ROBUST HIGH-RESOLUTION DOA ESTIMATION WITH ARRAY PRE-CALIBRATION

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ABSTRACT

A robust high-resolution technique for DOA estimation in the presence of array imperfections such as sensor position errors and non-uniform sensor gain is presented. When the basis matrix of a sparse DOA estimation framework is derived from an ideal model, array errors cannot be handled which causes performance deterioration. Array pre-calibration via robust steering vector estimation yields an improved overcomplete basis matrix. It alleviates the delicate problem of selecting the regularization parameter of the optimization problem and improves the performance significantly. Thus, closely spaced sources can be resolved in the presence of severe array imperfections, even at low SNRs.

Index Terms— sparse regularization, array imperfections, robust DOA estimation, array calibration

1. INTRODUCTION

Sparse regularization is an emerging technique with high-resolution capabilities for direction-of-arrival (DOA) estimation and source localization problems [1, 2, 3, 4, 5]. However, an automatic determination of the regularization parameter is highly non-trivial and the focus of recent research. When the noise statistics are known, this knowledge can be exploited to yield proper regularization [6, 7]. One established approach is the discrepancy principle [1, 8]. For unknown noise statistics, techniques based on the L-curve [9] have been proposed. In practice, however, noise is not the only impairment. Limited fabrication accuracy or external influences introduce model errors such as gain/phase mismatch or uncertainties in the sensor locations. These errors lower the quality of the sparse basis and may result in severe performance deterioration. As a result, sparse regularization for model errors has been addressed recently [10]. Also Bayesian techniques have been reported [11]. Other approaches introduce additional constraints into the optimization problem [12] or apply diagonal loading [13, 14] for increased robustness against incorrect choices of the regularization parameter. Nevertheless, resolution and accuracy can hardly be improved by controlling the regularization only. Therefore, we propose the use of array calibration techniques in combination with sparse regularization in order to achieve this goal. In the past, efficient calibration methods such as the projection approach [15] and improved versions [16] have been presented. They obtain robust estimates of the perturbed steering vectors which can significantly improve the quality of the sparse basis. It allows for high-resolution DOA estimation at low SNRs in the presence of array imperfections.

Our original contribution is as follows. First, we alleviate the problem of finding the regularization parameter by using pre-calibration to obtain a robustified basis matrix for sparse DOA estimation. Second, we show the high-resolution capabilities of the combined method and its robustness against array errors and noise. Third, we compare our proposed scheme to the conventional case where the sparse basis is derived under ideal conditions.

The remainder of this article is organized as follows. Section 2 introduces the sparse DOA estimation framework. In Section 3, we demonstrate how a robustified sparse basis is obtained. Simulation results for performance evaluation are provided in Section 4 and Section 5 concludes this work.

2. SPARSE SIGNAL MODEL

We follow the sparse signal model presented in [1] which is briefly reviewed. Consider a uniform linear array (ULA) with L elements and a spacing of $d = \lambda/2$, where λ is the signal wavelength. Let us assume K farfield signals $u_k(t)$ with powers σ_k^2. The directions of arrival (DOAs) of the corresponding point sources are denoted by θ_k, $k = 1, \ldots, K$. If we discretize the complete angular range into N segments, the signals $\{u_k(t)\}_{k=1}^K$ represent the non-zero entries of a sparse vector $\mathbf{s} = [s_1, \ldots, .s_N]^T$, where $K \ll N$. A number of T snapshots is taken at times t_i, $i = 1, \ldots, T$, and collected in $\mathbf{S} = [\mathbf{s}(t_1), \ldots, \mathbf{s}(t_T)] \in \mathbb{C}^{N \times T}$. Similarly, the vectors $\mathbf{y}(t_i)$ in $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_T)] \in \mathbb{C}^{L \times T}$ represent T snapshots of the sensor measurements. They can be modeled by

$$\mathbf{Y} = \mathbf{A}\mathbf{S} + \mathbf{N}\,, \tag{1}$$

where the columns of $\mathbf{A} = [\mathbf{a}_0(\theta_1), \ldots, \mathbf{a}_0(\theta_N)] \in \mathbb{C}^{L \times N}$ contain the array steering vectors for all N locations with

