

GENERALIZED MNS METHOD FOR PARALLEL MINOR AND PRINCIPAL SUBSPACE ANALYSIS

Viet-Dung Nguyen¹ Karim Abed-Meraim¹ Nguyen Linh-Trung² Rodolphe Weber¹

¹ PRISME Lab., University of Orléans, France, ² Faculty of Elec. and Telecom., VNU Univ. of Eng. & Tech., Vietnam

ABSTRACT

This paper introduces a generalized minimum noise subspace method for the fast estimation of the minor or principal subspaces for large dimensional multi-sensor systems. In particular, the proposed method allows parallel computation of the desired subspace when $K > 1$ computational units (DSPs) are available in a parallel architecture. The overall numerical cost is approximately reduced by a factor of K^2 while preserving the estimation accuracy close to optimality. Different algorithm implementations are considered and their performance is assessed through numerical simulation.

1. INTRODUCTION

Principal subspace analysis (PSA) and minor subspace analysis (MSA) are known techniques in multivariable signal processing needed in different application fields including high resolution parameter estimation [1], data compression [2], blind source separation [3] or radio frequency interference (RFI) mitigation [4]. Standard subspace estimation methods require SVD-like techniques which are computationally expensive especially for large dimensional systems (e.g large sensor networks [5], massive MIMO systems [6], large antenna arrays [7], etc). In such a case, the use of distributed algorithms [8] or parallel computation schemes [4] becomes of high interest and leads in general to large gains in terms of computational cost and memory requirements.

In this paper, we generalize the concept of minimum noise subspace (MNS) to achieve the desired subspace estimation in a parallel scheme. The MNS method has been initially introduced in [9] as a tool to reduce the cost of blind channel estimation of MIMO system using subspace techniques. Later on, the MNS has been extended to other array processing applications in [10] and to adaptive tracking of minor subspace in [11]. The proposed GMNS (Generalized MNS) consists of the following two contributions (i) we extend the concept of PCS (properly connected sequence) used in the MNS method in such a way one can extract the minor subspace with a fixed number K of DSPs in a parallel architecture or otherwise to improve the estimation of noise vectors in large dimensional systems, and (ii) we propose new algorithms for the computation of the principal subspace using again K properly chosen subsystems in a parallel scheme.

2. MINIMUM NOISE SUBSPACE: A REVIEW

Let consider a general linear system with p inputs and n outputs ($p < n$) so that the received system can be modeled as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad (1)$$

where $\mathbf{x}(t)$ is the observation vector, \mathbf{A} is the $n \times p$ full column rank system matrix, $\mathbf{s}(t)$ is a p -dimensional random vector and $\mathbf{n}(t)$ is the additive white noise vector with unknown variance σ^2 . The covariance matrix is then given by

$$\mathbf{R}_{xx} = E\{\mathbf{x}(t)\mathbf{x}(t)^H\} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I} \quad (2)$$

Now, having a parallel architecture with K computational units at hand, we would like to exploit it to reduce the numerical cost for minor or principal subspace extraction of the data covariance matrix \mathbf{R}_{xx} . For that, we generalize the principle of the MNS method in [9] to our particular case. The latter method has been proposed for fast computation of the least eigenvectors using PCS concept briefly reviewed below.

2.1. Concept of PCS

Let m_1, m_2, \dots, m_n be members of the system output. Consider a sequence of $n - p$ output subsets containing each $p + 1$ members, and denoted by $(p + 1)$ -tuple $t_i = (m_{i_1}, m_{i_2}, \dots, m_{i_{p+1}})$, $i = 1, \dots, n - p$. This sequence is said to be properly connected, if the following condition is satisfied

$$\begin{aligned} \{m_{i_1}, \dots, m_{i_p}\} &\subset \{m_{j_k} \mid j < i, 1 \leq k \leq p + 1\} \\ m_{i_{p+1}} &\notin \{m_{j_k} \mid j < i, 1 \leq k \leq p + 1\} \end{aligned}$$

It means that each tuple in the sequence has p common members with its preceding tuples plus another member which is not shared by its preceding tuples. The PCS concept has been introduced to guarantee that the noise vectors computed from these subsystems form a basis of the noise subspace.

