BONE MICROSTRUCTURE RECONSTRUCTIONS FROM FEW PROJECTIONS WITH STOCHASTIC NONLINEAR DIFFUSION

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ABSTRACT

In this work, we use a stochastic diffusion equation for the reconstruction of binary tomography cross-sections obtained from a small number of projections. The aim of this new method is to escape from local minima by changing the shape of the boundaries of the image. First, an initial binary image is reconstructed with a deterministic Total Variation regularization method, and then this binary reconstructed image is refined by a stochastic partial differential equation with singular diffusivity and a gradient dependent noise. This method is tested on a 256×256 experimental micro-CT trabecular bone image with different additive Gaussian noises. The reconstruction images are clearly improved.

Index Terms— X-ray imaging, TV regularization, binary tomography, bone microstructure

1. INTRODUCTION

The tomographic reconstruction problem with a limited number projection angles has many applications in medical imaging and material science. Binary tomography methods can be used to formulate it as a simpler inverse problem [1]. In this framework, the inverse problem is associated with an underdetermined linear system of equations with the linear Radon projection operator R and binary constraints:

$$Rf = p^\delta \quad f = (f_1, ... f_n) \in \{0,1\}^n \quad (1)$$

relating the pixel values $(f_i)_{1 \le i \le n}$ of the image and the measured projection values p^δ which is some approximation of the real projection data p, corresponding to the true solution f^* with $Rf^* = p$. Numerous approaches have been proposed to solve this tomography reconstruction problem [2,3]. Statistic methods are based on Bayesian estimation and Markov random fields [4]. Some studies are based on the minimization of a functional that incorporates a data term and a binary constraint, with stochastic techniques [5] or convex analysis optimization [6, 7]. The Total Variation regularization introduced by Rudin et al. [8] can be very useful to obtain stable solutions [9, 10]. But there are still obvious errors existing in the boundaries of the binary reconstructed images [10]. It may be interesting to escape this local minimum with a global

optimization search. Stochastic algorithms based on stochastic partial differential equations have often been proposed for the global optimization of non-convex functions [11–15]. The main idea of this method is to escape the traps of local minima by changing the shape of boundaries of the binary reconstruction images with some stochastic noise.

The main contribution of this work is to use a nonlinear diffusion derived from the TV regularization method for the discrete tomography problem. A gradient dependent noise is used in this stochastic approach. We start from a deterministic TV regularization scheme and obtain an initial reconstructed image and then improve this reconstructed image with an intermittent diffusion in which the stochastic TV approach and the TV regularization methods are used alternatively.

This paper is organized as follows. After the introduction, the inverse problem formulation of the binary tomography problem is presented together with the TV regularization method, and the Alternate Direction of Minimization Method (ADMM) used to minimize the regularization functional. The next section describes the nonlinear diffusion based on the TV regularization and a gradient dependent noise term. The numerical results obtained on a noisy bone CT cross-sections are reported and discussed in the last section. We then give the main conclusions and perspectives of our work.

2. TOTAL VARIATION REGULARIZATION AND ADMM APPROACH

The binary tomography is highly ill-posed problem and a regularization is necessary to obtain stable solutions. In this work, a Total Variation regularization with convex constraints $f \in [0,1]^n$ is considered with the following minimization problem:

$$\text{minimize} \quad \frac{\mu}{2}\|p^\delta - Rf\|_2^2 + J_{TV}(f) \quad s.t. \quad f \in [0,1]^n \quad (2)$$

where μ is the regularization parameter,and $J_{TV}(f)$ the Total Variation semi-norm of f. This regularization is based on computing the L_1 norm of the gradient:

$$J_{TV}(f) = \int_\Omega |\nabla f(r)| dr \quad (3)$$

When a discrete image f is considered, the $J_{TV}(f)$ can be expressed as f: $T_{TV}(f) = \sum_i \| D_i f \|$, where D_i is the gradient operator at pixel i. In this study, the isotropic TV regularization is used. A fast, efficient TV/L_2 minimization algorithm based on an augmented Lagrangian function and the Alternating Direction of Minimization Method (ADMM) [16–20] is used to minimize the TV regularization functional. In this work, the following augmented Lagrangian including convex constraints, $f \in C = [0,1]^n$ is considered:

$$\mathcal{L}(f, (g_i), h, (\lambda_i), \lambda_C) = \sum_i (\|g_i\|_2 - \lambda_i^t(g_i - D_i f)$$
$$+ \frac{\beta}{2}\|g_i - D_i f\|_2^2 + \frac{\mu}{2}\|g - Rf\|_2^2 \quad (4)$$
$$+ I_C(h) + \frac{\beta}{2}\|h - f\|_2^2 - \lambda_C^t(h - f)$$