$\|\mathbf{a}_0\|_2^2 = L$. The sensor noise $\mathbf{N}=[\mathbf{n}(t_1),\ldots,\mathbf{n}(t_T)] \in \mathbb{C}^{L\times T}$ is zero-mean Gaussian distributed. To determine $\mathbf{S}$, given $\mathbf{Y}$ and $\mathbf{A}$, we formulate the following optimization problem [1]:

$$\min \|\mathbf{s}^{(\ell_2)}\|_1 \quad s.t. \quad \|\mathbf{Y} - \mathbf{AS}\|_f^2 \leq \beta \,. \tag{2}$$

The entries of the objective function $\mathbf{s}^{(\ell_2)}= [s_1^{(\ell_2)},\ldots,s_N^{(\ell_2)}]^T$ have the form $s_n^{(\ell_2)} := \|[s_n(t_1),\ldots,s_n(t_T)]\|_2$, $n=1,\ldots,N$. The expressions $\|\cdot\|_p$ and $\|\cdot\|_f$ denote the l_p-norm and the Frobenius norm, respectively. Since the data is complex-valued, it can be solved by second-order cone programming (SOCP) [1, 17]. However, finding an automated process to determine an optimal value of β is still a problem. For known noise statistics, β can be obtained as follows. Let us assume i.i.d. zero-mean, circular Gaussian noise with power σ_n^2. Since $\frac{2}{\sigma_n^2}\|\mathbf{N}\|_f^2 \sim \mathcal{X}_{2TL}^2$ with $2TL$ degrees of freedom, we can estimate the $\alpha\%$ confidence interval with $\mathcal{X}_{u,\alpha}^2$, $\mathcal{X}_{l,\alpha}^2$ being the upper and lower bound, respectively. As $\alpha\%$ of all realizations will fall below $\mathcal{X}_{u,\alpha}^2$, we find [1]

$$E[\|\mathbf{N}\|_f^2] \leq \beta^{(\alpha)} = \mathcal{X}_{u,\alpha}^2 \frac{\|\mathbf{N}\|_f^2}{\mathcal{X}_{l,\alpha}^2} \approx TL\,\mathcal{X}_{u,\alpha}^2 \frac{\hat{\sigma}_n^2}{\mathcal{X}_{l,\alpha}^2} \,, \tag{3}$$

where $\hat{\sigma}_n^2$ is an estimate of σ_n^2. However, α is an empirically selected user parameter. A thoroughly justified rule is hard to obtain but desirable to avoid over-regularization (i.e. suppression of peaks). Section 4 shows that our proposed approach significantly alleviates this problem. In the presence of array errors, the choice of β becomes even more delicate.

Model Errors:

Ideally, the sensors have a uniform spacing of $d = \lambda/2$. For simplicity, we describe position errors only along the array axis (x-axis). Errors in y-direction can be treated in a straightforward manner. Let us define a maximum sensor position error of $\pm d/2$ which corresponds to a delay error of $\Delta\tau_l(\theta_k)$ at the l-th sensor with $\theta_k, k=1,\ldots,K$ being the DOA of the k-th source. The maximum error occurs when $\theta_k = \pm 90°, k=1,\ldots,K$. Hence, the maximum phase error due to uncertainty in the position of the l-th sensor becomes

$$\Delta\phi_l \;=\; \pm\, p_\phi \cdot \omega \frac{d}{2c} \;=\; \pm\, p_\phi \cdot \frac{\pi}{2} \,, \qquad l = 1,\ldots,L \,, \tag{4}$$

where c is the signal propagation speed and $p_\phi,\ 0 \leq p_\phi \leq 1$, is the actual error, i.e. a fraction of the maximum error which can be selected from the system specifications. Also the sensor gain may be non-uniform along the array. We consider uncorrelated sensor gain errors, $\tilde{g}_l$, $l=1,\ldots,L$, distributed around the ideal gain value g_0 with maximum standard deviation $\sigma_{g,\max} := \frac{g_0}{2}$. Typically, we set $g_0 = 1$. Again, the actual deviation at hand can be specified by a fraction $p_g,\ 0 \leq p_g \leq 1$, i.e. $\sigma_g = p_g \cdot \frac{g_0}{2}$. The gain variance σ_g^2 is a function of different physical parameters of the system according to the specifications.