2.2. MNS Implementation

The covariance matrix \mathbf{R}_i related to i th tuple t_i , $1 \leq i \leq n - p$, is written as

$$\mathbf{R}_i = E\{\mathbf{x}_i(t)\mathbf{x}_i(t)^H\} = \mathbf{A}_i\mathbf{R}_{ss}\mathbf{A}_i^H + \sigma^2\mathbf{I} \quad (3)$$

where $\mathbf{x}_i(t) = [x_{i_1}, x_{i_2}, \dots, x_{i_{p+1}}]^T$ is the partial observation vector and \mathbf{A}_i is its corresponding system response matrix. For each \mathbf{R}_i^1 , we compute the least eigenvector $\hat{\mathbf{v}}_i$ and construct its relevant vector \mathbf{v}_i as follows

$$v_i(j) = \begin{cases} 0 & \text{if the } j\text{th output of the system} \\ & \text{does not belong to the } i\text{th tuple} \\ \hat{v}_i(j') & \text{if the } j\text{th output of the system} \\ & \text{is the } j'\text{th entry of } i\text{th tuple} \end{cases}$$

where $1 \leq j \leq n$. It is proved in [9, 10] that, the resulting set of vectors $\{\mathbf{v}_i\}_{1 \leq i \leq n-p}$ forms a basis of the noise subspace.

3. GMNS FOR MSA

In the MNS method, each noise vector is estimated with the minimum number of system outputs, i.e. $p + 1$ which might lead to non-negligible performance loss if $n \gg p$. On the other hand, to achieve the parallel computation of the noise vectors, we need $n - p$ DSPs in parallel, a number which depends on the impinging source number p (usually a non-controllable system parameter).

In this paper, we propose a generalization of the MNS method that overcomes the previous mentioned shortcomings. Let assume we have K computational units at hand which will be used in a parallel scheme to compute the $n - p$ noise vectors. For simplicity, let assume that $(n - p)/K = d$ is integer valued (the general case is discussed below in remark 1). We generalize the PCS concept in the following way:

Definition 1 *The generalized PCS (GPCS) is a sequence of K $(p + d)$ -tuples $t_i = (m_{i_1}, \dots, m_{i_{p+d}})$, $1 \leq i \leq K$, satisfying:*

$$\begin{aligned} \{m_{i_1}, \dots, m_{i_p}\} &\subset \{m_{j_k} \mid j < i, 1 \leq k \leq p + d\} \\ \{m_{i_{p+1}}, \dots, m_{i_{p+d}}\} &\notin \{m_{j_k} \mid j < i, 1 \leq k \leq p + d\} \end{aligned}$$

In other words, each tuple in the sequence has p common members with its preceding tuples plus d other members which are not shared by its preceding tuples.

Based on the GPCS, the noise vectors are computed as follows: For each subsystem, one computes the $(p + d) \times (p + d)$ covariance matrix \mathbf{R}_i and its d least eigenvectors represented by matrix $\hat{\mathbf{V}}_i$. Now we construct the desired noise matrix \mathbf{V}_i by zero-padding $\hat{\mathbf{V}}_i$ according to

$$\mathbf{V}_i(j, :) = \begin{cases} \mathbf{0} & \text{if the } j\text{th output of the system} \\ & \text{does not belong to the } i\text{th tuple} \\ \hat{\mathbf{V}}_i(j', :) & \text{if the } j\text{th output of the system} \\ & \text{is the } j'\text{th entry of } i\text{th tuple} \end{cases}$$

Finally, we obtain the noise matrix $\mathbf{V} = [\mathbf{V}_1, \dots, \mathbf{V}_K]$ which forms a basis of the noise subspace under the following conditions (proof is omitted due to space limitation).

¹In practice, \mathbf{R}_i is replaced by its sample averaged estimate $\hat{\mathbf{R}}_i = \frac{1}{T} \sum \mathbf{x}_i(t) \mathbf{x}_i^H(t)$, T being the sample size.

