin{array}{ll} \min & ||\mathbf{w}||_1 \\ \text{subject to} & ||\mathbf{p}_r - \mathbf{w}^{\triangleleft}\mathbf{S}||_2 \leq \alpha\ . \end{array} \tag{17}$$

This formulation is effective for the design of sparse arrays with real-valued weight coefficients. However, when considering quaternionic coefficients the problem has to be reformulated to ensure the real and three imaginary parts of the quaternion are simultaneously minimised. This is achieved by following a scheme similar to that used when considering the l_1 minimisation of complex data [29]. When all four parts of a quaternionic coefficient are equal to zero, the crossed dipoles can be considered inactive/not present, and as a result sparsity is introduced.

First we rewrite (17) as

$$\begin{array}{ll} \min & t\ \epsilon\ \mathbb{R}^+ \\ \text{subject to} & ||\mathbf{p}_r - \mathbf{w}^{\triangleleft}\mathbf{S}||_2 \leq \alpha\ , \quad |\langle\mathbf{w}\rangle|_1 \leq t \end{array}$$

where

$$|\langle\mathbf{w}\rangle|_1 = \sum_{m=1}^{M} \left|\left| \begin{bmatrix} R(w_m) \\ I(w_m) \\ J(w_m) \\ K(w_m) \end{bmatrix} \right|\right|_2 . \tag{18}$$

Now we decompose t to $t = \sum_{m=1}^{M} t_m,\ t_m \epsilon\ \ \mathbb{R}^+$. In vector form, we have

$$t = [1,\cdots,1] \begin{bmatrix} t_1 \\ \vdots \\ t_M \end{bmatrix} = \mathbf{1}^T\mathbf{t}. \tag{19}$$

Then (18) can be rewritten as

$$\begin{array}{ll} \min\limits_{\mathbf{t}} & \mathbf{1}^T\mathbf{t} \\ \text{subject to} & ||\mathbf{p}_r - \mathbf{w}^{\triangleleft}\mathbf{S}||_2 \leq \alpha \\ & \left|\left| \begin{bmatrix} R(w_m) \\ I(w_m) \\ J(w_m) \\ K(w_m) \end{bmatrix} \right|\right|_2 \leq t_m, \quad m = 1,\cdots,M. \end{array} \tag{20}$$

Now define

$$\begin{aligned} \hat{\mathbf{w}} &= [t_1, R(w_1), -I(w_1), -J(w_1), -K(w_1), \cdots, \\ &\quad R(w_M), -I(w_M), -J(w_M), -K(w_M)]]^T \quad (21) \\ \hat{\mathbf{c}} &= [1,0,0,0,0,\cdots,1,0,0,0,0]^T\ , \quad (22) \\ \hat{\mathbf{p}}_r &= [R(\mathbf{p}_r), I(\mathbf{p}_r), J(\mathbf{p}_r), K(\mathbf{p}_r)] \quad (23) \end{aligned}$$

and

$$\hat{\mathbf{S}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ R(\mathbf{s}_1) & I(\mathbf{s}_1) & J(\mathbf{s}_1) & K(\mathbf{s}_1) \\ -I(\mathbf{s}_1) & R(\mathbf{s}_1) & -K(\mathbf{s}_1) & J(\mathbf{s}_1) \\ -J(\mathbf{s}_1) & K(\mathbf{s}_1) & R(\mathbf{s}_1) & -I(\mathbf{s}_1) \\ -K(\mathbf{s}_1) & -J(\mathbf{s}_1) & I(\mathbf{s}_1) & R(\mathbf{s}_1) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ R(\mathbf{s}_M) & I(\mathbf{s}_M) & J(\mathbf{s}_M) & K(\mathbf{s}_M) \\ -I(\mathbf{s}_M) & R(\mathbf{s}_M) & -K(\mathbf{s}_M) & J(\mathbf{s}_M) \\ -J(\mathbf{s}_M) & K(\mathbf{s}_M) & R(\mathbf{s}_M) & -I(\mathbf{s}_M) \\ -K(\mathbf{s}_M) & -J(\mathbf{s}_M) & I(\mathbf{s}_M) & R(\mathbf{s}_M) \end{pmatrix}, \tag{24}$$

where $\mathbf{s}_m$ contains the designed contribution of the m^{th} vector sensor to the array's steering vector for all combinations of θ,ϕ,γ and η, of interest.