3. ROBUSTIFIED OVERCOMPLETE BASIS

To obtain a sparse basis that is robust against array errors, we apply a pre-processing step to estimate the perturbed steering vectors. They can be written in terms of the steering vectors obtained from the ideal model, $\mathbf{a}_0(\theta)$, and a perturbation term, $\mathbf{\Delta}$, that accounts for gain and phase mismatches as well as uncertainties in the sensor positions, i.e. $\mathbf{a}(\theta) = \mathbf{a}_0(\theta) + \mathbf{\Delta}$. A commonly used technique [13, 16] for estimating $\mathbf{\Delta}$ is to introduce an ellipsoidal uncertainty set $(\mathbf{a}(\theta)-\mathbf{a}_0(\theta))^H\mathbf{C}^{-1}(\mathbf{a}(\theta)-\mathbf{a}_0(\theta)) \leq 1$, where $\mathbf{C}$ contains the coefficients of the ellipsoid. After applying an adequate linear transformation [13], it can generally be written by $\mathbf{C} = \epsilon\mathbf{I}$. Thus, the error between the ideal and the perturbed steering vectors is bounded as

$$\|\mathbf{a}(\theta) - \mathbf{a}_0(\theta)\|_2^2 \;:=\; \|\mathbf{\Delta}\|_2^2 \;\leq\; \epsilon \,. \tag{5}$$

When no *a priori* information is available, a too high value of ϵ decreases the resolution and closely-spaced sources may no longer be resolved. Hence, it is desirable to estimate $\mathbf{\Delta}$ directly without knowledge of ϵ. This can be done by employing the estimation technique proposed in [15, 16] which will be briefly reviewed. Let us consider the covariance matrix of the sensor outputs which is modeled by [13]

$$\mathbf{R} = \sum_{k=1}^{K} \sigma_k^2 \mathbf{a}_0(\theta_k)\mathbf{a}_0^H(\theta_k) + \mathbf{Q} \,, \tag{6}$$

with $\mathbf{Q}$ being the noise covariance. The singular value decomposition (SVD) yields $\mathbf{R} = \mathbf{U}(\mathbf{L}\mathbf{D}_K + \mathbf{L}\tilde{\mathbf{D}}_K)\mathbf{V}^H$ with unitary matrices $\mathbf{U}$, $\mathbf{V}$, and diagonal matrix $\mathbf{L}$ containing the singular values. $\mathbf{D}_K = [\mathbf{I}_K, \mathbf{0}]^T$ and $\tilde{\mathbf{D}}_K = [\mathbf{0}, \mathbf{I}_{L-K}]^T$ are the subspace selection matrices. Hence, the signal and noise subspaces can be written by $\mathbf{U}_s = \mathbf{U}\mathbf{L}\mathbf{D}_K$ and $\mathbf{U}_n = \mathbf{U}\mathbf{L}\tilde{\mathbf{D}}_K$, respectively [1]. As in [16], let the true noise subspace be $\mathbf{U}_n = \hat{\mathbf{U}}_n + \delta\mathbf{U}_n$, and let $\hat{\mathbf{U}}_n$ be an estimate of $\mathbf{U}_n$ when the sample covariance $\hat{\mathbf{R}} = \frac{1}{T}\mathbf{Y}\mathbf{Y}^H$ is used. The small deviation $\delta\mathbf{U}_n$ represents a stochastic estimation error. Ideally, the true steering vectors are orthogonal to $\mathbf{U}_n$, i.e. [16]

$$(\hat{\mathbf{U}}_n + \delta\mathbf{U}_n)^H(\mathbf{a}_0 + \mathbf{\Delta}) = 0 \,, \tag{7}$$

However, this does not hold for $\hat{\mathbf{U}}_n$ alone. The mean-squared error (MSE) using only $\hat{\mathbf{U}}_n$ can be approximated by [16]

$$E\Big[\|\hat{\mathbf{U}}_n^H(\mathbf{a}_0(\theta)+\mathbf{\Delta})\|_2^2\Big] \approx \mathbf{a}_0(\theta)^H\mathbf{C}_{\delta U}\mathbf{a}_0(\theta) := \gamma^2 \,. \tag{8}$$

