EFFICIENT SPECTRAL ANALYSIS IN THE MISSING DATA CASE USING SPARSE ML METHODS

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ABSTRACT

Given their wide applicability, several sparse high-resolution spectral estimation techniques and their implementation have been examined in the recent literature. In this work, we further the topic by examining a computationally efficient implementation of the recent SMLA algorithms in the missing data case. The work is an extension of our implementation for the uniformly sampled case, and offers a notable computational gain as compared to the alternative implementations in the missing data case.

***Index Terms*—** Spectral estimation theory and methods, Sparse Maximum Likelihood methods, fast algorithms.

1. INTRODUCTION

High-resolution spectral estimation algorithms find applications in a wide range of fields, and the topic have been frequently examined in the recent literature. In particular a range of non-parametric high-resolution spectral estimation algorithms have been examined, as well as various ways to efficiently implement these estimators. Among the discussed methods, the data-adaptive approaches are known to yield a preferable performance (see, e.g., [1–4]), although these methods will typically require large data sets to offer reliable estimates of the second-order statistics, a requirement that is hard to satisfy in practice. In order to alleviate this, recent approaches often impose sparsity constraints on the estimators, and methods such as the sparse learning via iterative minimization (SLIM) method [5], the iterative adaptive approach (IAA) [6], and more recently a set of iterative sparse maximum likelihood-based approaches (SMLA) [7,8], which have all been shown to offer significant performance improvements as compared to the traditional methods [9–12]. Regrettably, these methods all suffer from being computationally cumbersome, resulting in a series of studies aiming to formulate computationally efficient implementations for these algorithms [12–14]. In this work, we further this development, presenting a computationally efficient implementation of the SMLA algorithms in the missing data case, extending on our recent contribution on how to efficiently implement the uniformly sampled case.

2. AN OVERVIEW OF THE SMLA APPROACH

Let $\{y_n\}_{n=0}^{N-1} \in \mathbb{C}$ denote a uniformly sample sequence of observations affected by the additive complex circular Gaussian white noise for which one wish to compute a power spectral estimate. Form the data and frequency vectors

$$\mathbf{y}_N \triangleq \begin{bmatrix} y_0 & \dots & y_{N-1} \end{bmatrix}^T \tag{1}$$

$$\mathbf{f}_N(\omega_k) \triangleq \begin{bmatrix} 1 & e^{\jmath\omega_k} & \dots & e^{\jmath\omega_k(N-1)} \end{bmatrix}^T \tag{2}$$

where $\omega_k = 2\pi k/K$, for $k = 0, \dots, K$, with $K > N+1$, and denote the power of the signal $p_k \triangleq |x(\omega_k)|^2$, where $x(\omega_k)$ is the complex-valued spectral amplitude at frequency ω_k. An estimate of the complex covariance matrix of $\mathbf{y}_N$ is then obtained as $\mathbf{R}_N \triangleq \sum_{k=0}^{K-1} p_k \mathbf{f}_N(\omega_k)\mathbf{f}_N^H(\omega_k) + \mathbf{\Sigma}_N$ where $\mathbf{\Sigma}_N \triangleq \sigma\mathbf{I}_N$, with σ denoting the variance of the additive noise process.

In [7, 8], a family of four different SMLA power estimation algorithms was derived, where, for the all frequencies of interest, the SMLA power estimates are formed by iteratively computing the estimates p_k, $\mathbf{R}_N$, and $\mathbf{\Sigma}_N$, until practical convergence. The resulting algorithm is computationally cumbersome, and, in earlier work, we have strived to alleviate this by proposing more efficient implementations for the case of uniformly sampled data sets [15]. In this paper, we extend on this work to the case of arbitrarily sampled data sets. Consider the vector of available (or given) data and the corresponding frequency vector

$$\mathbf{y}_{N_g} = \mathbf{S}_{N_g N}\mathbf{y}_N, \quad \mathbf{f}_{N_g}(\omega_k) = \mathbf{S}_{N_g N}\mathbf{f}_N(\omega_k) \tag{3}$$

where $\mathbf{S}_{N_g N}$ is a $N_g \times N$ selection matrix, with zeros and ones in proper places (see also [16]), and where $N_g \leq N$ denotes the number of available data samples. Also, let $\mathbf{S}_{N_m N}$ be the selection matrix corresponding to the missing data. Of the algorithms presented in [7, 8], we here consider the implementation of the most involved of the there presented algorithms, the so-called SMLA-3 estimator, noting that the others can be implemented similarly. We term the here presented estimator the missing data SMLA-3 (MSMLA-3) estimator. This estimate may be formed

