CORRELATION TEST FOR HIGH DIMENSIONAL DATA WITH APPLICATION TO SIGNAL DETECTION IN SENSOR NETWORKS

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ABSTRACT

The problem of correlation detection of multivariate Gaussian observations is considered. The problem is formulated as a binary hypothesis test, where the null hypothesis corresponds to a diagonal correlation matrix with possibly different diagonal entries, whereas the alternative would be associated to any other form of positive covariance. Using tools from random matrix theory, we study the asymptotic behavior of the Generalized Likelihood Ratio Test (GLRT) under both hypothesis, assuming that both the sample size and the observation dimension tend to infinity at the same rate. It is shown that the GLRT statistic always converges to a Gaussian distribution, although the asymptotic mean and variance will strongly depend the actual hypothesis. Numerical simulations demonstrate the superiority of the proposed asymptotic description in situations where the sample size is not much larger than the observation dimension.

Index Terms— Hypothesis testing, correlation matrix, random matrix theory, central limit theorem.

1. INTRODUCTION

The detection of correlation between multiple signals is a mathematical problem that arises in multiple fields and applications of very different nature. This is the case, for instance, of sensor networks, multi-antenna radar or cognitive radio applications, where the presence of a directional signal results in strong correlations between the signals received at the multiple antennas or sensors. In all these applications, the absence of an external source leads to spatially uncorrelated signals, whereas the presence of directional sources results in high correlation of the signals received at the different sensors/antennas. Therefore, one may formulate the signal detection problem in all these contexts as a binary hypothesis test on the signal correlation matrix: the null hypothesis (absence of signal) would correspond to a diagonal correlation matrix, whereas the alternative (presence of signal with indefinite spatial structure) would be associated to the presence of a signal with non-diagonal correlation matrix.

Let us denote by $\mathbf{y}_n$ an $M \times 1$ column vector that contains the measurements collected by the M sensors or antennas at the nth time instant. We will model these measurements as Gaussian random vectors:

$(\mathbf{As1})$ The set of M-dimensional observations $\mathbf{y}_n$, $n = 1, \ldots, N$, $N > M$, are independent and identically distributed complex random vectors, such that their real and imaginary parts are zero mean Gaussian distributed with covariance $0.5\mathbf{R}_M$. This will be denoted as $\mathbf{y}_n \sim \mathcal{CN}(0, \mathbf{R}_M)$.

Let $\mathbf{D}_M$ denote an $M \times M$ diagonal matrix that contains the diagonal elements of the correlation matrix $\mathbf{R}_M$. The main interest of this paper is on the following hypothesis testing problem:

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{y}_n \sim \mathcal{CN}(0, \mathbf{R}_M), \mathbf{R}_M = \mathbf{D}_M \\ \mathcal{H}_1 &: \mathbf{y}_n \sim \mathcal{CN}(0, \mathbf{R}_M), \mathbf{R}_M \neq \mathbf{D}_M. \end{aligned}$$

It should be stressed here that, contrary to the well-known sphericity test [1, p.431], the null hypothesis here does not require the correlation matrix to be proportional to the identity matrix. This allows for more general settings in sensor networks or uncalibrated antenna arrays, where the background noise power is not necessarily the same at all receivers.

There exist several tests in the literature dealing with the above problem, although perhaps the most prominent one is the Generalized Likelihood Ratio Test (GLRT), see [2]. In order to formulate this test, let us consider the sample correlation matrix $\hat{\mathbf{R}}_M$, which is constructed as

$$\hat{\mathbf{R}}_M = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^H$$

and let $\hat{\mathbf{D}}_M$ denote the diagonal matrix constructed from the diagonal entries of $\hat{\mathbf{R}}_M$. We will denote by $\hat{\mathbf{C}}_M$ the sample coherence matrix, defined as

$$\hat{\mathbf{C}}_M = \hat{\mathbf{D}}_M^{-1/2} \hat{\mathbf{R}}_M \hat{\mathbf{D}}_M^{-1/2}$$

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where $\hat{\mathbf{D}}_M^{1/2}$ is the positive square root of $\hat{\mathbf{D}}_M$. One can easily see that, after replacing the covariance matrices under both hypothesis with their maximum likelihood estimates, the logarithm of the quotient of the likelihood functions is proportional to the statistic

$$\hat{\eta}_M = \frac{-1}{M} \log\det\left(\hat{\mathbf{C}}_M\right) \tag{1}$$

where the normalization factor $-M^{-1}$ is introduced for reasons that will be come apparent below. Hence, the GLRT is constructed for a given threshold α as

$$\hat{\eta}_M \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \alpha.$$

The threshold α is in practice selected in order to guarantee a certain probability of false alarm (Type I error probability), which is obtained from the distribution of $\hat{\eta}_M$ under the null hypothesis. In order to obtain such a result, the literature typically relies on the asymptotic regime obtained when the sample size N increases to infinity for a fixed M.