Finally we arrive at the final formulation, with the additional robustness constraint $\varepsilon||\mathbf{w}||_2 \leq \beta$, for the sparse vector sensor array design problem

$$\begin{array}{ll} \min\limits_{\hat{\mathbf{w}}} & \mathbf{c}^T\hat{\mathbf{w}} \\ \text{subject to} & ||\hat{\mathbf{p}}_r - \hat{\mathbf{w}}^T\hat{\mathbf{S}}||_2 \leq \alpha \\ & \left|\left| \begin{bmatrix} R(w_m) \\ I(w_m) \\ J(w_m) \\ K(w_m) \end{bmatrix} \right|\right|_2 \leq t_m, \quad m = 1,\cdots,M \\ & \varepsilon||\mathbf{w}||_2 \leq \beta\ , \end{array} \tag{25}$$

which can be solved using cvx, a package for specifying and solving convex problems [30], [31]. Note that without the robustness constraint, the above formulation will change to the design of standard sparse vector sensor arrays.

III. Design Examples

In this section design examples are provided to verify the effectiveness of the proposed method. For both cases

TABLE I
RESULTANT CROSSED-DIPOLE LOCATIONS WITHOUT ROBUSTNESS CONSTRAINT.

n	d_n/λ	n	d_n/λ
1	1.55	6	5.38
2	2.35	7	6.13
3	3.13	8	6.88
4	3.88	9	7.65
5	4.63	10	8.45

there are 201 potential active crossed-dipole locations over a maximum aperture of 10λ. Also, in the figures that follow positive values of θ indicate the value range $\theta \in [0^\circ,\ 90^\circ]$ for $\phi = 90^\circ$, while negative values of $\theta \in [-90^\circ,\ 0^\circ]$ indicate an equivalent range of $\theta \in [0^\circ,\ 90^\circ]$ with $\phi = -90^\circ$.

The average achieved response and variance of responses are calculated when assessing the robustness of an array. When doing so 1000 different error vectors are generated that meet the condition given in (14). For the n^{th} error vector the achieved response for angles θ_k and ϕ_k, $P_n(\theta_k, \phi_k)$, is found and the average achieved response is given by

$$\bar{P}(\theta_k, \phi_k) = \frac{1}{N} \sum_{n=0}^{N-1} P_n(\theta_k, \phi_k), \quad (26)$$

which is then used to find the normalised variance of the achieved array response,

$$var(\theta_k, \phi_k) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{|P_n(\theta_k, \phi_k) - \bar{P}(\theta_k, \phi_k)|^2}{|\bar{P}(\theta_k, \phi_k)|}, \quad (27)$$

where N is the number of different error vectors used. A close match between mean achieved and designed responses, along with low variance levels, would indicate that robustness has been achieved.

For the first design example, the mainlobe is defined by $\theta_{ML} = 0^\circ$ and $\phi_{ML} = 90^\circ$. The sidelobe regions are given by $\theta_{SL} = [10^\circ, 90^\circ]$ for both $\phi_{SL} = 90^\circ$ and $\phi_{SL} = -90^\circ$. The polarisations being considered are given by $(\gamma, \eta) = (0^\circ, 0^\circ)$. Finally the value of $\alpha = 0.95$ was used.

The resulting array has 10 active crossed dipoles spread over an aperture of 6.90λ with an average spacing of 0.77λ (see Table. I), indicating that some sparsity has been achieved. From Fig. 2 we can see that the mainlobe is in the desired location and sufficient sidelobe attenuation has been achieved.

