GENERALIZATION OF CAMPBELL'S THEOREM TO NONSTATIONARY NOISE

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ABSTRACT

Campbell's theorem is a fundamental result in noise theory and is applied in many fields of science and engineering. It gives a simple but very powerful expression for the mean and standard deviation of a stationary random pulse train. We generalize Campbell's theorem to the non-stationary case where the random process is space and time dependent. We also generalize it to a pulse train of waves, acoustic and electromagnetic, where the intensity is defined as the absolute square of the pulse train.

Index Terms— nonstationary noise, Campbell's theorem, random pulse train, reverberation, time-frequency

1. INTRODUCTION

One of the most fundamental results in noise theory is Campbell's theorem, a result that was given by Campbell [1, 2] in 1909 soon after the basic concepts of noise were developed [3]. Campbell's theorem applies to a random process that is a pulse train of the following form

$$s = \sum_{n=1}^{N} f(t - t_n) \tag{1}$$

where $f(t)$ is a deterministic function of time, t_n are random arrival times, and N is the number of constituents. Campbell's result is that the mean and standard deviation of the process are given by

$$\langle s \rangle = N \int_{-\infty}^{\infty} f(t) dt \tag{2}$$

$$\sigma^2 = \left\langle s^2 \right\rangle - \left\langle s \right\rangle^2 = N \int_{-\infty}^{\infty} f^2(t) dt \tag{3}$$

This is a simple but very powerful result that is easy to apply, is used in almost all fields and has found many applications. Furthermore, sometime later, Rowland [4] considered two series where in addition to Eq. (1) we have the series

$$q = \sum_{n=1}^{N} g(t - t_n) \tag{4}$$

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He showed that covariance between two pulse trains is given by

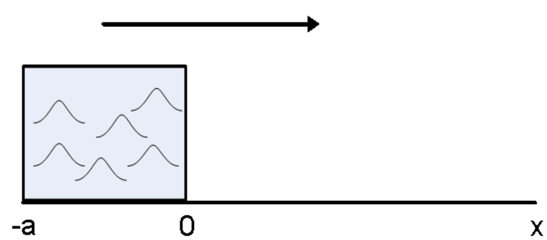
$$\langle (s - \langle s \rangle) (q - \langle q \rangle) \rangle = N \int_{-\infty}^{\infty} f(t) g(t) dt \tag{5}$$

We point out that a fruitful view point is to think of a pulse train as consisting of delta functions, $\sum_{n=1}^{N} \delta(t - t_n)$, and letting the sum go through a linear filter whose impulse response function is $f(t)$.

In deriving the above results a number of assumptions are made, the most important is that the series are stationary. In this paper we consider a non-stationary model where we have spatial and time dependence. We explicitly derive the mean and standard deviations analogous to Campbell's theorem and show that in the limit of infinite time we recover Campbell's theorem.

2. NON-STATIONARY PULSE TRAIN

Consider the following model of a non-stationary pulse train as illustrated in the figure. At $t = 0$, N "particles with width",



as represented by the $f(t)$, are generated at random positions x_n chosen from a uniform distribution ranging from $-a$ to 0. The particles move to the right with a constant velocity, c. At position x and time t we observe the random process given by

$$V(x, t) = \sum_{n=1}^{N} f(x - ct - x_n) \tag{6}$$

We define the ensemble average as the average over the initial positions x_n,

$$\langle V(x, t) \rangle = N \int_{-\infty}^{\infty} P(x_n) f(x - ct - x_n) dx_n \tag{7}$$

where $P(x_n)$ is the probability distribution of x_n. For the case of a uniform distribution

$$P(x_n) = \begin{cases} 1/a & -a < x_n < 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and therefore

$$\langle V(x,t)\rangle = \frac{N}{a}\int_{-a}^{0} f(x-ct-x_n)dx_n = \frac{N}{a}\int_{x-ct}^{x-ct+a} f(z)dz \tag{9}$$

where we have made the substitution $z = x - ct - x_n$ and changed the limits accordingly.