2. CLASSICAL ASYMPTOTICS

It is well known [2] that, under $\mathcal{H}_0$,

$$2MN\hat{\eta}_M \underset{N\to\infty}{\xrightarrow{\mathcal{L}_0}} \chi^2_{M^2-M}$$

where $\xrightarrow{\mathcal{L}_0}$ denotes convergence in law under[1] $\mathcal{H}_0$ and where χ^2_p is a central chi-square distribution with p degrees of freedom. Hence, one could in principle use this asymptotic law in order to fix the threshold α for a given false alarm probability. However, it was shown in [3] (real-valued case) and [4] (complex case) that this approximation is very poor for moderate values of the sample size, and a better asymptotic fit can be obtained by considering the following Box-Barlett correction:

$$\begin{aligned}&\mathbb{P}_0\left(2MN\rho\hat{\eta}_M \le x\right) = \mathbb{P}\left(\chi^2_{M^2-M} \le x\right) + \\ &+\omega_2\left[\mathbb{P}\left(\chi^2_{M^2-M+4} \le x\right) - \mathbb{P}\left(\chi^2_{M^2-M} \le x\right)\right] + \mathcal{O}(N^{-3})\end{aligned} \tag{2}$$

where $\mathbb{P}_0$ is the probability under $\mathcal{H}_0$, and where we have defined $\rho = 1 + (M+1)/3N$,

$$\omega_2 = \frac{-1}{6(\rho N)^2}\sum_{\ell=1}^{M}\left(B\left((1-\rho)N + 1 - \ell\right) - B\left((1-\rho)N\right)\right)$$

and $B(x) = x^3 - 1.5x^2 + 0.5x$. In parallel with this, it was recently shown in [5] that under the null hypothesis $\hat{\eta}_M$ can be represented as a product of independent beta-distributed random variables, a fact that was used in order to derive an alternative asymptotic approximation of the law of $\hat{\eta}_M$ for large sample sizes.

Regarding the power of the test, it was shown in [4] that for sufficiently large sample size (N) and fixed M, one may approximate

$$\mathbb{P}_1\left(2MN\rho\hat{\eta}_M \le x\right) \approx \mathbb{P}\left(\chi^2_{M^2-M}\left(2MN\rho\eta_M\right) \le x\right) \tag{3}$$

where $\chi^2_p(\lambda)$ is the non-central chi square distribution with p degrees of freedom and noncentrality parameter λ and where

$$\eta_M = \frac{-1}{M}\log\det\left(\mathbf{C}_M\right)$$

with $\mathbf{C}_M$ denoting the true coherence matrix, defined as $\mathbf{C}_M = \mathbf{D}_M^{-1/2}\mathbf{R}_M\mathbf{D}_M^{-1/2}$. Now, the above approximations turn out to be very accurate for moderately high sample volume N and relatively low observation dimension M. We will see below that when M becomes large enough and tends to be comparable in magnitude to the sample size N, the above approximations are no longer valid. In the next section, we propose to follow an alternative asymptotic approach that assumes that both M and N are large but comparable in magnitude. Using this approach, we will derive more accurate approximations of the level of significance and the power of the GLRT for the situation where the observation dimension is large.

3. PROPOSED APPROACH

In this section, we will derive an asymptotic characterization of the GLRT statistic under the assumption that both M and N increase without bound.

We will use the following assumptions:

(**As2**) The observation dimension M is a function of N and $\lim_{N\to\infty} M/N = c$, $0 < c < 1$.

(**As3**) If $\lambda_{\min}(\mathbf{R}_M)$ and $\lambda_{\max}(\mathbf{R}_M)$ denote the minimum and maximum eigenvalues of the Hermitian matrix $\mathbf{R}_M$, $\inf_M \lambda_{\min}(\mathbf{R}_M) > 0$ and $\sup_M \lambda_{\max}(\mathbf{R}_M) < \infty$.

