PERFORMANCE BOUNDS UNDER MISSPECIFICATION MODEL FOR MIMO RADAR APPLICATION

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ABSTRACT

Recent tools established on misspecified lower bound on the mean square error allow to predict more accurately the mean square error behavior than the classical lower bounds in presence of model.errors. These bounds are helpful since model errors exist in practice due to system imperfections. In this paper, we are interested in the direction of arrival and direction of departure estimation in MIMO radar context with array elements position error. A closed-form expression is derived for the misspecified Cramér-Rao bound (or Huber limit) for any antennas geometry. A comparison of the misspecified Cramér-Rao bound with the classical Cramér-Rao bound and with the maximum likelihood estimator mean square error highlights the tightness improvement resulting from the use of the proposed bound.

***Index Terms*—** Misspecified Cramér-Rao bound, Huber limit, error model, MIMO radar.

1. INTRODUCTION

Supported by the MIMO communication theory, Multiple Input Multiple Output (MIMO) radar has been widely investigated for target detection or for direction and range estimation problems during the last decade [1]. A MIMO radar system consists of multiple transmit antennas, where each transmitter can send different waveforms. On the other hand, multiple receiver antennas collect the scattered signals. Contrary to the well-known phased-array radar, the possibility to use a set of orthogonal waveforms with such system is known to improve estimation and detection performance [2].

Estimation performance are usually studied in the standard conditions that the data distribution is correctly specified. Indeed, optimal algorithms, such as the maximum likelihood technique, are based on a statistical observation model generally derived from a priori knowledge about the noise probability density function and the physical model of waveforms propagation. However, in practice, such assumed observation model for MIMO radar system can differ from the true data distribution. Several error models can occur due to the inaccuracy of array element positions [3], of the received signal modeling [4] or, of the noise assumption, [5, 6]. Even though a calibration procedure can correct some mismatches, this process is time consuming (the observation model must be complexified what adds new parameters to be estimated) and cannot generally eliminate all system's default. Therefore, several solutions have been proposed in the literature to minimize the impact of the error model, which are commonly referred as robust estimation methods [7, 8].

An alternative method is to investigate the influence of the model error directly on the performance analysis. Indeed, when the true data distribution is different from the assumed observation model, some asymptotic properties on the consistency, the normality and the asymptotic covariance matrix of Maximum Likelihood Estimator (MLE) have been established in [9, 10]. Moreover, due to the asymptotic efficiency of the MLE when the model is correctly specified, one can expect that a slight error on the model will moderately degrade the MLE performance [6]. Recently, lower bounds on the mean square error have been established in the case where the model is misspecified, *i.e.,* when the true data distribu-

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tion differs from the assumed observation model [11]. These bounds, called "misspecified lower bounds", are asymptotically achievable (under mild conditions) by the Mean Square Error (MSE) of the MLE and are applicable for a class of estimators including the MLE.

The aim of this paper is to study the impact of a modeling error of the array element positions on the MLE's MSE behavior and to provide a closed-form expression of the Mismatched Cramér-Rao Bound (MCRB) under misspecification model in a MIMO radar context. Unlike the work proposed in [3] in which the steering vector is assumed to be perturbed by a random vector with known probability density function (pdf), we consider here a more realistic situation where the MLE is based on an idealistic observation model (with no error) but the data used to perform estimation are driven by another pdf. Such methodology has been recently successfully applied in the context of array processing [11] and scattering matrix estimation in radar processing [6].