where μ is the regularization parameter, β the Lagrangian parameter, I_C the characteristic function of the convex set. The Lagrange multipliers (λ_i), λ_c are vectors in $\mathbb{R}^{2n^2}$ and $\mathbb{R}^{n^2}$. For each pixel i, $D_i f \in \mathbb{R}^2$ is the first-order finite difference at pixel i in both horizontal and vertical directions. With the alternating minimization algorithm, the sequences $(f^k, g_i^k, h^k, \lambda_i^k, \lambda_C^k)$ are constructed with the following iterative scheme: For each pixel i:

$$g_i^{k+1} = max\{\|D_i f^k + \frac{1}{\beta}(\lambda_i^k)\| - \frac{1}{\beta}, 0\}\frac{D_i f^k + \frac{1}{\beta}(\lambda_i^k)}{\|D_i f^k + \frac{1}{\beta}(\lambda_i^k)\|} \quad (5)$$

The h^k update is:

$$h^{k+1} = \pi_C(f^k + \frac{\lambda_C^k}{\beta}) \quad (6)$$

where π_C is the projection of the convex set C. The new iterate f^{k+1} is obained from the following linear system:

$$(\sum_i D_i^t D_i + \frac{\mu}{\beta}R^t R + I)f^{k+1} = \sum_i D_i^t(g_i^{k+1} - \frac{1}{\beta}\lambda_i^k\}$$
$$+ \frac{\mu}{\beta}R^t g + h^{k+1} - \frac{\lambda_C^k}{\beta} \quad (7)$$

where I is the identity operator. The Lagrange multipliers $(\lambda_i), \lambda_C$ are updated with:

$$\lambda_i^{k+1} = \lambda_i^k - \beta(g_i^{k+1} - D_i f^{k+1}) \quad (8)$$

$$\lambda_C^{k+1} = \lambda_C^k - \beta(h^{k+1} - f^{k+1}) \quad (9)$$

The sequence $(f^k, g_i^k, h^k, \lambda_i^k, \lambda_C^k)$ which is generated by the ADMM algorithm converges to a Kuhn-Tucker point of problem (P), $(f^*, g_i^*, h^*, \lambda_i^*, \lambda_C^*)$.

3. SINGULAR STOCHASTIC DIFFUSION EQUATION WITH GRADIENT DEPENDENT NOISE

In order to find the global minimum of a function: $g : \mathbb{R}^m \to \mathbb{R}^m$, a random trajectory $X(t)$ governed by a stochastic differential equation may be used [11–15].

$$dX(t) = -\nabla g(X(t))dt + \eta(t)dW(t) \quad t \in [0, \infty] \quad (10)$$

where $W = (W_1(t), ..., W_m(t))$ is the standard m dimensional Brownian motion and $\eta(t)$ the stochastic noise strength. The main idea is to combine the advantages of gradient flow and of a stochastic perturbation to escape the traps of local minimizers [11–15]. We have used the same type of methodology for our binary tomography reconstruction problem. To refine the solution obtained with the ADMM algorithm, we are interested here with a singular stochastic diffusion equation (E) [21, 22] of the type:

$$dX(t) = \text{div}\,(\text{sgn}(\nabla)(X(t))dt - R^*(RX(t) - p^\delta)$$
$$+ \sigma(\nabla X(t))dW(t)$$
$$(11)$$

The random noise σ chosen is a gradient dependent noise $\sigma(\nabla X_\epsilon(t))dW(t) = \frac{\partial X}{\partial s}dW_1(x,t) + \frac{\partial X}{\partial t}dW_2(y,t)$ where $W_k(x,t)$ are independent Wiener random fields with continuous covariance function $r_k(x,y)$ bounded by a constant r_0. This type of equation has been extensively studied with additive and multiplicative noise [23]. The equation with a gradient-dependent noise is considered in this work because the noise is located on the boundaries of the images reconstructed with the TV regularization. The existence of solutions is not considered in this study.

The global optimization is performed with an intermittent diffusion [24] method: the stochastic diffusion and the TV regularization are performed successively on random time intervals. The time interval lengths and the stochastic noise strength η_i are chosen randomly in the range $[0, T_{max}]$ and $[0, \eta_{max}]$. where T_{max} is the scale ofthe diffusion time and η_{max} is the scale of the stochastic noise strength.