To study the asymptotic behavior of $\hat{\eta}_M$ under the above assumptions, we will first formulate the statistic $\hat{\eta}_M$ in terms of quantities that can be easily analyzed using random matrix theory methods. For $z \in \mathbb{C}^+$ (upper complex semiplane), let us define the two complex functions $\hat{m}_M(z)$ and $\hat{b}_M(z)$ as

$$\begin{aligned}\hat{m}_M(z) &= \frac{1}{M}\operatorname{tr}\left[\left(\hat{\mathbf{R}}_M - z\mathbf{I}_M\right)^{-1}\right] \\ \hat{b}_M(z) &= \frac{1}{M}\operatorname{tr}\left[\left(\hat{\mathbf{D}}_M - z\mathbf{I}_M\right)^{-1}\right].\end{aligned}$$

These functions can be seen as the Stieltjes transforms of the empirical distribution of the eigenvalues of $\hat{\mathbf{R}}_M$ and its diagonal entries, respectively. We consider two real values a, b, defined as $a = \inf_M \lambda_{\min}(\mathbf{R}_M)(1-\sqrt{c})^2$ and $b =$

[1] When no subindex is specified in $\mathcal{L}$, it is understood that convergence in law occurs under both hypothesis.

$\sup_M \lambda_{\max}(\mathbf{R}_M)(1+\sqrt{c})^2$, and define $\mathcal{S}$ as an open interval containing $[a,b]$ and not $\{0\}$. It is well know that for sufficiently large M almost surely all the eigenvalues of $\hat{\mathbf{R}}_M$ (and, hence, all its diagonal values) are inside $\mathcal{S}$ [6]. Therefore, one can express the statistic $\hat{\eta}_M$ as

$$\hat{\eta}_M = \frac{1}{2\pi \mathrm{j}} \oint_{\mathcal{C}^-} f(z)\hat{m}_M(z)dz + \frac{1}{2\pi \mathrm{j}} \oint_{\mathcal{C}^-} g(z)\hat{b}_M(z)dz \quad (4)$$

where $f(z)$ and $g(z)$ are two appropriate holormorphic functions on $\mathbb{C}\backslash\mathbb{R}^+$ and where $\mathcal{C}^-$ is the clockwise oriented simple contour $\mathcal{C}$ that intersects the real axis at two points, i.e. $\mathcal{C}\cap\mathbb{R} = \{x^-, x^+\}$, such that $x^- \in (0,a)$ and $x^+ \in (b,\infty)$. More specifically, in order to retrieve (1), one needs to set $g(z) = -f(z) = \log z$ in the above expression. We will leave $f(z)$ and $g(z)$ unspecified in order to derive a more general asymptotic result.

3.1. First order behavior: almost sure convergence of $\hat{\eta}_M$

It is well known [7] that under $(\mathbf{As1})-(\mathbf{As3})$, $\hat{m}_M(z) - \bar{m}_M(z) \to 0$ almost surely for all $z \in \mathbb{C}^+$ as $M,N \to \infty$, where

$$\bar{m}_M(z) = \frac{N-M}{Mz} - \frac{N}{M}\frac{1}{\omega_M(z)}$$

and where $\omega_M(z)$ is the unique solution in $\mathbb{C}^+$ to the following polynomial equation

$$z = \omega_M(z)\left(1 - \frac{1}{N}\operatorname{tr}\left[\mathbf{R}_M\left(\mathbf{R}_M - \omega_M(z)\mathbf{I}_M\right)^{-1}\right]\right). \quad (5)$$

Using classical random matrix techniques, it is not difficult to see that $\hat{b}_M(z) - \bar{b}_M(z) \to 0$ where $\bar{b}_M(z)$ is defined as $\hat{b}_M(z)$, replacing $\hat{\mathbf{D}}_M$ with $\mathbf{D}_M$. Loosely speaking, this means that the asymptotic distribution of the diagonal entries of $\hat{\mathbf{R}}_M$ is the same as the asymptotic distribution of the diagonal entries of $\mathbf{R}_M$ (assuming that it exists). This is in stark contrast with the asymptotic distribution of the eigenvalues of $\hat{\mathbf{R}}_M$ and $\mathbf{R}_M$, which do not coincide at all under this asymptotic regime.