2. PROBLEM SETUP

Let us consider a MIMO radar system with widely separated arrays. Let M denotes the number of transmit antennas and N the number of receive antennas. The transmit antennas send M orthogonal coded signals which are collected at the receive antennas after scattering on a single target. We assume that the target velocity can be neglected during the observation process (slow moving target). Then, the received data after a matched filtering process are modelled by [12]

$$\mathbf{y} = \sqrt{K}\beta\mathbf{a}\left(\varphi_D\right) \otimes \mathbf{b}\left(\varphi_A\right) + \mathbf{n}, \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^{MN}$, $K > 0$, $\beta \in \mathbb{C}$ and $\otimes$ denote, respectively, the output observations, the power of signal source, the complex attenuation incorporating propagation losses and radar cross section, and the Kronecker product. The Direction-Of-Arrival (DOA) w.r.t. the receive antenna and the Direction-Of-Departure (DOD) w.r.t. the transmit antenna are, respectively, denoted by φ_A and φ_D. The steering vectors are given by $\mathbf{a}\left(\varphi_D\right) = \left[e^{-j2\pi f_0 \tau_1(\varphi_D)} \ \ldots \ e^{-j2\pi f_0 \tau_M(\varphi_D)}\right]^T$ and $\mathbf{b}\left(\varphi_A\right) = \left[e^{-j2\pi f_0 \rho_1(\varphi_A)} \ \ldots \ e^{-j2\pi f_0 \rho_N(\varphi_A)}\right]^T$ for the transmit antennas and the receive antennas, respectively, in which the delays, $\left\{\tau_j\left(\varphi_D\right)\right\}_{j=1,..,M}$, depend on the DOD and the transmit array geometry , whereas, the delays, $\left\{\rho_j\left(\varphi_A\right)\right\}_{j=1,..,N}$, depend on the DOA and the receiver array geometry. f_0 is the carrier frequency of the emitted signals. Finally, the noise, denoted by $\mathbf{n} \in \mathbb{C}^{MN}$, is assumed to be complex Gaussian circular with zero mean and covariance matrix $\sigma^2\mathbf{I}_{MN}$. Assuming that the real unknown parameters $\boldsymbol{\theta} = \left[\mathrm{Re}\left(\beta\right) \ \mathrm{Im}\left(\beta\right) \ \varphi_D \ \varphi_A\right]^T$ are deterministic, the assumed data pdf is given by,

$$f_{\mathbf{y};\boldsymbol{\theta}}\left(\mathbf{y}\right) = \frac{1}{\pi^{MN}\sigma^2} e^{-\frac{1}{\sigma^2}\left\|\mathbf{y} - \sqrt{K}\beta\mathbf{a}(\varphi_D)\otimes\mathbf{b}(\varphi_A)\right\|^2}. \tag{2}$$

Based on these assumptions, the MLE is obtained by

$$\widehat{\boldsymbol{\theta}}_{\mathrm{MLE}} = \arg\max_{\boldsymbol{\theta}} f_{\mathbf{y};\boldsymbol{\theta}}\left(\mathbf{y}\right). \tag{3}$$

Nevertheless, in pratice, due to modeling errors, the assumed pdf $f_{\mathbf{y};\boldsymbol{\theta}}$ may differ from the true pdf, denoted here by $g_{\mathbf{y};\boldsymbol{\theta}_0}$. Such errors induce a degradation of estimator performance and classical lower bounds on the MSE (based on $f_{\mathbf{y};\boldsymbol{\theta}}$) are no longer asymptotically achievable by the MSE of the MLE computed by Eqn. (3). In the remaining of the paper, we will study the impact of array element position errors on the estimation performance. Therefore we assume that the true data distribution $\mathbf{y}$ is given by $g_{\mathbf{y};\boldsymbol{\theta}_0}$ where

$$g_{\mathbf{y};\boldsymbol{\theta}_0}\left(\mathbf{y}\right) = \frac{1}{\pi^{MN}\sigma^2} e^{-\frac{1}{\sigma^2}\left\|\mathbf{y} - \sqrt{K}\beta_0\widetilde{\mathbf{a}}(\varphi_{D_0})\otimes\widetilde{\mathbf{b}}(\varphi_{A_0})\right\|^2}, \tag{4}$$