$$p_k = \frac{|\mathbf{f}^H_{N_g}(\omega_k)\tilde{\mathbf{R}}^{-1}_{N_g}\mathbf{y}_{N_g}|^2}{(\mathbf{f}^H_{N_g}(\omega_k)\mathbf{R}^{-1}_{N_g}\mathbf{f}_{N_g}(\omega_k))^2} \quad (4)$$

$$\tilde{p}_k = \frac{1}{\mathbf{f}^H_{N_g}(\omega_k)\mathbf{R}^{-1}_{N_g}\mathbf{f}_{N_g}(\omega_k)} \quad (5)$$

$$\mathbf{R}_{N_g} = \sum_{k=0}^{K-1} p_k \mathbf{f}_{N_g}(\omega_k)\mathbf{f}^H_{N_g}(\omega_k) + \sigma \mathbf{I}_{N_g} \quad (6)$$

$$\tilde{\mathbf{R}}_{N_g} = \sum_{k=0}^{K-1} \tilde{p}_k \mathbf{f}_{N_g}(\omega_k)\mathbf{f}^H_{N_g}(\omega_k) + \sigma \mathbf{I}_{N_g} \quad (7)$$

$$\sigma = \frac{\|\mathbf{R}^{-1}_{N_g}\mathbf{y}_{N_g}\|^2}{\mathrm{Tr}[\mathbf{R}^{-2}_{N_g}]} \quad (8)$$

where (4)-(8) are iteratively computed until practically convergence, with, usually, 10-15 iterations being sufficient for convergence. Direct implementation of the MSMLA-3 estimate is computational intensive, requiring of the order of $\mathcal{C}^{BF} = m\left(3N_g^3 + 4N_g^2K\right)$ operations, where m denotes the number of the MSMLA iterations performed. Although fast algorithms are available for the computation of the SMLA power spectral estimates in the complete data case, these are not directly applicable in the missing data scenario, as the pertinent covariance matrices are no longer Toeplitz matrices. However, as it has been reported in [12, 13], savings in the computational burden is still possible, noting that

$$\mathbf{R}_{N_g} = \mathbf{S}_{N_gN}\mathbf{R}_N\mathbf{S}^T_{N_gN}, \quad \tilde{\mathbf{R}}_{N_g} = \mathbf{S}_{N_gN}\tilde{\mathbf{R}}_N\mathbf{S}^T_{N_gN} \quad (9)$$

where

$$\mathbf{R}_N = \sum_{k=0}^{K-1} p_k \mathbf{f}_N(\omega_k)\mathbf{f}^H_N(\omega_k) + \sigma \mathbf{I}_N \quad (10)$$

$$\tilde{\mathbf{R}}_N = \sum_{k=0}^{K-1} \tilde{p}_k \mathbf{f}_N(\omega_k)\mathbf{f}^H_N(\omega_k) + \sigma \mathbf{I}_N \quad (11)$$

are Toeplitz covariance matrices, formed as in the complete data case. The resulting scheme, discussed in [17], has a complexity approximately given by $\mathcal{O}(N_g^3 + K\log_2(K))$, which is notably less than the brute force approach. However, it may be noted that when $N_m \ll N_g$, an assumption that may be valid in many cases of interest, the gain offered by this approach is marginal. In order to improve the performance also for this case, we here develop a fast implementation of the MSMLA-3 spectral estimation algorithm in the case when $N_m < N_g$, by resorting to the technique recently proposed in [18, 19] in the context of missing data IAA spectral estimation.