Theorem 1 *Under the assumption that every p rows of \mathbf{A} are linearly independent, the noise matrix \mathbf{V} is full column rank i.e. $\text{rank}(\mathbf{V}) = n - p$ and hence its columns span the desired noise subspace of the data covariance matrix \mathbf{R}_{xx} .*

Remarks:

1. In the general case, $(n - p)/K$ is non integer-valued, i.e. $n - p = dK + r$, $0 \leq r < K$. In that case, we will use r -tuples of length $(p + d')$ with $d' = d + 1$ and $K - r$ tuples of length $(p + d)$.
2. Note that GPCS is just a practical way to guarantee that the set of noise vectors forms a basis of the desired subspace. In other words, the GPCS does not present necessary conditions to meet but only sufficient conditions.
3. For large dimensional systems where $n \gg p$, using only $p + 1$ system outputs (as in the original MNS) to compute a noise vector may result in non-negligible performance loss. Now if $\lceil (n - p)/K \rceil = d$ is relatively large, we will use $(p + d)$ (instead of $p + 1$) outputs to estimate a given noise vector which improves its estimation accuracy.

4. GMNS FOR PSA

The original MNS was dedicated to MSA. Here, we introduce new methods to deal with the PSA problem using K subsystems in parallel scheme. Next, two algorithms are proposed for overlapping and non-overlapping subsystem respectively.

4.1. Subsystems without overlapping parts

Let assume that we have a large dimensional system such that $d = n/K > p$ and, for simplicity, integer-valued. We divide the n system outputs into K subsystems of length n/K each represented by

$$(m_{(i-1)d+1}, \dots, m_{id}), i = 1, \dots, K.$$

Now, for each subsystem, we compute the corresponding covariance matrix \mathbf{R}_i and its p principal subspace matrix $\mathbf{W}_i = \mathbf{A}_i \mathbf{Q}_i$ (\mathbf{Q}_i is an unknown non singular $p \times p$ matrix). To have a global estimate of the signal subspace (a $n \times p$ matrix, $\mathbf{W} = \mathbf{A} \mathbf{Q}$ where \mathbf{Q} is any $p \times p$ non-singular matrix and columns of \mathbf{W} span the range space of \mathbf{A}), one needs to get rid of the unknown matrices \mathbf{Q}_i . For that, we exploit the fact that all subsystems receive the same source signals since

$$\mathbf{X}_i = \mathbf{A}_i \mathbf{S} + \mathbf{N}_i \quad i = 1, \dots, K \quad (4)$$

the $p \times T$ matrix \mathbf{S} is shared by all the subsystems. Let define

$$\mathbf{S}_i = \mathbf{W}_i^\dagger \mathbf{X}_i = \mathbf{Q}_i^{-1} \mathbf{S} + \mathbf{W}_i^\dagger \mathbf{N}_i \quad (5)$$

where \dagger denotes pseudo-inverse. In the noiseless case, one can observe that

$$\mathbf{S}_i = \mathbf{T}_i \mathbf{S}_1 \quad (6)$$

where $\mathbf{T}_i = \mathbf{Q}_i^{-1} \mathbf{Q}_1$ can be estimated as the LS solution of

$$\min_{\mathbf{T}} \|\mathbf{S}_i - \mathbf{T}_i \mathbf{S}_1\|_2^2 \Rightarrow \hat{\mathbf{T}}_i = \mathbf{S}_i \mathbf{S}_1^\dagger \quad (7)$$

Finally, the principal subspace weight matrix is obtained as

$$\mathbf{W} = [\mathbf{W}_1^T, (\mathbf{W}_2 \mathbf{T}_2)^T, \dots, (\mathbf{W}_K \mathbf{T}_K)^T]^T = \mathbf{A} \mathbf{Q}_1 \quad (8)$$

In the noisy case, the previous estimate of the principal subspace weight matrix is biased due to the ‘biased’ estimate of matrix \mathbf{T}_i . In fact, one can observe that

$$\hat{\mathbf{T}}_i = \left(\frac{\mathbf{S}_i \mathbf{S}_1^H}{T} \right) \left(\frac{\mathbf{S}_1 \mathbf{S}_1^H}{T} \right)^{-1} \quad (9)$$

$$\simeq (\mathbf{Q}_i^{-1} \mathbf{R}_{ss} \mathbf{Q}_1^{-H}) (\mathbf{Q}_1^{-1} \mathbf{R}_{ss} \mathbf{Q}_1^{-H} + \sigma^2 \mathbf{I})^{-1} \quad (10)$$

where T is sample size. Here we used the fact that the subsystems being non-overlapping, their noise terms are uncorrelated (spatially white noise assumption) and the fact that \mathbf{W}_i are unitary matrices, $i = 1 \dots K$ (most subspace estimation methods compute an orthogonal basis of the desired subspace, e.g. [12]). Because of the additive term $\sigma^2 \mathbf{I}$, $\hat{\mathbf{T}}_i$ deviates from its desired value and leads to an estimation bias for the global weight matrix, especially, at low SNR values.