in which $\boldsymbol{\theta}_0 = \left[\mathrm{Re}\left(\beta_0\right) \ \mathrm{Im}\left(\beta_0\right) \ \varphi_{D_0} \ \varphi_{A_0}\right]^T$ are the true value of the parameters. Since array geometry errors occur, the real steering vectors will be denoted by $\widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) = \left[e^{-j2\pi f_0 \widetilde{\tau}_1(\varphi_{D_0})} \ \ldots \ e^{-j2\pi f_0 \widetilde{\tau}_M(\varphi_{D_0})}\right]^T$ and $\widetilde{\mathbf{b}}\left(\varphi_{A_0}\right) = \left[e^{-j2\pi f_0 \widetilde{\rho}_1(\varphi_{A_0})} \ \ldots \ e^{-j2\pi f_0 \widetilde{\rho}_N(\varphi_{A_0})}\right]^T$ where $\left\{\widetilde{\tau}_j\left(\varphi_{D_0}\right)\right\}_{j=1,..,M}$ and $\left\{\widetilde{\rho}_j\left(\varphi_{A_0}\right)\right\}_{j=1,..,N}$ are the true delays depending on DOD, DOA and on the true array element positions. In practice, the true data pdf given by (4) is not accessible, and thus estimation algorithms are usually based on the assumed model given by (1). Therefore, classical lower bounds on the MSE cannot be correctly applied. Consequently, in the following, we fill this lack by using the so-called misspecified lower bound on the MSE introduced in [6] and [11].

3. BACKGROUND ON THE MISSPECIFIED CRAMÉR-RAO BOUND

Let us recall the estimation context in which misspecified lower bounds can be applied: let $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta} \subset \mathbb{R}^p$ an unknown parameter vector to estimate (in our MIMO application $P = 4$) from the observations $\mathbf{y} \in \mathbb{C}^{MN}$ and let $g_{\mathbf{y};\boldsymbol{\theta}_0}$ be the true data distribution of $\mathbf{y}$. However, the true pdf does not belong to the set of assumed pdf $f_{\mathbf{y};\boldsymbol{\theta}}$, *i.e.*, $g_{\mathbf{y};\boldsymbol{\theta}_0} \notin \left\{f_{\mathbf{y};\boldsymbol{\theta}} | \boldsymbol{\theta} \in \boldsymbol{\Theta}\right\}$. Note that the MLE is still built from the assumed pdf model $\left\{f_{\mathbf{y};\boldsymbol{\theta}} | \boldsymbol{\theta} \in \boldsymbol{\Theta}\right\}$ even if the data $\mathbf{y}$ are distributed under the true pdf $g_{\mathbf{y};\boldsymbol{\theta}_0}$. In this context, one should note that the nice asymptotic behavior of MLE has been studied in Huber [9] and White [10] works in which consistency and asymptotic normality has been proved under certain conditions (fulfilled here). Recently, misspecified lower bounds have been proposed in [6] and [11]. The following inequality holds

$$\begin{aligned} &\mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left[\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_A\right)\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_A\right)^T\right] \\ &\succeq \mathbf{C}^{-1}\left(\boldsymbol{\theta}_A\right)\mathbf{J}\left(\boldsymbol{\theta}_A\right)\mathbf{C}^{-1}\left(\boldsymbol{\theta}_A\right), \end{aligned} \tag{5}$$

where $\boldsymbol{\theta}_A$ is the point minimizing the Kullback-Leibler divergence between the pdf $g_{\mathbf{y};\boldsymbol{\theta}_0}$ and $\{f_{\mathbf{y};\boldsymbol{\theta}}|\boldsymbol{\theta}\in\boldsymbol{\Theta}\}$, *i.e.,* $\boldsymbol{\theta}_A = \arg\min_{\boldsymbol{\theta}} \int_{\mathbb{C}^{MN}} g_{\mathbf{y};\boldsymbol{\theta}_0}(\mathbf{y}) \log\left(\frac{g_{\mathbf{y};\boldsymbol{\theta}_0}(\mathbf{y})}{f_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}\right) d\mathbf{y}$, and for any measurable function $h(\mathbf{y})$, the conditional expectation is defined by $\mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}[h(\mathbf{y})] = \int_{\mathbb{C}^{MN}} h(\mathbf{y})\, g_{\mathbf{y};\boldsymbol{\theta}_0}(\mathbf{y})\, d\mathbf{y}$. Unfortunately, $\boldsymbol{\theta}_A$ does not provide a closed form expression and should be evaluated through independent Monte Carlo trials. The right-hand side of the Eqn. (5) is defined as the MCRB[1] [11] evaluated at $\boldsymbol{\theta}_A$ with