3. TECHNICAL BACKGROUND

In order to facilitate the derivation of the fast implementation, we here examine some of the needed technical material. Consider a Hermitian matrix $\mathbf{A}_N \in \mathcal{C}^{N\times N}$ and define the lower shifting matrix

$$\mathbf{Z}_N = \begin{bmatrix} \mathbf{0}^T & 0 \\ \mathbf{I}_{N-1} & \mathbf{0} \end{bmatrix}. \quad (12)$$

The displacement of $\mathbf{A}_N$, with respect to $\mathbf{Z}_N$ and $\mathbf{Z}_N^T$ is defined as $\nabla_{\mathbf{Z}_N,\mathbf{Z}_N^T}\mathbf{A}_N \triangleq \mathbf{A}_N - \mathbf{Z}_N\mathbf{A}_N\mathbf{Z}_N^T$. Suppose that there exist integers ρ and $\sigma_i \in \{-1, 1\}$, for $i = 1, 2, \ldots, \rho$, such that (see also [20–22])

$$\nabla_{\mathbf{Z}_N,\mathbf{Z}_N^T}\mathbf{A}_N = \sum_{i=1}^{\rho}\sigma_i\mathbf{t}_N^{(i)}\mathbf{s}_N^{(i)H} = \mathbf{T}_{N,\rho}\mathbf{\Sigma}_\rho\mathbf{S}^H_{N,\rho} \quad (13)$$

where $\mathbf{\Sigma}_\rho = \mathrm{diag}\left\{\begin{bmatrix}\sigma_1 & \cdots & \sigma_\rho\end{bmatrix}^T\right\}$ and

$$\mathbf{T}_{N,\rho} = \begin{bmatrix}\mathbf{t}_N^{(1)} & \cdots & \mathbf{t}_N^{(\rho)}\end{bmatrix}, \mathbf{S}_{N,\rho} = \begin{bmatrix}\mathbf{s}_N^{(1)} & \cdots & \mathbf{s}_N^{(\rho)}\end{bmatrix}$$

with $\mathrm{diag}(\mathbf{x})$ denoting the diagonal matrix formed with the vector $\mathbf{x}$ along its diagonal, and with $\mathbf{t}_N^{(i)}$ and $\mathbf{s}_N^{(i)}$ being the ith so-called generator vector. The displacement representation allows for the development of useful results for computation of expressions including $\mathbf{A}_N$.

Lemma 1 ([20,21]) *The Gohberg-Semencul (GS) factorization of* $\mathbf{A}_N$ *may be expressed as*

$$\mathbf{A}_N = \sum_{i=1}^{\rho}\sigma_i\mathcal{L}\left(\mathbf{Z}_N, \mathbf{t}_N^{(i)}\right)\mathcal{L}^H\left(\mathbf{Z}_N, \mathbf{s}_N^{(i)}\right) \quad (14)$$

where $\mathcal{L}(\mathbf{Z}_N, \mathbf{b}_N)$ *denotes a Krylov matrix of the form* $\mathcal{L}(\mathbf{Z}_N, \mathbf{b}_N) = [\mathbf{b}_N \; \mathbf{Z}_M\mathbf{b}_N \; \mathbf{Z}_N^2\mathbf{b}_N \; \ldots \; \mathbf{Z}_N^{N-1}\mathbf{b}_N]$.

Lemma 2 ([15,23]) *The trace of* $\mathbf{A}_N$ *can be computed as*

$$\mathrm{Tr}\left[\mathbf{A}_N\right] = \sum_{k=1}^{N-1}(N+1-k)\boldsymbol{\delta}_N[k] \quad (15)$$

where $\boldsymbol{\delta}_N = \sum_{i=1}^{\rho}\sigma_i\mathbf{t}_N^i \odot \mathbf{s}_N^{i*}$ *with* $\boldsymbol{\delta}_N[k]$ *denoting the* k*-th element of* $\boldsymbol{\delta}_N$ *and* $\odot$ *the Hadamard (pointwise) product.*

Lemma 1 implies that given the displacement representation of matrix $\mathbf{A}_B$, matrix-vector products may be computed at a cost of $\mathcal{O}\left(\rho N\log_2(N)\right)$ operations using the Fast Fourier Transform (FFT). Moreover, its trace $\mathrm{Tr}\left[\mathbf{A}_N\right]$ can be computed at a cost of $\mathcal{O}\left(\rho N\right)$, as it is suggested by Lemma 2.