Now, suppose we take N and a large in such a manner that

$$\frac{N}{a} \to \rho \qquad \text{for } N \to \infty \text{ and } a \to \infty \tag{10}$$

where ρ is the constant density. Hence, we have that

$$\langle V(x,t)\rangle = \rho\int_{x-ct}^{\infty} f(z)dz \tag{11}$$

This is the average at position x and time t for N and $a \to \infty$. If we further take $t \to \infty$ which can be thought of as the steady state limit then we have

$$\langle V(x,t)\rangle = \rho\int_{-\infty}^{\infty} f(z)dz \tag{12}$$

which is the first part of Campbell's theorem Eq. (2). In Eq. (9) and (11) the result is position dependent but as time goes to infinity we obtain a stationary process and the resulting average, Eq. (12), is independent of position.

To calculate the standard deviation we first consider

$$V^2(x,t) = \sum_{n=1}^{N}\sum_{k=1}^{N} f(x-ct-x_n)f(x-ct-x_k) \tag{13}$$

and following the same procedure as above we obtain

$$\begin{aligned}\left\langle V^2(x,t)\right\rangle = {} & \frac{N}{a}\int_{x-ct}^{x-ct+a} f^2(z)dz \\ & + \frac{N(N-1)}{a^2}\left(\int_{x-ct}^{x-ct+a} f(z)dz\right)^2 \end{aligned} \tag{14}$$

Taking the limit as in Eq. (10) and noting that $N/a^2 \to 0$ we have

$$\left\langle V^2(x,t)\right\rangle = \rho\int_{x-ct}^{\infty} f^2(z)dz + \rho^2\left(\int_{x-ct}^{\infty} f(z)dz\right)^2 \tag{15}$$

The standard deviation $\sigma^2 = \left\langle V^2(x,t)\right\rangle - \langle V(x,t)\rangle^2$ is

$$\begin{aligned}\sigma^2 = {} & \frac{N}{a}\int_{x-ct}^{x-ct+a} f^2(z)dz + \frac{N(N-1)}{a^2}\left(\int_{x-ct}^{x-ct+a} f(z)dz\right)^2 \\ & - \left(\frac{N}{a}\int_{x-ct}^{x-ct+a} f(z)dz\right)^2 \end{aligned} \tag{16}$$

$$= \frac{N}{a}\int_{x-ct}^{x-ct+a} f^2(z)dz - \frac{N}{a^2}\left(\int_{x-ct}^{x-ct+a} f(z)dz\right)^2 \tag{17}$$

If we take the limit as per Eq. (10) we obtain

$$\sigma^2 = \rho\int_{x-ct}^{\infty} f^2(z)dz \tag{18}$$

and if we further take $t \to \infty$ then we have

$$\sigma^2 = \rho\int_{-\infty}^{\infty} f^2(z)dz \tag{19}$$

which is the second part of Campbell's theorem, Eq. (3).

We consider Eq. (9) and Eq. (11) as a nonstationary position and time dependent Campbell's theorem for the mean; Eq. (17) and (18) to be a generalization of the standard deviation.

3. NON-UNIFORM DENSITY

In the above considerations we have taken the initial density to be uniform as given by Eq. (8). For a non uniform initial distribution given by a probability density $P(x_n)$, but still constant velocity motion, we have

$$\langle V(x,t)\rangle = N\int_{-\infty}^{\infty} f(x-ct-x_n)P(x_n)dx_n \tag{20}$$

and

$$\begin{aligned}\left\langle V^2(x,t)\right\rangle = {} & N\int_{-\infty}^{\infty} f^2(x-ct-x_n)P(x_n)dx_n \\ & + N(N-1)\left(\int_{-\infty}^{\infty} f(x-ct-x_n)P(x_n)dx_n\right)^2 \end{aligned} \tag{21}$$

The standard deviation is given by

$$\begin{aligned}\sigma^2 = {} & N\int_{-\infty}^{\infty} f^2(x-ct-x_n)P(x_n)dx_n \\ & - N\left(\int_{-\infty}^{\infty} f(x-ct-x_n)P(x_n)dx_n\right)^2 \end{aligned} \tag{22}$$

It is possible for the second term to be significant depending on $P(x_n)$ and $f(x)$.