To overcome this problem, we replace the previous estimate of \mathbf{T}_i by the following asymptotically unbiased estimate

$$\tilde{\mathbf{T}}_i = \left(\frac{\mathbf{S}_i \mathbf{S}_1^H}{T} \right) \left(\frac{\mathbf{S}_1 \mathbf{S}_1^H}{T} - \hat{\sigma}^2 \mathbf{I} \right)^{-1} \quad (11)$$

$$= (\mathbf{W}_i^H \mathbf{R}_{i,1} \mathbf{W}_1) (\mathbf{W}_1^H \mathbf{R}_1 \mathbf{W}_1 - \hat{\sigma}^2 \mathbf{I})^{-1} \quad (12)$$

$$\hat{\sigma}^2 = (Tr(\mathbf{R}_1) - Tr(\mathbf{W}_1^H \mathbf{R}_1 \mathbf{W}_1)) / (d - p) \quad (13)$$

where $\mathbf{R}_{i,1} = E(\mathbf{x}_i(t) \mathbf{x}_1(t)^H)$. We refer to these algorithms as GMNS-N-PSA (N stands for non-overlapping) and GMNS-NU-PSA (NU stands for non-overlapping and unbiased), respectively.

4.2. Subsystems with overlapping parts

In the non-overlapping case, we assumed that $n/K > p$. To relax this assumption, we consider here the case of overlapping subsystems of size $p + q$ sharing $q > p$ system outputs, and represented by the K tuples:

$$(m_1, \dots, m_{p+q}), (m_{p+1}, \dots, m_{2p+q}), \dots, (m_{n-p+q+1}, \dots, m_n)$$

In other words, the p last members of the i th subsystem are the p first member of the $i+1$ -th subsystem. Now, for each subsystem, we compute the covariance \mathbf{R}_i and its corresponding weight matrix \mathbf{W}_i which can be written as:

$$\mathbf{W}_i = \begin{bmatrix} \mathbf{W}'_i \\ \mathbf{W}_{i \text{ overlap}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}'_i \\ \mathbf{A}_{i \text{ overlap}} \end{bmatrix} \mathbf{Q}_i \quad (14)$$

$$\mathbf{W}_{i+1} = \begin{bmatrix} \mathbf{W}'_{i+1} \\ \mathbf{W}'_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{i \text{ overlap}} \\ \mathbf{A}'_{i+1} \end{bmatrix} \mathbf{Q}_{i+1} \quad (15)$$

To get rid of matrices \mathbf{Q}_i , one exploits the overlap between two successive subsystems by assuming that $q \times p$ submatrix of \mathbf{A} is of full column rank. In that case, the global weight matrix is estimated as:

$$\mathbf{W} = [\tilde{\mathbf{W}}_1^T, \tilde{\mathbf{W}}_2^T, \dots, \tilde{\mathbf{W}}_K^T]^T \quad (16)$$

where $\tilde{\mathbf{W}}_1 = \mathbf{W}_1$ and for $i = 1, \dots, K - 1$

$$\tilde{\mathbf{W}}_{i+1} = \mathbf{W}_{i+1} \mathbf{T}_i \quad (17)$$

$$\mathbf{T}_i = (\mathbf{W}_{i+1}^{\text{overlap}})^\dagger (\tilde{\mathbf{W}}_i^{\text{overlap}}) \quad (18)$$

This algorithm is referred to as GMNS-O-PSA (where O stands for overlapping).

5. DISCUSSION

We provide here some important comments on the proposed algorithms and their comparative performance.