Now the robustness constraint is added with the values $\beta = 0.01$ and $\varepsilon = 1$ used. This resulted in 12 active crossed dipole locations over an aperture of 6.90λ, as detailed in Table. II, giving a mean adjacent separation of 0.63λ. We can see the cost of meeting the robustness constraint is that extra crossed dipoles are required. The final beam responses are shown in Fig. 3 and the corresponding normalised variance levels shown in Fig. 4. We can see there for both designed and mean achieved responses the mainlobe is in the correct location and there is sufficient sidelobe attenuation. There is also a reasonable match between the two, along with the low normalised variance levels this indicates some degree of robustness has been achieved.

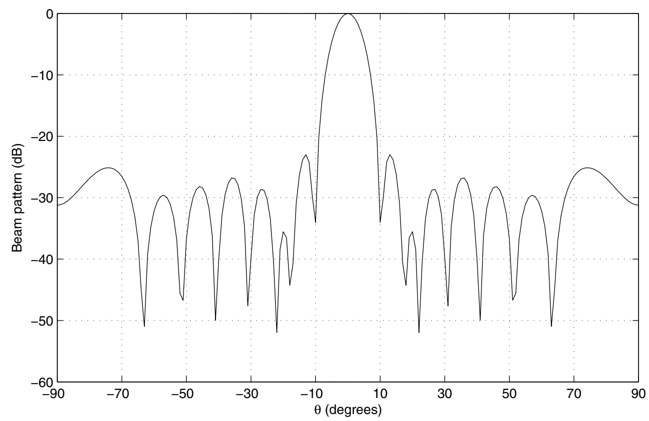

Fig. 2. Narrowband design example with no robustness constraint.

TABLE II
RESULTANT CROSSED-DIPOLE LOCATIONS WITH ROBUSTNESS CONSTRAINT.

n	d_n/λ	n	d_n/λ	n	d_n/λ
1	1.55	5	4.63	9	6.13
2	2.35	6	4.7	10	6.88
3	3.13	7	5.33	11	7.65
4	3.88	8	5.4	12	8.45

IV. CONCLUSIONS

In this paper a method for designing sparse quaternionic arrays using compressive sensing has been proposed. The real valued l_1 minimisation problem is reformulated as a modified l_1 minimisation, in order to ensure that the real and three imaginary parts of the quaternionic weight coefficients are simultaneously minimised. An extra constraint is also added to ensure that a robust solution can be reached when required. Design examples are provided to verify the effectiveness

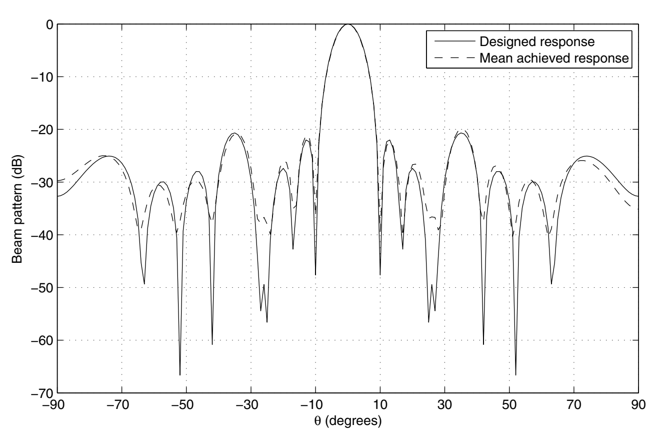

Fig. 3. Narrowband design example with robustness constraint.