4. GENERAL MOTION

In the above, we have taken motion to be constant velocity; however, there are situations where non-constant velocity is appropriate. Suppose the motion is governed by

$$x = x_n + \eta(t) \tag{23}$$

To take that into account, replace ct by a time function that describes the motion. We have

$$\langle V(x,t)\rangle = N\int_{-\infty}^{\infty} f(x-\eta(t)-x_n)P(x_n)dx_n \quad (24)$$

and further

$$\left\langle V^2(x,t)\right\rangle = N\int_{-\infty}^{\infty} f^2(x-\eta(t)-x_n)P(x_n)dx_n$$

$$+ N(N-1)\left(\int_{-\infty}^{\infty} f(x-\eta(t)-x_n)P(x_n)dx_n\right)^2 \quad (25)$$

The standard deviation is then

$$\sigma^2 = N\int_{-\infty}^{\infty} f^2(x-\eta(t)-x_n)P(x_n)dx_n$$

$$- N\left(\int_{-\infty}^{\infty} f(x-\eta(t)-x_n)P(x_n)dx_n\right)^2 \quad (26)$$

For the case of uniform density, as per Eq. (8) and taking the same limits as in Section 2 we have for the mean and standard deviation that

$$\langle V(x,t)\rangle = \rho\int_{x-\eta(t)}^{\infty} f(z)dz \quad (27)$$

$$\sigma^2 = \rho\int_{x-\eta(t)}^{\infty} f^2(z)dz \quad (28)$$

5. TWO-TIME AUTOCORRELATION FUNCTION

We now calculate the two-time correlation function at position x,

$$R(t_1,t_2;x) = \langle V(x,t_1)V(x,t_2)\rangle \quad (29)$$

For the constant velocity case we have

$$\langle V(x,t_2)V(x,t_1)\rangle = \sum_{n=1}^{N}\sum_{k=1}^{N}\langle f(x-ct_2-x_n)f(x-ct_1-x_k))\rangle \quad (30)$$

and following the same procedure as in the previous sections one obtains

$$\langle V(x,t_2)V(x,t_1)\rangle = \frac{N}{a}\int_{x-ct_2}^{x-ct_2+a} f(z)f(z+c(t_2-t_1))dz$$

$$+ \frac{N(N-1)}{a^2}\left(\int_{x-ct_2}^{x-ct_2+a} f(z)dz\right)\left(\int_{x-ct_1}^{x-ct_1+a} f(z)dz\right) \quad (31)$$

Taking the constant density limit one obtains

$$\langle V(x,t_2)V(x,t_1)\rangle = \rho\int_{x-ct_2}^{\infty} f(z)f(z+c(t_2-t_1)dz$$

$$+ \rho^2\left(\int_{x-ct_2}^{\infty} f(z)dz\right)\left(\int_{x-ct_1}^{\infty} f(z)dz\right) \quad (32)$$

Letting

$$t_2-t_1=\tau \quad (33)$$

we have

$$\langle V(x,t_2)V(x,t_1)\rangle = \rho\int_{x-ct_2}^{\infty} f(z)f(z+c\tau)dz$$

$$+ \rho^2\left(\int_{x-ct_2}^{\infty} f(z)dz\right)\left(\int_{x-ct_1}^{\infty} f(z)dz\right) \quad (34)$$

As expected, this is not a function τ, and hence the process is not stationary. However, if we take the limit of large times

$$t_2, t_1 \ \to \infty \quad (35)$$

but keeping (t_2-t_1) a constant, then

$$\langle V(x,t_2)V(x,t_1)\rangle = \rho\int_{-\infty}^{\infty} f(z)f(z+c\tau)dz$$

$$+ \rho^2\left(\int_{-\infty}^{\infty} f(z)dz\right)^2 \qquad t_2, t_1 \ \to \infty\, ;\, t_2-t_1=\tau \quad (36)$$

and we now see that the process becomes stationary.