$$\mathbf{C}(\boldsymbol{\theta}_A) = \mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left[\left.\frac{\partial^2 \ln f_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_A}\right]^{-1}, \quad (6)$$

and

$$\mathbf{J}(\boldsymbol{\theta}_A) = \mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left[\left.\frac{\partial \ln f_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}{\partial\boldsymbol{\theta}}\frac{\partial \ln f_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}{\partial\boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_A}\right]. \quad (7)$$

Note that, the left hand side of Eqn. (5) is generally not the classical MSE since $\boldsymbol{\theta}_A \neq \boldsymbol{\theta}_0$. Fortunately, for any estimator such as $\mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left(\widehat{\boldsymbol{\theta}}\right) \simeq \boldsymbol{\theta}_A$ one has

$$\begin{aligned}&\mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left[\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)^T\right] \quad (8)\\ \succeq\ &(\boldsymbol{\theta}_0-\boldsymbol{\theta}_A)(\boldsymbol{\theta}_0-\boldsymbol{\theta}_A)^T + \mathbf{C}^{-1}(\boldsymbol{\theta}_A)\,\mathbf{J}(\boldsymbol{\theta}_A)\,\mathbf{C}^{-1}(\boldsymbol{\theta}_A).\end{aligned}$$

Eqn. (8) is an inequality on the MSE for any estimator, $\widehat{\boldsymbol{\theta}}$, sharing the same constraint as the MLE [11]. Additionally, Slepian-Bangs like expression has been provided in misspecified gaussian observation model with parameterized by mean [11]. Precisely, when the true data distribution is given by $\mathbf{y};\boldsymbol{\theta}_0 \sim \mathbb{C}\mathcal{N}(\mathbf{d}(\boldsymbol{\theta}_0), \mathbf{B})$ and the assumed data distribution is given by $\mathbf{y};\boldsymbol{\theta} \sim \mathbb{C}\mathcal{N}(\mathbf{m}(\boldsymbol{\theta}), \mathbf{R})$ where $\mathbf{d}(\boldsymbol{\theta}_0) \neq \mathbf{m}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ and the covariance matrix $\mathbf{B}$ and $\mathbf{R}$ are independent of $\boldsymbol{\theta}$**,** then the matrices $\mathbf{C}$ and $\mathbf{J}$**,** given in Eqn. (5), are given by the following formulas:

$$\mathbf{C}_{ij}(\boldsymbol{\theta}) = \begin{array}{c}\frac{\partial^2\mathbf{m}^H(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\mathbf{R}^{-1}(\mathbf{d}(\boldsymbol{\theta}_0)-\mathbf{m}(\boldsymbol{\theta}))\\ +(\mathbf{d}(\boldsymbol{\theta}_0)-\mathbf{m}(\boldsymbol{\theta}))^H\mathbf{R}^{-1}\frac{\partial^2\mathbf{m}(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\\ -\frac{\partial\mathbf{m}^H(\boldsymbol{\theta})}{\partial\theta_i}\mathbf{R}^{-1}\frac{\partial\mathbf{m}(\boldsymbol{\theta})}{\partial\theta_j} - \frac{\partial\mathbf{m}^H(\boldsymbol{\theta})}{\partial\theta_j}\mathbf{R}^{-1}\frac{\partial\mathbf{m}(\boldsymbol{\theta})}{\partial\theta i},\end{array} \quad (9)$$