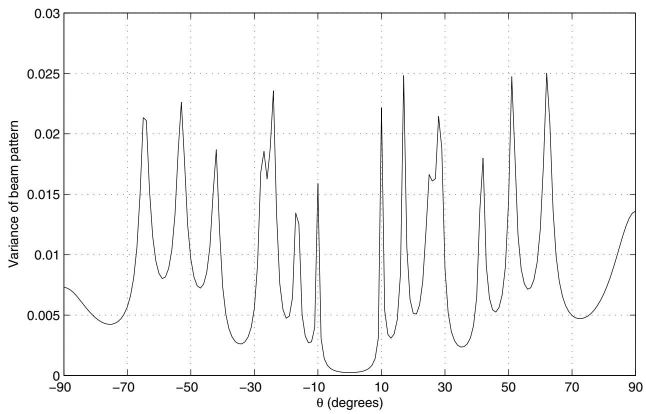

Fig. 4. Normalised variance levels.

of the proposed design method, with the levels of sparsity being determined by the value of the limits placed on the constraints. In other words the smaller the value of the limits that is placed on the constraints the less sparse the solution will be.

REFERENCES

[1] H. L. Van Trees, *Optimum Array Processing, Part IV of Detection, Estimation, and Modulation Theory*, John Wiley & Sons, Inc., New York, U.S.A., 2002.

[2] W. Liu and S. Weiss, *Wideband Beamforming: Concepts and Techniques*, John Wiley & Sons, Chichester, UK, 2010.

[3] P. Jarske, T. Saramaki, S. K. Mitra, and Y. Neuvo, "On properties and design of nonuniformly spaced linear arrays," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 36, no. 3, pp. 372 –380, March 1988.

[4] R. L. Haupt, "Thinned arrays using genetic algorithms," *IEEE Transactions on Antennas and Propagation*, vol. 42, no. 7, pp. 993–999, July 1994.

[5] Keen-Keong Yan and Yilong Lu, "Sidelobe reduction in array-pattern synthesis using genetic algorithm," *IEEE Transactions on Antennas and Propagation*, vol. 45, no. 7, pp. 1117 –1122, July 1997.

[6] K. Chen, Z. He, and C. Han, "Design of 2-dimensional sparse arrays using an improved genetic algorithm," in *Proc. IEEE Workshop on Sensor Array and Multichannel Processing*, July 2006, pp. 209–213.

[7] Ling Cen, Wee Ser, Zhu Liang Yu, and S. Rahardja, "An improved genetic algorithm for aperiodic array synthesis," in *Proc. IEEE International Conference on Acoustics, Speech, and Signal Processing*, April 2008, pp. 2465 –2468.

[8] M.B. Hawes and Wei Liu, "Location optimization of robust sparse antenna arrays with physical size constraint," *IEEE Antennas and Wireless Propagation Letters*, vol. 11, pp. 1303 –1306, 2012.

[9] A. Trucco and V. Murino, "Stochastic optimization of linear sparse arrays," *IEEE Journal of Oceanic Engineering*, vol. 24, no. 3, pp. 291–299, July 1999.

[10] E.J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489 – 509, February 2006.

[11] L. L. Li, W. J. Zhang, and F. Li, "The design of sparse antenna array," *CoRR*, vol. arXiv.org/abs/0811.0705, 2008.

[12] G. Prisco and M. D'Urso, "Exploiting compressive sensing theory in the design of sparse arrays," in *Proc. IEEE Radar Conference*, May 2011, pp. 865 –867.

[13] L. Carin, "On the relationship between compressive sensing and random sensor arrays," *IEEE Antennas and Propagation Magazine*, vol. 51, no. 5, pp. 72 –81, October 2009.

[14] M. B. Hawes and W. Liu, "Compressive sensing based approach to the design of linear robust sparse antenna arrays with physical size constraint," *IET Microwaves, Antennas & Propagation*, 2014, DOI: 10.1049/iet-map.2013.0469.

[15] M. B. Hawes and W. Liu, "Sparse array design for wideband beamforming with reduced complexity in tapped delay-lines," *IEEE Trans. Audio, Speech and Language Processing*, 2014, DOI: 10.1109/TASLP.2014.2327298.