6. TIME-VARYING SPECTRUM

Another fruitful approach is to calculate the time varying spectrum. Here we use the Wigner spectrum and relate it to the autocorrelation function. The time-frequency Wigner distribution at position x, is defined by

$$W(t,\omega,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V^*(x,t-\tfrac{1}{2}\tau)\,V(x,t+\tfrac{1}{2}\tau)e^{-i\tau\omega}\,d\tau \quad (37)$$

To deal with a nonstationary process [6–10] one takes the ensemble average of Eq. (37)

$$\overline{W}(t,\omega;x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle V^*(x,t-\tfrac{1}{2}\tau)\,V(x,t+\tfrac{1}{2}\tau)\right\rangle e^{-i\tau\omega}\,d\tau \quad (38)$$

where $\overline{W}(t,\omega;x)$ is called the Wigner *spectrum* which can be thought of as the instantaneous spectrum at position x and time t. The Wigner spectrum can be written as

$$\overline{W}(t,\omega,x) \ = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(t+\tau/2,t-\tau/2;x)e^{-i\tau\omega}d\tau \quad (39)$$

and inversely we have,

$$R(t+\tau/2,t-\tau/2;x) = \int_{-\infty}^{\infty}\overline{W}(t,\omega,x)e^{i\tau\omega}d\omega \quad (40)$$

and

$$R(t_1,t_2) = \int_{-\infty}^{\infty}\overline{W}\left(\frac{t_1+t_2}{2},\omega;x\right)e^{-i(t_2-t_1)\omega}d\omega \quad (41)$$

Consider now the Wigner distribution for the pulse train given by Eq. (1). Substituting Eq. (1) into Eq. (29) we have

$$\overline{W}(t,\omega;x) = \sum_{n=1}^{N}\sum_{m=1}^{N}\overline{W}_{nm}(t,\omega;x) \tag{42}$$

$$= N\overline{W}_{nn}(t,\omega;x) + N(N-1)\overline{W}_{nm} \tag{43}$$

where

$$\overline{W}_{nn} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x-ct-\tfrac{1}{2}\tau - x_n) f(x-ct+\tfrac{1}{2}\tau - x_n)P(x_n)\, e^{-i\tau\omega}\, d\tau dx_n \tag{44}$$

and

$$\overline{W}_{nm} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x-ct-\tfrac{1}{2}\tau - x_n)$$

$$f(x-ct+\tfrac{1}{2}\tau - x_m)P(x_n)P(x_m)\, e^{-i\tau\omega}\, d\tau dx_n dx_m \tag{45}$$

Consider $\overline{W}_{nn}$ and where we consider the uniform distribution, Eq. (8). We obtain

$$\overline{W}_{nn} = \frac{1}{2\pi}\int_{x-ct}^{x-ct+a}\int_{-\infty}^{\infty} f^*(z-\tfrac{1}{2}\tau) \tag{46}$$

$$f(z+\tfrac{1}{2}\tau)\, e^{-i\tau\omega}\, d\tau dz \tag{47}$$

The inner integral is simply the Wigner spectrum of the function $f(z)$. Therefore we have

$$\overline{W}_{nn} = \frac{1}{a}\int_{x-ct}^{x-ct+a} W(z,\omega)dz \tag{48}$$

Now consider

$$\overline{W}_{nm} = \frac{1}{2\pi}\frac{1}{a^2}\int_{x-ct}^{x-ct+a}\int_{x-ct}^{x-ct+a}\int_{-\infty}^{\infty} f^*(z-\tfrac{1}{2}\tau)f(z'+\tfrac{1}{2}\tau)e^{-i\tau\omega}\, d\tau dz dz' \tag{49}$$

$$= \frac{1}{a^2}\int_{x-ct}^{x-ct+a}\int_{x-ct}^{x-ct+a} W\left(\frac{z+z'}{2},\omega\right)e^{i(z'-z)\omega}\, dz dz' \tag{50}$$

If we substitute these results into Eq. (43) and apply the limit as in the previous sections we obtain

$$\overline{W}(t,\omega;x) = \rho\int_{x-ct}^{\infty} W(z,\omega)dz + \rho^2\int_{x-ct}^{\infty}\int_{x-ct}^{\infty} W\left(\frac{z+z'}{2},\omega\right)e^{i(z'-z)\omega}\, dz dz' \tag{51}$$