4. RESULTS AND DISCUSSION

4.1. Simulation details

The TV and stochastic TV based methods were applied to simulated projections of an experimental bone cross-section obtained with synchrotron micro-CT (voxel size: 15 μm) [25]. Fig.1 is a 256×256 bone cross-section image reconstructed from Filtered Back Projections (FBP) with 400 projections and 400 X-rays per projection and subsequently thresholded. This reference image is the solution f^* of $Rf^* = p$. In the following, the discrete approximation of the projection operator R is the Radon transform implemented on Matlab Image Toolbox. Our optimization method

is tested with $M = 10$ equally projections angles and noisy projections p^δ. The number of rays per projection angle is $NP = 367$. Several additive gaussian noise noise levels were studied with the standard deviations $\sigma = 0$ (PSNR=0dB), $\sigma = 20$ (PSNR=6dB), $\sigma = 30$ (PSNR=3.5dB).

Fig. 1. Reconstruction of the bone cross-section from 400 projections with the FBP algorithm, as the "ground-truth" image. The bone fraction is 14.20%

First, the TV regularization method is used to obtain a first reconstruction image f_0. The Lagrange parameter β controls the speed of convergence, and the final reconstructed image $f^m(\mu)$ obtained in the optimization process only depends on the regularization parameter μ. The iteration are stopped when $\| f_{m+1} - f_m \| < 0.01$. We tested many pairs of parameters for the TV algorithm, and our the choice of the parameters is based on Morozov discrepancy principle [26]. The best parameters (μ, β) which are chosen satisfy $\| Rf^m(\mu) - p^\delta \| \approx \delta$ at the final iteration m, where δ is the noise level which can be estimated with $\delta = \sqrt{M \times NP}\sigma$. An example of reconstruction image obtained after binarization and the corresponding error map obtained are diplayed in Fig.2a and Fig.2b for $\sigma = 20$. Some errors are still present on the boundary of the image. This reconstruction errors are worse if the parameters are not chosen according to the Morozov principle.

In a second step, a nonlinear intermittent diffusion is applied. The discretization of the SPDE and of the Brownian motions was performed with classical finite difference methods [27].The stochastic noise was added at each iteration to the iterate f_k given by the ADMM method. The r-Wiener processes with a correlation function r were obtained with the Fourier transform of the covariance function $\tilde{r}(k) = \eta(|k|^2 + 1)^{-2}$ where η controls the noise strength. The stochastic noise strength η and the stochastic iteration number T are chosen randomly with uniform distribution in $[0.01, 0.1]$ and $[1, 100]$. The TV deterministic iterations are stopped when the data term stagnates. Before the next stochastic step, the reconstruction image is binarized. At each iteration, $\| Rf_{binary}^k - p^\delta \|$ is calculated, where f_{binary} is the binarization of the grey-level image. For comparison, some simulations have been performed, starting from the same image f_0, in which the stochastic diffusion is replaced by a TV regularization minimization.
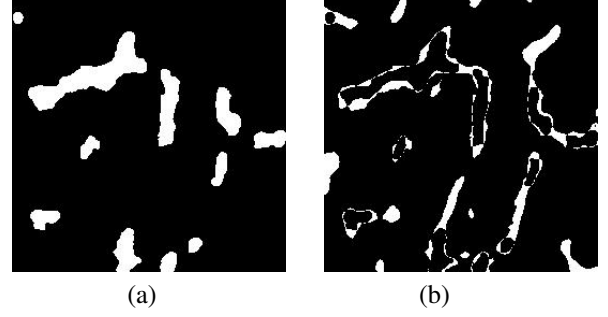

Fig. 2. (a) Reconstruction image f_0 obtained with the TV regularization for $\sigma = 20$ and corresponding error map (b).

4.2. Numerical results

The image obtained with the stochastic diffusion with the lowest value of $\| Rf_{binary}^k - p^\delta \|$ and the corresponding error map are displayed in Fig.3a and Fig.3b for $\sigma = 20$. The reconstruction errors on the boundaries of the homogeneous regions are reduced. Similar results are obtained for the other noise levels.
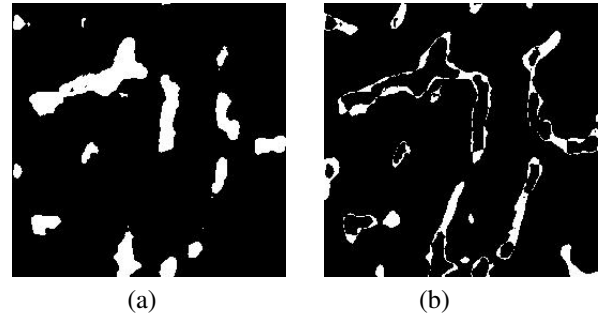

Fig. 3. (a) Reconstruction images obtained with the the nonlinear diffusion equation for $\sigma = 20$ and corresponding error map (b).