- A main advantage of the proposed methods is their reduced computational cost. For the MSA, the numerical cost is of order $O((p + (n - p)/K)^2 T)$ flops for the computation of the subsystems covariance matrices in a parallel scheme plus $O(p + (n - p)/K)^2 (n - p)/K$ flops for the estimation of the least (noise) eigenvectors. Comparatively, the estimation of the global covariance matrix costs $O(n^2 T)$ flops plus $O(n^2 (n - p))$ flops for the noise vectors extraction. If $n \gg p$, the overall cost is almost reduced by a factor K^2 for the covariance matrix estimation and a factor of K^3 for the noise subspace estimation. Now, for the PSA, algorithm GMNS-NU-PSA costs $O((n/K)^2 p + p^2 (n/K + p))$ flops for the computation of the p signal subspace vectors and $O(2(n/K)^2 T)$ flops for the computation of the covariance and correlation matrices \mathbf{R}_i and $\mathbf{R}_{i,1}$, $i = 1, \dots, K$. This overall cost is approximately K^2 less than the cost, equal to $O(n^2 (T + p))$ flops, of a direct computation of the signal subspace using the global covariance matrix. On the other hand, algorithm GMNS-O-PSA costs $O((p + q)^2 p + p^2 (2p + q))$ flops for the computation of the p signal subspace vectors and $O((p + q)^2 T)$ flops for the parallel computation of the covariance matrices. If $n \gg p$, $n \gg K$ and $T \gg 1$, we have $q \approx n/K$, in which case GMNS-O-PSA is slightly cheaper than GMNS-NU-PSA since it does not require the computation of the correlation matrices² $\mathbf{R}_{i,1}$.
- The proposed methods have some drawbacks including significant performance loss (as compared to a direct computation using the overall system outputs) when one or some subsystems are ‘ill conditioned’ while the overall system (i.e. matrix \mathbf{A}) is well conditioned (as illustrated

²For certain applications, e.g. radioastronomy, the global covariance matrix is available or computed for other needs, in which case GMNS-NU-PSA becomes more advantageous than GMNS-O-PSA in term of numerical cost.

| Experiment (i.e. Figure) | n | p | K | T | q |
|--------------------------|-----|-----|-----|---------|-------|
| 1 | 14 | 2 | 4 | 200 | |
| 2 | 16 | 2 | 4 | 200 | 4 |
| 3 | 16 | 2 | 4 | 2:5:200 | 4 |
| 4 | 24 | 2 | 4 | 200 | 1:1:6 |

Table 1. Particular parameters are set in our experiments.

in Figure 5). Also, the estimated weight matrix of the desired subspace is non-orthogonal and its orthogonalization, if needed, would require extra numerical cost.

- A key hypothesis for the GMNS method is the spatial whiteness of the noise vector, i.e. $E(\mathbf{n}(t)\mathbf{n}(t)^T) = \sigma^2\mathbf{I}$. This assumption might be too restrictive in certain applications and can be relaxed if $E(\mathbf{n}(t)\mathbf{n}(t)^T) = \sigma^2\mathbf{Q}$ where \mathbf{Q} is a priori known or by using instrumental variable methods. However, an open problem would be to study the robustness of the proposed methods against slight deviations from the considered noise model.
- In [11], adaptive subspace tracking based on the MNS have been proposed leading to significant performance gain both in terms of convergence rate and estimation accuracy. Similar adaptive implementations can be considered for the proposed GMNS methods for both PSA and MSA problems (this would be the focus of future works).

6. SIMULATIONS

The simulation-based performance comparison of the proposed algorithms are assessed in this section. In all our experiments, the system matrix \mathbf{A} is generated to be normal random matrix. The sources are independent i.i.d. Gaussian processes of unit power and the noise is spatially white with different Signal-to-Noise ratios (SNR) specified for each experiment. We use (19) to evaluate the performance

$$SEP(i) = \frac{1}{R} \sum_{i=1}^R \frac{tr\{\mathbf{W}_i^H(\mathbf{I} - \mathbf{W}_{ex}\mathbf{W}_{ex}^H)\mathbf{W}_i\}}{tr\{\mathbf{W}_i^H(\mathbf{W}_{ex}\mathbf{W}_{ex}^H)\mathbf{W}_i\}} \quad (19)$$

where SEP stands for subspace estimation performance, R is the number of simulation runs, and \mathbf{W}_{ex} is the exact subspace weight matrix computed by orthogonalizing \mathbf{A} . Table 1 lists the particular parameters used for each experiment.