7. WAVE TRAINS

In the above considerations we have taken the intensity to be given by Eq. (1) but if $f(t)$ are waves then the intensity and the intensity squared are given, respectively, by

$$I(x,t) = \left|\sum_{n=1}^{N} f(x-ct-x_n)\right|^2 \tag{52}$$

and

$$I^2(x,t) = \left|\sum_{n=1}^{N} f(x-ct-x_n)\right|^4 \tag{53}$$

Furthermore, $f(t)$ can be complex. In this paper we consider the real case. One can show that [14]

$$\langle I\rangle = N\left[\langle f^2\rangle + (N-1)\langle f\rangle^2\right] \tag{54}$$

$$\langle I^2\rangle = N(N-1)(N-2)(N-3)\langle f\rangle^4 + 6N(N-1)(N-2)\langle f\rangle^2\langle f^2\rangle + 4N(N-1)\langle f\rangle\langle f^3\rangle + 3N(N-1)\langle f^2\rangle^2 + N\langle f^4\rangle] \tag{55}$$

Consider first $\langle I\rangle$. We can use the results obtained above to immediately write, as per Eq. (17), that

$$\langle I\rangle = \frac{N}{a}\int_{x-ct}^{x-ct+a} f^2(z)dz + \frac{N(N-1)}{a^2}\left(\int_{x-ct}^{x-ct+a} f(z)dz\right)^2 \tag{56}$$

If we take the limit as given by Eq. (10) we have

$$\langle I\rangle = \rho\int_{x-ct}^{\infty} f^2(z)dz + \rho^2\left(\int_{x-ct}^{\infty} f(z)dz\right)^2 \tag{57}$$

and further, the long time limit is

$$\langle I\rangle = \rho\int_{-\infty}^{\infty} f^2(z)dz + \rho^2\left(\int_{-\infty}^{\infty} f(z)dz\right)^2 \tag{58}$$

The calculation for $\langle I^2\rangle$ is more involved and we just give the final answer,

$$\langle I^2\rangle = \rho^4\left(\int_{x-ct}^{\infty} f(z)dz\right)^4 + 6\rho^3\left(\int_{x-ct}^{\infty} f(z)dz\right)^2\left(\int_{x-ct}^{\infty} f^2(z)dz\right) + \rho^2\left[\begin{array}{c} 4\left(\int_{x-ct}^{\infty} f(z)dz\right)\left(\int_{x-ct}^{\infty} f^3(z)dz\right) \\ +3\left(\int_{x-ct}^{\infty} f^2(z)dz\right)^2 \end{array}\right] + \rho\left(\int_{x-ct}^{\infty} f^4(z)dz\right) \tag{59}$$

giving

$$
\begin{aligned}
\sigma^2 = {} & \rho \left(\int_{x-ct}^{\infty} f^4(z)dz \right) \\
& + 2\rho^2 \left[\begin{array}{c} 2 \left(\int_{x-ct}^{\infty} f(z)dz \right) \left(\int_{x-ct}^{\infty} f^3(z)dz \right) \\ + \left(\int_{x-ct}^{\infty} f^2(z)dz \right)^2 \end{array} \right] \\
& + 4\rho^3 \left(\int_{x-ct}^{\infty} f(z)dz \right)^2 \left(\int_{x-ct}^{\infty} f^2(z)dz \right)
\end{aligned} \qquad (60)
$$

8. CONCLUSION

We derived an extension of Campbell's theorem to non-stationay situations. As with Campbell's theorem, there are a number of sensitive mathematical issues regarding convergence and that will be addressed in a future paper. One of the applications of the model we have given is to the problem of reverberation which is often inherently non-stationary [11–14]. This is the case if the scatterers, which are spatially distributed, are excited by a pulse and therefore the returns at a particular spatial point is a random process where the function $f(t)$ is proportional to the return from each of the scatterers if indeed all the scatterers are the same. If the scatterers are different then the stochastic process has to be replaced by [11–14]

$$
V(x,t) = \sum_{n=1}^{N} f_n(x - ct - x_n) \qquad (61)
$$

where f_n reflects the scattering of each scatterer. This presents additional and very interesting issues which are currently being investigated.

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