Lemma 3 *The coefficients of the trigonometric polynomial associated with* $\mathbf{A}_N$

$$\psi(\omega) \triangleq \mathbf{f}^H_N(\omega)\mathbf{A}_N\mathbf{f}_M(\omega) = \sum_{\kappa=-N+1}^{N-1}c_\kappa e^{-j\kappa\omega} \quad (17)$$

can be estimated at a cost of $\mathcal{O}(\rho N\log_2(2N))$ *using the method detailed in [25].*

Algorithm 1 Proposed implementation

1: Given p_k and $\tilde{p}_k$, for $k = 0, 1, \dots K-1$, compute the first column of $\mathbf{R}_N$ and $\tilde{\mathbf{R}}_N$, using the FFT, at a cost of $\mathcal{O}(K \log 2(K))$.
2: Compute the displacement representation of $\mathbf{R}_N^{-1}$ and $\tilde{\mathbf{R}}_N^{-1}$ using Lemma 4, at a cost of $\mathcal{O}(N^2)$ operations. Compute the displacement representation of $\mathbf{R}_N^{-2}$ with an additional cost of $\mathcal{O}(N \log N)$ operations.
3: Reconstruct $\mathbf{R}_N^{-1}$ and $\tilde{\mathbf{R}}_N^{-1}$ using Trench's algorithm for the estimation of the inverse of Toeplitz matrices [24, p. 132], at a cost of $\mathcal{O}(N^2)$ operations. Compute the matrices $\mathbf{B}_N$ and $\tilde{\mathbf{B}}_N$, corresponding to $\mathbf{R}_N$ and $\tilde{\mathbf{R}}_N$, as in Lemma 6. This may be done at a cost of $\mathcal{O}(N_m^3 + N_m N \log(N))$ operations using the displacement representations of $\mathbf{R}_N^{-1}$ and $\tilde{\mathbf{R}}_N^{-1}$.
4: Compute the numerators and the denominators of (4) and (5) by using the decompositions (21) and (22). The first part may be computed in $\mathcal{O}(K \log K)$ operations [15] and the low rank part in $\mathcal{O}(N_m N \log(N) + K \log K)$ operations [18, 19].
5: Compute $\mathrm{Tr}[\mathbf{R}_{N_g}^{-2}]$ using Corollary 7. By Lemma 2, the first term of (20) may be computed in $\mathcal{O}(N)$ operations. The bottleneck for the second term is the multiplication $\mathbf{R}_N^{-1}\mathbf{B}_N$ which may be done at a cost of $\mathcal{O}(N_m N \log(N))$ operations. By noting that

$$\mathbf{B}_N^H \mathbf{B}_N = \mathbf{L}^H \mathbf{S}_{N_m N}^T \left(\mathbf{R}_N^{-1}\mathbf{B}_N\right) \tag{16}$$

the final term may be computed at an additional cost of $\mathcal{O}(N_m^3)$ operations.

Let $\mathbf{R}_N$ denotes a Hermetian and positive definite Toeplitz matrix. A displacement representation of $\mathbf{R}_N^{-1}$ and $\mathbf{R}_N^{-2}$ may be estimated as follows.

Lemma 4 ([4, 20, 21]) *A displacement representation of $\mathbf{R}_N^{-1}$ with respect to $\mathbf{Z}_N$ and $\mathbf{Z}_N^T$ and with $\rho = 2$, is given by*

$$\mathbf{t}_N^1 = \mathbf{R}_N^{-1}\mathbf{e}_N^1 \sqrt{\mathbf{e}_N^{1T}\mathbf{t}_N^1}, \quad \mathbf{t}_N^2 = \mathbf{Z}_N \left(\mathbf{J}_N \mathbf{t}_N^1\right)^*$$

where $(\cdot)^*$ denotes the complex conjugate, with $\mathbf{s}_N^1 = \mathbf{t}_N^1$, $\mathbf{s}_N^2 = \mathbf{t}_N^2$ and $\sigma_1 = 1$, $\sigma_2 = -1$, and where $\mathbf{e}_N^1$ denotes a $N \times 1$ vector with one in the first element and zeros elsewhere and with $\mathbf{J}_N$ denoting the exchange matrix.

Lemma 5 *Given Lemma 4, a displacement representation of $\mathbf{R}_N^{-2}$, with respect to $\mathbf{Z}_N$ and $\mathbf{Z}_N^T$ and with displacement rank $\rho = 4$, is given by*

$$\begin{aligned}
\boldsymbol{\tau}_N^1 &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{R}_{N-1}^{-1} \end{bmatrix} \mathbf{t}_N^1, \quad \boldsymbol{\tau}_N^3 = \mathbf{Z}_N \begin{bmatrix} \mathbf{R}_{N-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{J}_N \mathbf{t}_N^{1*} \\
\mathbf{s}_N^2 &= \mathbf{R}_N^{-1}\mathbf{t}_N^1, \quad \mathbf{s}_N^4 = \mathbf{Z}_N \mathbf{R}_N^{-1} \mathbf{J}_N \mathbf{t}_N^{1*}
\end{aligned}$$