The expectation is taken over the estimation error $\delta\mathbf{U}_n$ with covariance matrix $\mathbf{C}_{\delta U} = E[\delta\mathbf{U}_n\delta\mathbf{U}_n^H]$. Since $\mathbf{C}_{\delta U}$ and $\mathbf{\Delta}$ are small, their products are neglected in (8). In order to estimate the true steering vectors $(\mathbf{a}_0 + \mathbf{\Delta})$, we are interested in the error term $\mathbf{\Delta}$ which can be found by solving the quadratically constraint quadratic program (QCQP) [16]:

$$\min \|\mathbf{\Delta}\|_2^2 \qquad s.t. \qquad \|\hat{\mathbf{U}}_n^H(\mathbf{a}_0(\theta) + \mathbf{\Delta})\|_2^2 \leq \gamma^2 \tag{9}$$

Using the method of Lagrange multipliers, the authors in [16] obtained a closed form solution as

$$\hat{\mathbf{\Delta}} = \left[\gamma \left(\mathbf{a}_0(\theta)^H \hat{\mathbf{U}}_n^H \hat{\mathbf{U}}_n \mathbf{a}_0(\theta)\right)^{-\frac{1}{2}} - 1\right] \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \mathbf{a}_0(\theta). \quad (10)$$

For a sufficiently high number of snapshots T, the estimation error $\delta\mathbf{U}$ will be small, so γ tends to zero. Then, Eq. (10) becomes $\hat{\mathbf{\Delta}} = -\hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \mathbf{a}_0(\theta)$. Finally, problem (2) can be solved using the robustified sparse basis matrix obtained by

$$\mathbf{A}^{(R)} = \left(\mathbf{I} - \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H\right) \mathbf{A} \quad (11)$$

The computational complexity to obtain $\mathbf{A}^{(R)}$ is dominated by the subspace decomposition of $\hat{\mathbf{R}}$, i.e. $O(L^3)$, while the sparse optimization problem can be solved with $O((K \times N)^3)$ operations and less than $O((K \times N)^{0.5})$ iterations [1, 18]. Commonly, $K \times N > L$, so the computational complexity is not increased. However, additional computation time is required for the pre-calibration step whenever newly acquired information from system monitoring is processed.

4. SIMULATION RESULTS

We compare the performance of the sparse techniques using (a) the standard basis derived from the ideal model (SPARSE), and (b) the robustified basis obtained via pre-calibration (R-SPARSE). The simulation setup contains $L = 9$ sensors, $K = 2$ sources, closely spaced at locations $\phi_1 = -45°$ and $\phi_2 = -35°$, which is inside the Rayleigh resolution limit. The sources may be correlated with correlation coefficient ζ. The angular accuracy of the system was set to $1°$. For sparse DOA estimation, we used $T = 10$ snapshots, while the covariance matrix for pre-calibration was estimated using $T = 200$ snapshots. In all simulations we set $p_\phi = p_g = p$. Figure 1 shows the detection rates for varying SNR (lower subfigures) and the averaged absolute angular errors (upper subfigures) which were calculated using the peaks closest to the true source positions. All results have been averaged over 500 Monte Carlo trials. First, we determined the upper confidence bound of the noise distribution using $\alpha_S = 0.996$ for SPARSE and $\alpha_R = 0.5$ for R-SPARSE. In the case of uncorrelated sources ($\zeta = 0$), R-SPARSE provides reliable detection rates (2 peaks detected) for the given α_R at all SNR values while SPARSE has a narrow optimal SNR region where reliable detection rates are obtained using α_S (α_S was chosen to shift the optimal SNR into the shown SNR region). At lower SNRs, its regularization is too high resulting in suppression of peaks, i.e. higher missed detection rate. At higher SNRs, insufficient regularization leads to spurious peaks, i.e. higher false alarm rate. The absolute angular error for R-SPARSE goes smoothly down to the minimum achievable value which is proportional to the gain/phase errors. For correlated sources, higher SNRs are required to reach this point. The error of SPARSE, however, increases abruptly at a certain SNR threshold. When the source correlation increases (second row of Figure 1), the performance of R-SPARSE