Using the convergence of the above functions together with the dominated convergence theorem, one can show that under $(\mathbf{As1})-(\mathbf{As3})$, $\hat{\eta}_M - \bar{\eta}_M \to 0$ almost surely, where $\bar{\eta}_N$ is defined as

$$\bar{\eta}_M = \frac{1}{2\pi \mathrm{j}} \oint_{\mathcal{C}^-} f(z)\bar{m}_M(z)dz + \frac{1}{2\pi \mathrm{j}} \oint_{\mathcal{C}^-} g(z)\bar{b}_M(z)dz.$$

For the particular choice $g(z) = -f(z) = \log z$, one can use the integration technique developed in [8] to show that

$$\bar{\eta}_M = \eta_M + 1 - \frac{N-M}{M}\log\left(\frac{N}{N-M}\right).$$

It is interesting to observe that the GLRT statistic does not converge to η_M when we allow the observation dimension to increase with N. Instead, a bias term appears that critically depends on the quotient $c_N = M/N$. Observe that this bias term disappears when $c_N \to 0$, agreeing with the fact that $\hat{\eta}_M \to \eta_M$ when $N \to \infty$ for a fixed M. We next provide a more interesting result that characterizes the asymptotic fluctuations of $\hat{\eta}_M$ around $\bar{\eta}_M$ in this asymptotic regime.

3.2. Second order behavior: asymptotic fluctuations of $\hat{\eta}_M$

We next present the central contribution of this paper, which describes the asymptotic second order behavior of $\hat{\eta}_M$ around $\bar{\eta}_M$ for two general complex functions $g(z) = f(z)$ that are holomorphic on $\mathbb{C}\backslash\mathbb{R}^+$.

Theorem 1. *Let μ_M and σ_M^2 be two deterministic quantities defined as $\mu_M = \frac{1}{2N}\operatorname{tr}\left[\mathbf{D}_M^2 g''(\mathbf{D}_M)\right]$ and*

$$\begin{aligned}\sigma_M^2 = {} & \frac{1}{N}\operatorname{tr}\left[\left(\mathbf{R}_M g'(\mathbf{D}_M)\right)^2\right] \\ & - \frac{1}{4\pi^2}\oint_{\omega_M(\mathcal{C}^-)}\oint_{\omega_M(\mathcal{C}^-)} \frac{F_M(\omega_1)F_M(\omega_2)}{(\omega_1-\omega_2)^2}d\omega_1 d\omega_2 \\ & - \frac{1}{\pi \mathrm{j}}\oint_{\omega_M(\mathcal{C}^-)} \frac{F_M(\omega)}{N}\operatorname{tr}\left[\mathbf{R}_M^2\left(\mathbf{R}_M - \omega\mathbf{I}_M\right)^{-2} g'(\mathbf{D}_M)\right]d\omega\end{aligned}$$

where g' and g'' are the first and second order derivatives of g and where $F_M(\omega) = f(z)$ with z replaced by the right hand side of (5) as a function of $\omega = \omega_M(z)$. Assume that $\sup_M |\mu_M| < \infty$ and that $0 < \inf_M |\sigma_M^2| \le \sup_M |\sigma_M^2| < \infty$. Then, under $(\mathbf{As1})-(\mathbf{As3})$,

$$\sigma_M^{-1}\left(M\left(\hat{\eta}_M - \bar{\eta}_M\right) - \mu_M\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

Proof. To prove this result, we may follow the same approach as in [9]. More specifically, we define $\Psi_M(u)$ as the characteristic function of the random variable $M(\hat{\eta}_M - \bar{\eta}_M)$, that is $\Psi_M(u) = \mathbb{E}\left[\exp\left(\mathrm{j}\,uM\left(\hat{\eta}_M - \bar{\eta}_M\right)\right)\right]$. The main idea of the proof consists in showing that

$$\frac{d\Psi_M(u)}{du} = \left(\mathrm{j}\,\mu_M - u\sigma_M^2\right)\Psi_M(u) + o(1) \quad (6)$$

where μ_M and σ_M^2 are as in the statement of the theorem, and where the term $o(1)$ converges to zero uniformly in u over compact subsets. This can be shown by using the integration by parts formula for Gaussian functionals together with the Poincaré-Nash inequality (see [9, Section III.B] for further details). The derivations are quite standard, although somewhat long and tedious, so we choose to omit them here for the sake of simplicity. The only tricky point in the application of these tools comes from the fact that the moments of $\hat{m}_M(z)$ and $\hat{b}_M(z)$ do not need to exist when z lies on the positive real axis. To solve this, one may use a regularization function such as the one in [10, eq.(27)] multiplying $\hat{m}_M(z)$ and $\hat{b}_M(z)$ to guarantee that all the realizations of the regularized functions are bounded for any $z \in \mathcal{C}$.