[16] Jr. Compton, R.T., "On the performance of a polarization sensitive adaptive array," *IEEE Transactions on Antennas and Propagation*, vol. 29, no. 5, pp. 718–725, 1981.

[17] J. Li and R. T. Compton Jr, "Angle and polarization estimation using esprit with a polarization sensitive array," *IEEE Transactions on Antennas and Propagation*, vol. 39, no. 9, pp. 1376–1383, 1991.

[18] D. B. Williams and V. Madisetti, Eds., *Digital Signal Processing Handbook*, CRC Press, Inc., Boca Raton, FL, USA, 1st edition, 1997.

[19] S. Miron, N. le Bihan, and J.I. Mars, "High resolution vector-sensor array processing using quaternions," in *Proc. IEEE Workshop on Statistical Signal Processing*, 2005, pp. 918–923.

[20] S. Miron, N. Le Bihan, and J.I. Mars, "Quaternion-music for vector-sensor array processing," *IEEE Transactions on Signal Processing*, vol. 54, no. 4, pp. 1218–1229, 2006.

[21] Jin-Jun Xiao and Arye Nehorai, "Optimal polarized beampattern synthesis using a vector antenna array," *IEEE Transactions on Signal Processing*, vol. 57, no. 2, pp. 576–587, 2009.

[22] Xiaoming Gou, Yougen Xu, ZhiWen Liu, and Xiaofeng Gong, "Quaternion-capon beamformer using crossed-dipole arrays," in *IEEE International Symposium on Microwave, Antenna, Propagation, and EMC Technologies for Wireless Communications (MAPE)*, 2011, pp. 34–37.

[23] Jian-Wu Tao and Wen-Xiu Chang, "A novel combined beamformer based on hypercomplex processes," *IEE Transactions on Aerospace and Electronic Systems*, vol. 49, no. 2, pp. 1276–1289, 2013.

[24] X. R. Zhang, W. Liu, Y. G. Xu, and Z. W. Liu, "Quaternion-valued robust adaptive beamformer for electromagnetic vector-sensor arrays with worst-case constraint," *Signal Processing*, vol. 104, pp. 274–283, November 2014.

[25] X. R. Zhang, Z. W. Liu, Y. G. Xu, and W. Liu, "Adaptive tensorial beamformer based on electromagnetic vector-sensor arrays with coherent interferences," *Multidimensional Systems and Signal Processing*, 2014, DOI: 10.1007/s11045-014-0281-8.

[26] M. D. Jiang, W. Liu, and Y. Li, "A zero-attracting quaternion-valued least mean square algorithm for sparse system identification," in *Proc. of IEEE/IET International Symposium on Communication Systems, Networks and Digital Signal Processing*, Manchester, UK, July 2014.

[27] M. D. Jiang, W. Liu, and Y. Li, "A general quaternion-valued gradient operator and its applications to computational fluid dynamics and adaptive beamforming," in *Proc. of the International Conference on Digital Signal Processing*, Hong Kong, August 2014.

[28] W. Liu, "Antenna array signal processing for a quaternion-valued wireless communication system," in *The Benjamin Franklin Symposium on Microwave and Antenna Sub-systems (BenMAS)*, Philadelphia, US, September 2014.

[29] S. Winter, H. Sawada, and S. Makino, "On real and complex valued l_1-norm minimization for overcomplete blind source separation," in *Proc. of IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, October 2005, pp. 86 – 89.

[30] CVX Research, "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, September 2012.

[31] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, V. Blondel, S. Boyd, and H. Kimura, Eds., Lecture Notes in Control and Information Sciences, pp. 95–110. Springer-Verlag, 2008.

[32] I. Kantor, A.S. Solodovnikov, and Abe Shenitzer, *Hypercomplex numbers: an elementary introduction to algebras*, Springer Verlag, New York, 1989.