with $\mathbf{s}_N^1 = \mathbf{t}_N^1$, $\mathbf{s}_N^3 = \mathbf{t}_N^2$, $\boldsymbol{\tau}_N^2 = \mathbf{t}_N^1$, $\boldsymbol{\tau}_N^4 = \mathbf{t}_N^2$, and where $\hat{\gamma}_1 = \hat{\gamma}_2 = 1$ and $\hat{\gamma}_3 = \hat{\gamma}_4 = -1$.

The displacement vectors involved in Lemma 4 can be efficiently estimated using the Levinson-Durbin algorithm at a cost of $\mathcal{O}(N^2)$, while those involved in Lemma 5 require some extra $\mathcal{O}(N \log 2(N))$ operations.

Lemma 6 ([18]) *Let $\mathbf{R}_N > 0$ and let $\mathbf{S}_{N_g N}$ and $\mathbf{S}_{N_g N}$ be the selection matrices corresponding to the given and missing data, respectively. Then, the following decomposition holds*

$$\mathbf{S}_{N_g N}^T \mathbf{R}_{N_g}^{-1} \mathbf{S}_{N_g N} = \mathbf{R}_N^{-1} - \mathbf{B}_N \mathbf{B}_N^H \tag{18}$$

where $\mathbf{B}_N = \mathbf{R}_N^{-1}\mathbf{S}_{N_m N}\mathbf{L}$, with

$$\mathbf{L}\mathbf{L}^* \triangleq \left(\mathbf{S}_{N_m N}\mathbf{R}_N^{-1}\mathbf{S}_{N_m N}^T\right)^{-1} \tag{19}$$

This decomposition allows for decomposing the trace of $\mathbf{R}_{N_g}^{-2}$ into three parts, given by the following corollary, each of which may be computed efficiently.

Corollary 7 *Given the decomposition described by Lemma 6 it holds*

$$\begin{aligned}
\mathrm{Tr}\left[\mathbf{R}_{N_g}^{-2}\right] \quad &= \quad \mathrm{Tr}\left[\mathbf{R}_N^{-2}\right] - 2\mathrm{Tr}\left[\mathbf{R}_N^{-1}\mathbf{B}_N\mathbf{B}_N^H\right] + \\
&\qquad \mathrm{Tr}\left[\mathbf{B}_N\mathbf{B}_N^H\mathbf{B}_N\mathbf{B}_N^H\right]
\end{aligned} \tag{20}$$

4. FAST IMPLEMENTATION

Given this, we are now ready to proceed to present the proposed efficient implementation of MSMLA-3 algorithm. As noted in [13, 14], (10) and (11) are Toeplitz matrices which may be extracted from larger circulant matrices using the FFT, requiring about $K \log_2(K)$ operations. As a result, the GS factorization $\mathbf{R}_N^{-1}$, $\mathbf{R}_N^{-2}$ and $\tilde{\mathbf{R}}_N^{-1}$ may be performed in an efficient way. In the missing data case, the covariance matrices (9) no longer have a Toeplitz structure, and the presented GS factorization is not directly applicable. To deal with this, denote by $\Psi_{\mathrm{n}}(\omega)$ and $\Psi_{\mathrm{d}}(\omega)$ the trigonometric polynomials appearing in the numerators and the denominators of (4) and (5), respectively, and note that these may be decomposed according to Lemma 6 as

$$\begin{aligned}
\Psi_{\mathrm{n}}(\omega) &= \mathbf{f}_N^H(\omega_k)\mathbf{S}_{N_g N}^T \tilde{\mathbf{R}}_{N_g}^{-1}\mathbf{S}_{N_g N}\mathbf{y}_{N_g} \\
&= \mathbf{f}_N^H(\omega_k)\tilde{\mathbf{R}}_N^{-1}\mathbf{y}_{N_g} - \mathbf{f}_N^H(\omega_k)\tilde{\mathbf{B}}_N\tilde{\mathbf{B}}_N^H\mathbf{y}_{N_g}
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\Psi_{\mathrm{d}}(\omega) &= \mathbf{f}_N^H(\omega_k)\mathbf{S}_{N_g N}^T \mathbf{R}_{N_g}^{-1}\mathbf{S}_{N_g N}\mathbf{f}_N(\omega_k) \\
&= \mathbf{f}_N^H(\omega_k)\mathbf{R}_N^{-1}\mathbf{f}_N(\omega_k) - \mathbf{f}_N^H(\omega_k)\mathbf{B}_N\mathbf{B}_N^H\mathbf{f}_N(\omega_k)
\end{aligned} \tag{22}$$