For MSA, we compare the minor subspace estimated by GMNS-MSA with the one estimated by SVD-MSA algorithm which uses the Singular Value Decomposition (SVD) to extract the least eigenvectors (i.e. minor subspace) from the sample averaged covariance matrix of all observation data. The noise subspace is computed from the GPCS shown in Table 2. As a result of the first experiment, Figure 1 shows a very slight performance loss of GMNS-MSA.

For PSA, four algorithms including GMNS-N-PSA, GMNS-NU-PSA, GMNS-O-PSA and SVD-PSA are compared. Here,

| Tuples | |
|--------|--------------------------------------|
| t_1 | $(m_1, m_2, m_3, m_4, m_5)$ |
| t_2 | $(m_1, m_2, m_6, m_7, m_8)$ |
| t_3 | $(m_1, m_2, m_9, m_{10}, m_{11})$ |
| t_4 | $(m_1, m_2, m_{12}, m_{13}, m_{14})$ |

Table 2. GPCS used for first experiment.

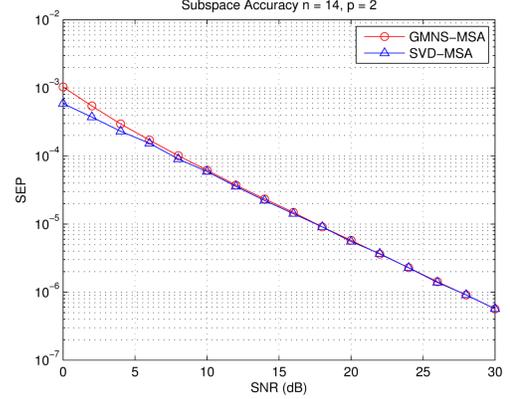


Fig. 1. Minor subspace estimation ($n = 14, p = 2$).

the SVD-PSA uses SVD to extract the principal eigenvectors (i.e. principal subspace) from the sample averaged covariance matrix of all observation data. As shown in Figure 2, GMNS-O-PSA outperforms algorithm GMNS-N-PSA and GMNS-NU-PSA, and reaches the same performance as SVD-PSA. Besides, GMNS-NU-PSA has better performance than GMNS-N-PSA due to the asymptotically unbiased estimate of $\hat{\mathbf{T}}_i$ (i.e. equation (12)). This result confirms our observation in the previous section.

We also consider the effect of certain parameters on the per-

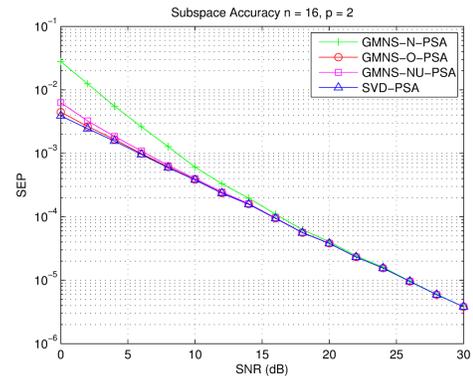


Fig. 2. Principal subspace estimation ($n = 16, p = 2$).

formance of the algorithms in the case of PSA. Firstly, we analyse the effect of the observation time T in a very noisy environment (i.e. SNR = 0 dB). As indicated in Figure 3, GMNS-O-PSA has the best estimation accuracy among the

proposal algorithms comparable to SVD-PSA and the estimation error decreases almost linearly (for large T) with respect to the observation size. Secondly, we use the SVD-PSA as a

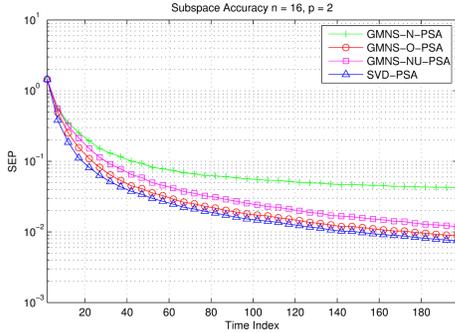


Fig. 3. Performance comparison of four algorithms when the observation duration T changes.

benchmark to evaluate performance of GMNS-O-PSA when the number of overlapping parts q varies for SNR = 0 dB. Figure 4 reveals that we can achieve close to SVD-PSA performance only with small increase of subsystems size q . Thus, the parameter q offers a flexible way to balance between the accuracy and the complexity.