and

$$\mathbf{J}_{ij}(\boldsymbol{\theta}) = \begin{array}{c}\frac{\partial\mathbf{m}^H(\boldsymbol{\theta})}{\partial\theta_i}\mathbf{R}^{-1}\mathbf{B}\left(\mathbf{R}^{-1}\right)^*\frac{\partial\mathbf{m}(\boldsymbol{\theta})}{\partial\theta_j}\\ +\frac{\partial\mathbf{m}^T(\boldsymbol{\theta})}{\partial\theta_i}\mathbf{R}^{-1}\mathbf{B}^*\left(\mathbf{R}^{-1}\right)^*\frac{\partial\mathbf{m}^*(\boldsymbol{\theta})}{\partial\theta_j},\end{array} \quad (10)$$

where * denotes the complex conjugate element. As Slepian Bangs formula for the classical CRB, an analytical expression is obtained for the MCRB in the gaussian observation model.

[1] Also known as the Huber limit [6].

Note that if $\mathbf{d}(\boldsymbol{\theta}_0) = \mathbf{m}(\boldsymbol{\theta})$ and $\mathbf{B} = \mathbf{R}$ (*i.e.,* no mismatch) then

$$\mathbf{C}^{-1}(\boldsymbol{\theta})\,\mathbf{J}(\boldsymbol{\theta})\,\mathbf{C}^{-1}(\boldsymbol{\theta}) = 2\,\mathrm{Re}\left(\frac{\partial\mathbf{m}^H(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\mathbf{R}^{-1}\frac{\partial\mathbf{m}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^T}\right), \quad (11)$$

where the right hand side of Eqn. (11) is the classical CRB for the Gaussian observation model with parameterized mean [13]. Therefore, the MCRB is an extension of the CRB which takes into account misspecified observation models.

4. APPLICATION TO PERFORMANCE ANALYSIS OF DOA AND DOD ESTIMATION

4.1. General expression of the MCRB for array position errors

In this part, we derive an analytical expression of the MCRB for the MIMO radar observation model with array elements position errors. The unknown parameters to estimate are $\boldsymbol{\theta} = [\mathrm{Re}(\beta)\ \ \mathrm{Im}(\beta)\ \ \varphi_D\ \varphi_A]$, the assumed observation model is given by (1) and the real data pdf is provided by (4). For notational convenience, we note $\mathbf{J} = \mathbf{J}(\boldsymbol{\theta})$ and $\mathbf{C} = \mathbf{C}(\boldsymbol{\theta})$. After some calculus and by using formulas (9) and (10), one obtains the MCRB where the diagonal elements of the matrix $\mathbf{J}$ are given by:

$$\mathbf{J}_{11} = \mathbf{J}_{22} = \frac{2KMN}{\sigma^2},\ \mathbf{J}_{33} = \frac{2KN\|\beta\|^2}{\sigma^2}\left\|\frac{d\mathbf{a}(\varphi_D)}{d\varphi_D}\right\|^2,$$

$$\text{and } \mathbf{J}_{44} = \frac{2KM\|\beta\|^2}{\sigma^2}\left\|\frac{d\mathbf{b}(\varphi_A)}{d\varphi_A}\right\|^2.$$

Since $\mathbf{J}$ is symmetric, the lower off diagonal elements of $\mathbf{J}$ are given by

$$\mathbf{J}_{12} = 0,\ \mathbf{J}_{13} = \frac{2KN}{\sigma^2}\mathrm{Re}\left(\beta^*\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D}\mathbf{a}(\varphi_D)\right),$$

$$\mathbf{J}_{14} = \frac{2KM}{\sigma^2}\mathrm{Re}\left(\beta^*\frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A}\mathbf{b}(\varphi_A)\right),$$

$$\mathbf{J}_{23} = -\frac{2KN}{\sigma^2}\mathrm{Im}\left(\beta^*\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D}\mathbf{a}(\varphi_D)\right),$$

$$\mathbf{J}_{24} = -\frac{2KM}{\sigma^2}\mathrm{Im}\left(\beta^*\frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A}\mathbf{b}(\varphi_A)\right),$$

and

$$\mathbf{J}_{34} = \frac{2K\|\beta\|^2}{\sigma^2}\mathrm{Re}\left(\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D}\mathbf{a}(\varphi_D)\,\mathbf{b}^H(\varphi_A)\frac{d\mathbf{b}(\varphi_A)}{d\varphi_A}\right).$$