In order to have more quantitative results, the evolution the discrepancy term $\| Rf - p^\delta \|$, of $\| Rf_{binary}^k - p^\delta \|$ and of the missclassification rate are displayed on Figure 4.a,4.b Figure 5.a,5.b , Figure 6.a,6.b, starting from f_0, for the stochastic intermittent diffusion method and for the standard deviations $\sigma = 0$, $\sigma = 20$, and $\sigma = 30$ respectively. On the same plot, the error curves for the iterated TV regularization are also displayed.

With the iterations, some decrease of the data term related to the binary image towards the noise level δ is obtained except for the iterated TV method with no noise. A decrease of the misclassification rate as a function of the number of iterations is also observed. The stochastic approach is more efficient that a TV regularization used repeatedly. The data term $\| Rf_{binary}^k - p^\delta \|$ remains higher than the noise levels

$\delta = 1220$ and $\delta = 1831$ for $\sigma = 20$ and $\sigma = 30$ but longer runs will be performed to improve further the results.
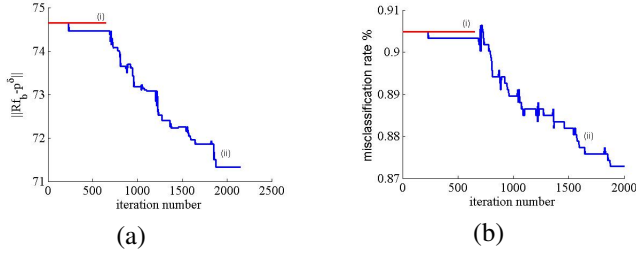

(a)

(b)

Fig. 4. (a) Evoluation with the iterations of $\|Rf_{binary} - p^\delta\|$ TV (i) and for stochastic diffusion (ii). (a) Evoluation with the iterations of the missclassification rate for TV (i) and for stochastic diffusion (ii). The standard deviation of the noise is $\sigma = 0$
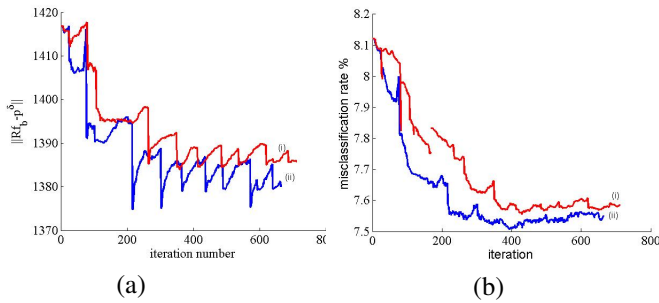

(a)

(b)

Fig. 5. (a) Evoluation with the iterations of $\|Rf_{binary} - p^\delta\|$ TV (i) and for stochastic diffusion (ii). (a) Evoluation with the iterations of the missclassification rate for TV (i) and for stochastic diffusion (ii). The standard deviation of the noise is $\sigma = 20$

5. CONCLUSION

This work proposes a new stochastic diffusion method with gradient dependent noise to reconstruct binary tomography cross-section from few projection angles. An initial binary reconstructed image is obtained with a deterministic TV regularization method. It is then refined by a stochastic partial differential equation. Usually, the reconstruction errors obtained with TV are localized on the boundary. The stochastic search algorithm based on TV with a gradient dependent noise is useful to escape from the local minima by changing the shape of boundary regions according to the stochastic noise. The new method leads to an obvious decrease of reconstruction errors and misclassification rate for 10 projection with different noise levels. The efficiency of this method will
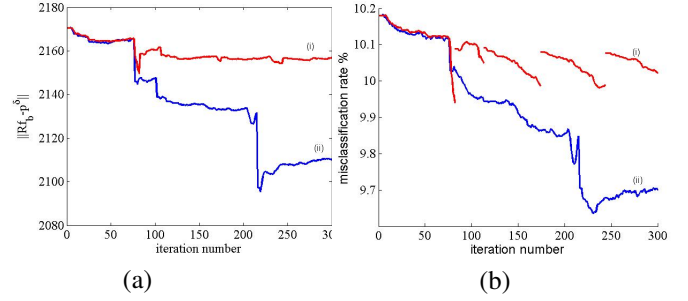

(a)

(b)

Fig. 6. (a) Evoluation with the iterations of $\|Rf_{binary} - p^\delta\|$ TV (i) and for stochastic diffusion (ii). (a) Evoluation with the iterations of the missclassification rate for TV (i) and for stochastic diffusion (ii). The standard deviation of the noise is $\sigma = 30$

be investigated in the future work on more projection numbers and noise levels, for longer simulations, larger images and clinical data sets with various structural parameters.

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