Figure 5 illustrates the performance loss of the proposed al-

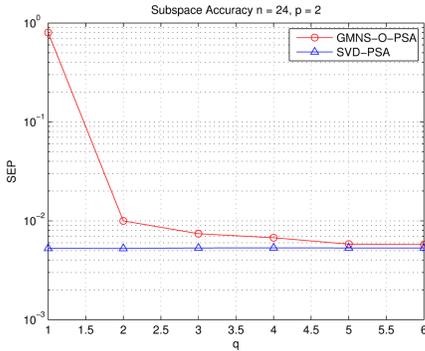


Fig. 4. Performance of GMNS-O-PSA when q changes.

gorithms when one of the K subsystems (here subsystem 1) is ill conditioned, i.e. $cond(\mathbf{A}_1) \geq 100$ while the condition number of matrix \mathbf{A} is relatively good.

7. CONCLUSION

The concept of MNS has been generalized and used for fast parallel computation of minor and principal subspaces. The overall cost is decreased by a large factor (K^2 or more) as compared to direct subspace estimation using all system outputs. In the MSA case, the proposed method allows more flexibility than the original MNS and leads to efficient estimation of the noise subspace. The GMNS is also extended to PSA case for which two fast algorithms are proposed. Again,

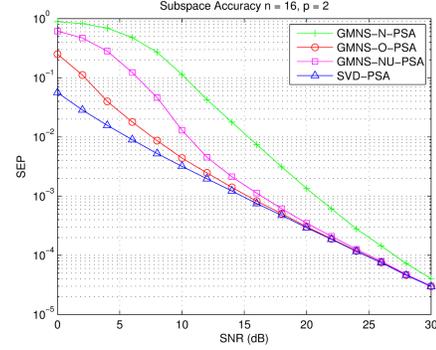


Fig. 5. Performance loss when condition numbers of first subsystem and overall system are 100 and 18.4 respectively.

efficient signal subspace estimation is observed through the simulation results close to the SVD-based performance.

REFERENCES

- [1] P. Stoica, R. Moses, *Spectral Analysis of Signals*, Prentice Hall, 2005.
- [2] Y. Hua and W. Liu, "Generalized karhunen-loeve transform," *IEEE SP Letters*, , no. 6, pp. 141–142, 1998.
- [3] P. Comon and Ch. Jutten, *Handbook of Blind Source Separation: ICA and Applications*, Elsevier, 2010.
- [4] G. Hellbourg, "RFI spatial processing for modern radio telescopes," *PhD Thesis, Univ. of Orléans*, Jan. 2014.
- [5] S. Haykin, K. J. Ray Liu, *Handbook on Array Processing and Sensor Networks*, Wiley-IEEE Press, 2010.
- [6] E. G. Larsson, F. Tufvesson, O. Edfors, and T. L. Marzetta, "Massive MIMO for next generation wireless systems," *IEEE Commun. Mag.*, 2013.
- [7] "Square kilometre array: Exploring the universe with the world's largest radio telescope," <https://www.skatelescope.org/>.
- [8] A. Bertrand and M. Moonen, "Distributed adaptive eigenvector estimation of the sensor signal covariance matrix in a fully connected sensor network," in *Proc. ICASSP*, May 2013, pp. 4236–4240.
- [9] K. Abed-Meraim and Y. Hua, "Blind identification of multi-input multi-output system using minimum noise subspace," *IEEE Tr-SP*, , no. 1, pp. 254–258, 1997.
- [10] K. Abed-Meraim, Y. Hua and A. Belouchrani, "Minimum noise subspace: concepts and applications," in *Proc. ICICS*, 1997, pp. 118–121.
- [11] M. Thameri, K. Abed-Meraim and A. Belouchrani, "Minor subspace tracking using MNS technique," in *Proc. ICASSP*, 2012, pp. 2433–2436.
- [12] G.H. Golub, C. F. Van Loan, *Matrix computations*, JHU Press, 2012.