The first term of (22) may be evaluated efficiently using the Toeplitz structure of $\mathbf{R}_N$ using Lemma 3 and 4 in $\mathcal{O}(N^2 + N\log_2(N) + K\log_2(K))$ operations, as is done

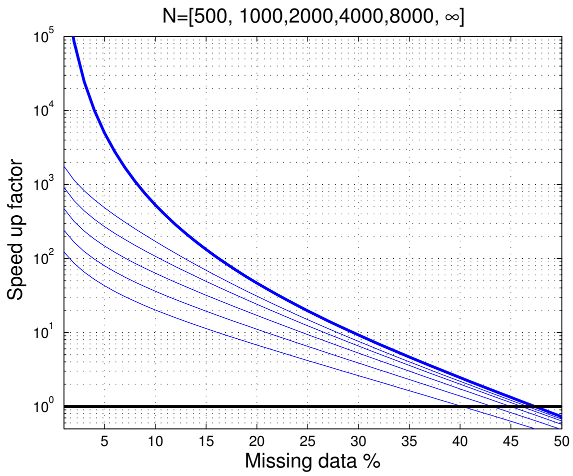
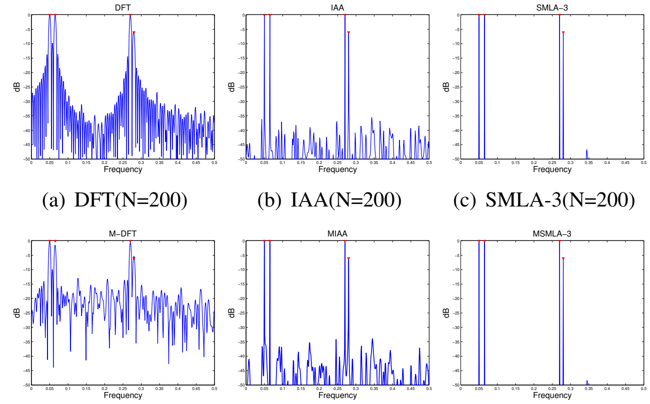


Fig. 1. The theoretical speed up: the number of flops for the missing data SMLA algorithm divided by the number of flops required for inverting $\mathbf{R}_{N_g}$ and $\tilde{\mathbf{R}}_{N_g}$, for $N = (500, 1000, 2000, 4000, 8000, \infty)$. The asymptotic speed up ($N = \infty$) is depicted with bold line.

in [13, 14]. The second term of (22) may be evaluated in $\mathcal{O}(N_m^3 + N_m N \log N + N^2 + K \log K)$ operations, and the matrix vector product that appears in the first term of (21) is indirectly performed using the GS factorization of the pertinent Toeplitz matrix and Lemma 6; note that this decomposition is only beneficiary if $N_m < N_g$, in which case the bottleneck which is the inversion of $\mathbf{R}_{N_g}$ is replace by the inversion of $\left(\mathbf{S}_{N_m N}\mathbf{R}_N^{-1}\mathbf{S}_{N_m N}^T\right)$. The required computations are summarized in Algorithm 1 (see [18, 19] for further details). A detailed study of the proposed algorithm shows that the leading terms of the computational cost are

$$\frac{11}{6}{N_\mathrm{m}}^3 + 20 N_\mathrm{m} N \log_2 N + 3N^2 + 2K\log(K) \tag{23}$$