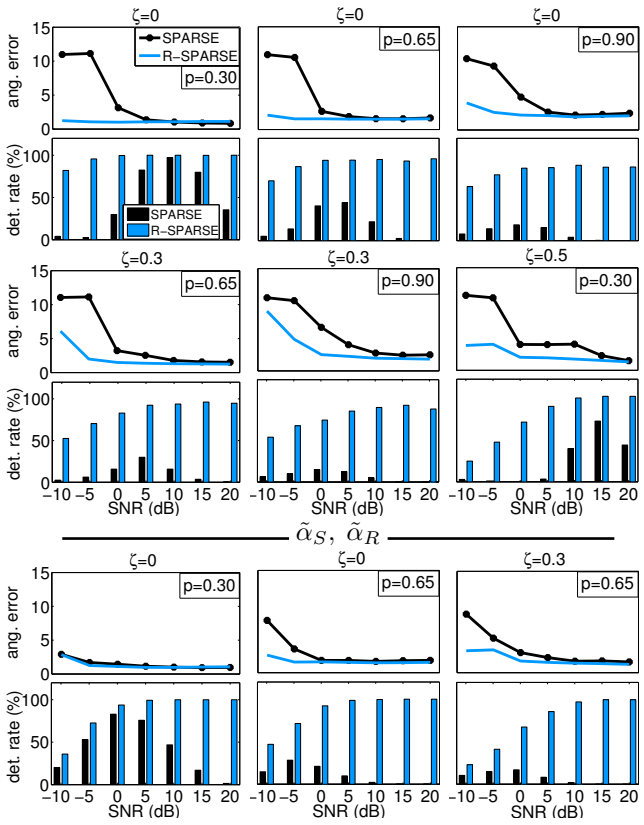

Fig. 1. Det. rates and ang. error vs. SNR for different p, ζ. Noise confidence bounds: $[\alpha_S, \alpha_R]$ (up) and $[\tilde{\alpha}_S, \tilde{\alpha_R}]$ (down)

decreases slowly but it is hardly affected by array errors. SPARSE, in contrast, is strongly affected by the joint influences of source correlation and array errors, rendering lower detection rates. The last row of Figure 1 shows the results for $\zeta = \{0.0, 0.3\}$ with a small change $\tilde{\alpha}_S = \alpha_S + \Delta\alpha_S = 0.91$ and a large change $\tilde{\alpha}_R = \alpha_R + \Delta\alpha_R = 0.8$. This corresponds to $\Delta\alpha_S = 0.086$, $\Delta\alpha_R = 0.3$, and a fraction $\frac{\Delta\alpha_R}{\Delta\alpha_S}$ of around 11 dB. We observe that the optimal SNR range of SPARSE is significantly shifted while R-SPARSE is only slightly affected at low SNRs. Finally, Figure 2 illustrates the results shown in Figure 1 by comparing the DOA spectra. The spectrum of a standard delay-and-sum beamformer (DSB) is shown as a reference. R-SPARSE is highly robust

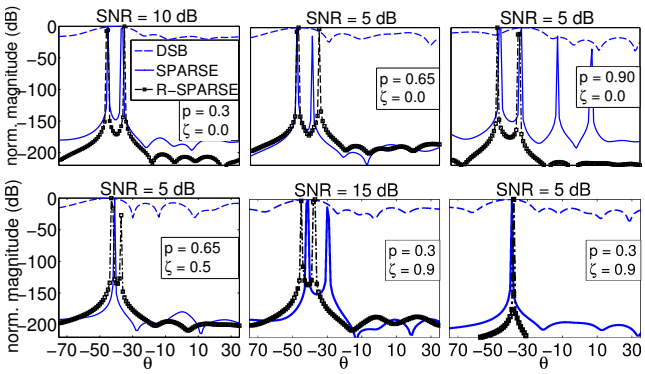

Fig. 2. DOA spectra for different values of SNR, p and ζ.

against array errors and resolves uncorrelated sources even at low SNRs where SPARSE fails or yields incorrect results. However, the limit of R-SPARSE is reached for almost coherent sources ($\zeta \rightarrow 1$) in the presence of array errors. The high detection rate and accuracy of R-SPARSE at low SNRs owes to the fact that most of the errors introduced by noise and array perturbations are mitigated by pre-calibration while the impact of source correlation remains.

5. CONCLUSION

A robust sparse DOA estimation method was presented. By array pre-calibration, an improved sparse basis is obtained which exhibits high robustness against array errors and noise. Thus, high resolution is still achieved for severe array errors and at low SNRs. Moreover, the problem of empirically selecting α to upper bound the noise distribution is significantly alleviated. For SPARSE, its choice is critical since slight deviations lead to abrupt changes in the optimal SNR range, while R-SPARSE exhibits high robustness against incorrect choices. Although standard SPARSE can handle source correlation without array errors, the combined effects of source correlation and array perturbations strongly debate the performance. Since R-SPARSE mitigates noise and array errors, it can perform better in the case of correlated sources.

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