Before calculating each element of the matrix $\mathbf{C}$**,** one can note that, if the covariance matrix of the observations is correctly specified then $\mathbf{B} = \mathbf{R} = \sigma^2\mathbf{I}_{MN}$, and

$$\begin{aligned}\mathbf{C}_{ij} \ &=\ \frac{\partial^2\mathbf{m}^H(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\mathbf{R}^{-1}(\mathbf{d}(\boldsymbol{\theta}_0)-\mathbf{m}(\boldsymbol{\theta}))\\ &\quad+(\mathbf{d}(\boldsymbol{\theta}_0)-\mathbf{m}(\boldsymbol{\theta}))^H\mathbf{R}^{-1}\frac{\partial^2\mathbf{m}(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} - \mathbf{J}_{ij}\end{aligned}$$

Therefore, after some algebra, the diagonal elements of the matrix $\mathbf{C}$ are given by

$$\mathbf{C}_{11} = \mathbf{C}_{22} = -\mathbf{J}_{11}$$

$$\mathbf{C}_{33} = \begin{array}{c} \frac{2K}{\sigma^2} \operatorname{Re}\left(\beta_0 \frac{d^2\mathbf{a}^H(\varphi_D)}{d\varphi_D^2} \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \mathbf{b}^H\left(\varphi_A\right) \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \\ -\frac{2KN}{\sigma^2} \operatorname{Re}\left(\beta^* \frac{d^2\mathbf{a}^H(\varphi_D)}{d\varphi_D^2} \mathbf{a}\left(\varphi_D\right)\right) - \mathbf{J}_{33} \end{array}$$

$$\mathbf{C}_{44} = \begin{array}{c} \frac{2K}{\sigma^2} \operatorname{Re}\left(\beta_0 \mathbf{a}^H\left(\varphi_D\right) \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \frac{d^2\mathbf{b}^H(\varphi_A)}{d\varphi_A^2} \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \\ -\frac{2KM}{\sigma^2} \operatorname{Re}\left(\beta^* \frac{d^2\mathbf{b}^H(\varphi_A)}{d\varphi_A^2} \mathbf{b}\left(\varphi_A\right)\right) - \mathbf{J}_{44} \end{array}$$

And, by the symmetry of $\mathbf{C}$, the lower off diagonal elements are given by

$$\mathbf{C}_{12} = 0,$$

$$\mathbf{C}_{13} = \begin{array}{c} \frac{2K}{\sigma^2} \operatorname{Re}\left(\beta_0 \frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \mathbf{b}^H\left(\varphi_A\right) \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \\ -\frac{4KN}{\sigma^2} \operatorname{Re}(\beta) \operatorname{Re}\left(\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \mathbf{a}\left(\varphi_D\right)\right) \end{array},$$

$$\mathbf{C}_{14} = \begin{array}{c} \frac{2K}{\sigma^2} \operatorname{Re}\left(\beta_0 \mathbf{a}^H\left(\varphi_D\right) \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \\ -\frac{4KM}{\sigma^2} \operatorname{Re}(\beta) \operatorname{Re}\left(\frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \mathbf{b}\left(\varphi_A\right)\right) \end{array},$$

$$\mathbf{C}_{23} = \begin{array}{c} \frac{4KN}{\sigma^2} \operatorname{Im}(\beta) \operatorname{Im}\left(\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \mathbf{a}\left(\varphi_D\right)\right) \\ -\frac{2K}{\sigma^2} \operatorname{Im}\left(\beta_0 \frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \mathbf{b}^H\left(\varphi_A\right) \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \end{array},$$

$$\mathbf{C}_{24} = \begin{array}{c} \frac{4KM}{\sigma^2} \operatorname{Im}(\beta) \operatorname{Im}\left(\frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \mathbf{b}\left(\varphi_A\right)\right) \\ -\frac{2K}{\sigma^2} \operatorname{Im}\left(\beta_0 \mathbf{a}^H\left(\varphi_D\right) \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \end{array},$$