which should be compared to $(4/3){N_\mathrm{g}}^3 + 2K\log(K)$ which is required by the implementation in [15]. The improvement for N ranging from 500 to 8000 is depicted in Figure 1, for $K = 8N$. It should be noted that the choice of K is irrelevant for the plot, since the last term of (23) is negligible for relevant cases ($K < 15N$). The complexity given in (23) is for a single iteration. In the first term, $(4/3){N_\mathrm{m}}^3$ comes from calculating $\mathbf{L}$ and $\tilde{\mathbf{L}}$, the inverses of the Cholesky factors of $\mathbf{S}_{N_m N}\mathbf{R}^{-1}\mathbf{S}_{N_m N}{}^T$ and $\mathbf{S}_{N_m N}\tilde{\mathbf{R}}^{-1}\mathbf{S}_{N_m N}{}^T$. The remaining $(1/2){N_\mathrm{m}}^3$ is from the multiplication (16). The second term is for calculating the terms $\mathbf{B}_N = \mathbf{R}^{-1}(\mathbf{S}_{N_m N}^T\mathbf{L})$, $\tilde{\mathbf{B}}_N = \tilde{\mathbf{R}}^{-1}(\mathbf{S}_\mathrm{m}^T\tilde{\mathbf{L}})$, and $\mathbf{R}^{-1}\mathbf{B}_N$, which each requires $6N_\mathrm{m}$ FFT's of size $2N$, as well as for calculating the Fourier coefficients of $\mathbf{f}_N^H(\omega_k)\mathbf{B}_N\mathbf{B}_N^H\mathbf{f}_N(\omega)$ from $\mathbf{B}_N$, which requires $2N_\mathrm{m}$ FFT's of size $2N$ (see [19] for details). The third term results from the Levinson-Durbin, requiring N^2 operations (see, e.g., [4]), and the reconstruction of $\mathbf{R}^{-1}$ and $\tilde{\mathbf{R}}_N^{-1}$ using Trench's algorithm for the estimation of the inverse of Toeplitz matrices [24, p. 132]. The final term results from the four FFT/IFFT calculations of size K, namely (i) $\mathbf{R}_{1:N,1} = K(\mathcal{F}^{-1}(p))_{1:N}$, (ii) $\tilde{\mathbf{R}}_{1:N,1} = K(\mathcal{F}^{-1}(\tilde{p}))_{1:N}$, (iii) Ψ_n, and (iv) Ψ_d.



(a) DFT(N=200) (b) IAA(N=200) (c) SMLA-3(N=200)

(d) DFT(N=200) (e) MIAA(N=200) (f) MSMLA-3(N=200)

Fig. 2. (a-c) Complete data case for N=200. (d-f) Incomplete data case (30% of missing data)

Figure 2 illustrates the performance of the SMLA-3 method implemented using the proposed fast scheme for the case of complete and incomplete data sets (see [10] for further details on the experimental setup), where the power spectral estimates using the DFT and IAA (MIAA) are also given for reasons of comparison.

Figure 3 shows a matlab comparison of the computational time between the proposed algorithm and the inversion of the two matrices $\mathbf{R}_{N_\mathrm{g}}$ and $\tilde{\mathbf{R}}_{N_\mathrm{g}}$. As can be seen from the figure, the speedup is considerable when the number of missing data is much smaller than the total number of data points.

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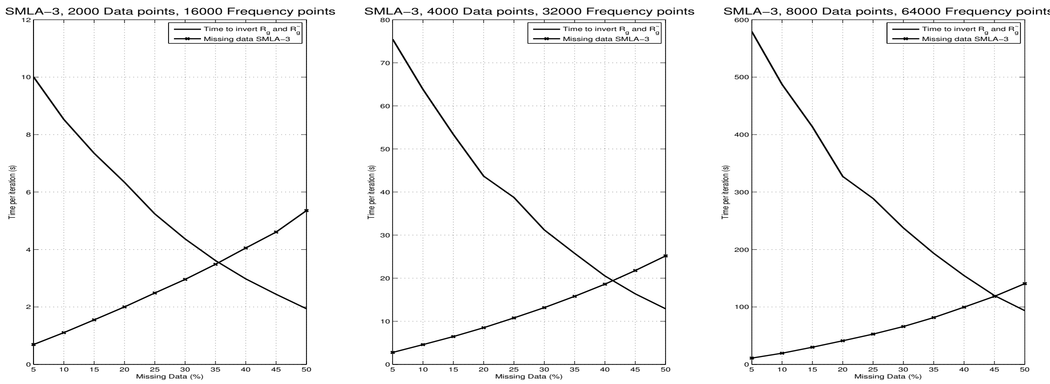

Fig. 3. The computation time for the fast missing data algorithm as compared to the computation time for inverting the matrices $\mathbf{R}_{N_g}$ and $\tilde{\mathbf{R}}_{N_g}$.

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