Having shown (6), one can readily solve the corresponding differential equation to conclude that

$$\Psi_M(u) - \exp\left(\mathrm{j}\,\mu_M u - \frac{u^2}{2}\sigma_M^2\right) \to 0$$

where the second term on the left hand side of the above equation is the characteristic function of a Gaussian $\mathcal{N}\left(\mu_M, \sigma_M^2\right)$. A straightforward modification of [9, Proposition 6] to account for the presence of μ_M leads to the desired result. □

We can particularize the above result to the problem at hand by computing the expressions of the asymptotic mean μ_M and variance σ_M^2 under the choice $g(z) = -f(z) = \log z$. Considering the second order derivative of $\log z$, it is trivial to see that $\mu_M = \frac{-1}{2}\frac{M}{N}$. On the other hand, a closed form expression for σ_M^2 can be obtained by using again the technique developed in [8], which shows that

$$\sigma_M^2 = \log\left(\frac{N}{N-M}\right) - 2\frac{M}{N} + \frac{1}{N}\|\mathbf{C}_M\|_F^2 \qquad (7)$$

where $\|\mathbf{C}_M\|_F^2$ is the squared Frobenius norm of $\mathbf{C}_M$, i.e.

$$\|\mathbf{C}_M\|_F^2 = \operatorname{tr}\left[\mathbf{R}_M\mathbf{D}_M^{-1}\mathbf{R}_M\mathbf{D}_M^{-1}\right].$$

Note that $|\mu_M|$ is upper bounded by a positive quantity independent of M by convergence of M/N as described in $(\mathbf{As2})$. The fact that $\inf_M \sigma_M^2 > 0$ follows from the inequality

$$\sigma_M^2 \geq \log\left(\frac{N}{N-M}\right) - \frac{M}{N} > 0$$

again because we assumed that $M < N$ and $\lim_{M\to\infty} M/N = c > 0$. On the other hand, $\sup_M \sigma_M^2 < \infty$ by convergence of M/N together with $(\mathbf{As3})$. Therefore, from Theorem 1 one readily obtains the following corollary.

Corollary 1. *Under assumptions* $(\mathbf{As1}) - (\mathbf{As3})$,

$$\sigma_M^{-1}\left\{M\left(\hat{\eta}_M - \bar{\eta}_M\right) + \frac{1}{2}\frac{M}{N}\right\} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

where σ_M^2 *is as defined in (7).*

According to this corollary, under both hypothesis the GLRT statistic asymptotically fluctuates around $\bar{\eta}_M$ like a Gaussian random variable, with mean $-0.5/N$ and variance σ_M^2/M^2. Hence, in practical applications one might approximate the asymptotic law of $\hat{\eta}_M$ under both hypothesis as Gaussian random variable $\mathcal{N}\left(\bar{\eta}_M - 0.5/N, \sigma_M^2/M^2\right)$, where the actual form of $\bar{\eta}_M$ and σ_M^2 will depend on the considered hypothesis. In the next section we will see that this provides a very good approximation of the actual law when both M,N are large.

The fact that the Frobenius norm of the true coherence matrix plays a determining role in the asymptotic distribution of $\hat{\eta}_M$ has been well known in the literature, and in fact a test based on this quantity has been proposed by multiple researchers [2, 4, 11]. An asymptotic characterization of these tests is, however, out of the scope of this paper.

4. NUMERICAL RESULTS

In this section we evaluate the accuracy of the asymptotic approximation presented in this paper. To that effect, we consider a large network of sensors that collect Gaussian circularly symmetric signals distributed according to $(\mathbf{As1})$. Under $\mathcal{H}_0$, the correlation matrix of the observation is diagonal $\mathbf{R}_M = \mathbf{D}_M$ and the diagonal correlation elements take values between 0 and 1. In the simulations, these values are randomly fixed according to a uniform law at the beginning of each experiment. Under the alternative hypothesis $\mathcal{H}_1$, the signals are assumed to be correlated according to $\mathbf{R}_M = \mathbf{D}_M + \mathbf{\Psi}_M$ where $\{\mathbf{\Psi}_M\}_{ij} = \varrho^{|i-j|}$, with ϱ a correlation coefficient that is fixed to $\varrho = 0.9$.

Figures 1 and 2 represent the empirical distribution of the statistic $\hat{\eta}_M$ under both hypothesis obtained with a total set of 10^5 independent simulation runs when the number of sensors was $M = 20$ and the sample volume was fixed to $N = 25$ and $N = 100$ respectively. Apart from the empirical distribution obtained by simulation, these figures also show the asymptotic distribution obtained for large M, N (as given by Corollary 1) and the asymptotic distribution obtained with the classical approximation of large N and fixed M as given in equations (2) and (3). We observe that in general the proposed approximation turns out to be much more accurate than the classical one, especially in situations where M, N are comparable in magnitude. This is also illustrated in Figure 3, which compares the empirical and the asymptotic Receiver Operating Characteristic (ROC) curves for different values of the sample size. Here again, we see that the asymptotic curve obtained with the proposed approximation is virtually indistinguishable from the empirical one. Furthermore, results are much more accurate than those obtained with the classical (large N, fixed M) approximation.

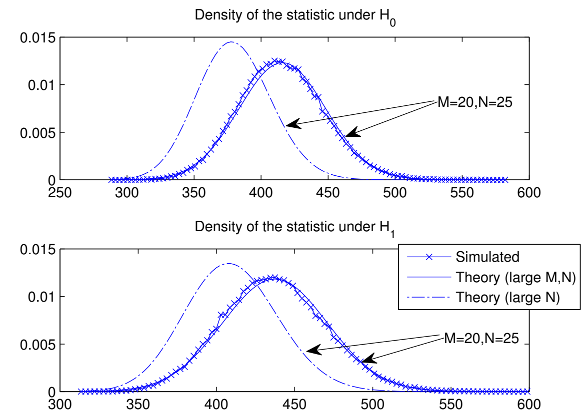

Fig. 1. Empirical and asymptotic density of the statistic $\hat{\eta}_M$ under the two hypothesis, $M = 20, N = 25$.

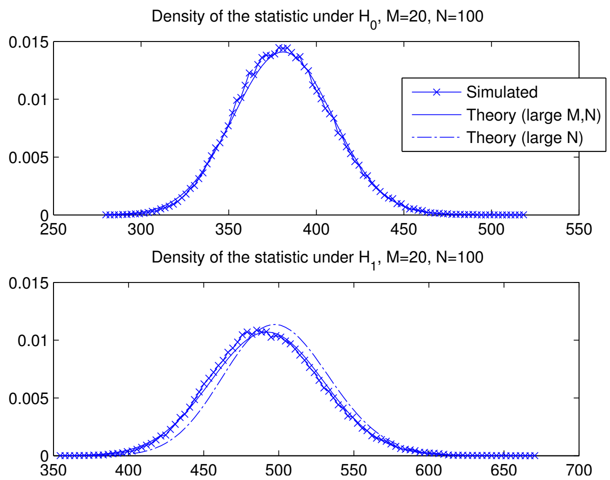

Fig. 2. Empirical and asymptotic density of the statistic $\hat{\eta}_M$ under the two hypothesis, $M = 20, N = 100$.

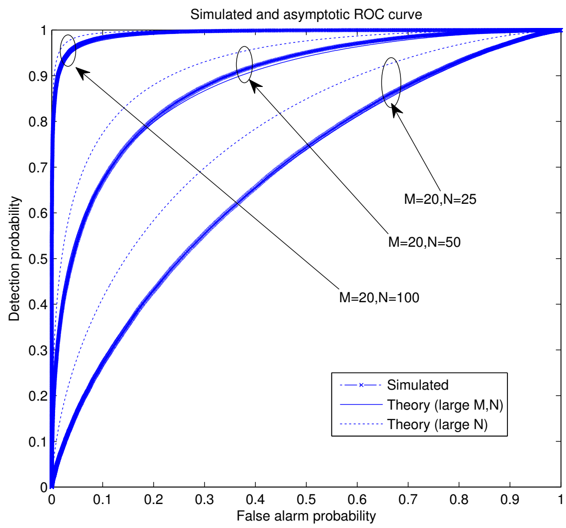

Fig. 3. Simulated vs. asymptotic ROC curve for different values of the sample size.

5. CONCLUSIONS

We have presented an asymptotic analysis of the GLRT for a binary hypothesis problem on the covariance matrix of a set of multivariate complex Gaussian observations. The null hypothesis corresponds to a diagonal correlation matrix, whereas the alternative one is associated with a general non-diagonal positive correlation matrix. It has been shown that when both the sample size (N) and the observation dimension (M) increase without bound at the same rate, the GLRT statistic converges in law to a Gaussian distribution, where the asymptotic variance and mean strongly depend on which hypothesis holds. Simulations indicate that the proposed asymptotic description provides much more accurate approximations than classical asymptotics when the sample volume is not much larger than the observation dimension.

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