and

$$\mathbf{C}_{34} = \begin{array}{c} \frac{2K}{\sigma^2} \operatorname{Re}\left(\beta_0 \frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \widetilde{\mathbf{a}}\left(\varphi_{D_0}\right) \frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \widetilde{\mathbf{b}}\left(\varphi_{A_0}\right)\right) \\ -\frac{4K}{\sigma^2} \beta^* \operatorname{Re}\left(\frac{d\mathbf{a}^H(\varphi_D)}{d\varphi_D} \mathbf{a}\left(\varphi_D\right)\right) \operatorname{Re}\left(\frac{d\mathbf{b}^H(\varphi_A)}{d\varphi_A} \mathbf{b}\left(\varphi_A\right)\right) \end{array}.$$

4.2. Simulation results

We consider a uniform linear array for the transmit antenna and the receive antenna. The inter-element space is given by $\frac{\lambda}{2}$ where λ the is wavelength of the transmit signals. Then the assumed observation model is given by Eqn.(1) where the assumed steering vector for the transmitter and the receiver become $\mathbf{a}\left(\varphi_D\right) = \left[1\; e^{-j\pi\sin(\varphi_D)}\; \dots\; e^{-j(M-1)\pi\sin(\varphi_D)}\right]$ and,
$\mathbf{b}\left(\varphi_A\right) = \left[1\; e^{-j\pi\sin(f_A)}\; \dots\; e^{-j(N-1)\pi\sin(f_A)}\right]$ respectively. However, for real scenario, the antennas are not perfectly uniform leading to the following true steering vectors $\widetilde{\mathbf{a}}\left(\varphi_D\right) = \left[1\; e^{-j\pi(1+e_1)\sin(\varphi_D)}\; \dots\; e^{-j(M-1+e_{M-1})\pi\sin(\varphi_D)}\right]$ and
$\widetilde{\mathbf{b}}\left(\varphi_A\right) = \left[1\; e^{-j\pi(1+r_1)\sin(\varphi_A)}\; \dots\; e^{-j(N-1+r_{N-1})\pi\sin(\varphi_A)}\right]$ where $\mathbf{e} = \left[e_1\; e_2\; \cdots\; e_{M-1}\right]^T$ and $\mathbf{r} = \left[r_1\; r_2\; \cdots\; r_{N-1}\right]^T$ denote the error position for the transmitter and the receiver antennas, respectively. The data $\mathbf{y}$ is generated with the true pdf $g_{\mathbf{y};\boldsymbol{\theta}_0}$ provided in Eqn. (4) where the true value of parameters are given by $\beta_0 = 0.1\,(1+j)$, $\varphi_{D_0} = 0.3$ rad and $\varphi_{A_0} = 0.2$ rad. The parameters are estimated with the MLE deriving from the assumed data pdf $f_{\mathbf{y};\boldsymbol{\theta}}$, defined by Eqn.(2), *i.e.,*

$$\widehat{\boldsymbol{\theta}}_{\mathrm{MLE}} = \underset{\beta\in\mathbb{C},\, \varphi_D\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right[,\varphi_A\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right[}{\arg\max} f_{\mathbf{y};\boldsymbol{\theta}}\left(\mathbf{y}\right). \qquad (12)$$

The setting for the simulations are $M = N = 8$, $K = 1$ and the errors $\mathbf{e}$ and $\mathbf{r}$ are randomly generated by a Gaussian law with zero mean and standard deviation 0.2. In the Fig.(1) (DOD) and Fig.(2) (DOA), we compare the empirical MSE of the MLE, assessed with 1000 Monte-Carlo trials, under the aforemenstionned misspecified assumption with the proposed MCRB given by (8), in which $\boldsymbol{\theta}_A$ is approximate by $\mathbb{E}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left(\widehat{\boldsymbol{\theta}}\right)$, with the true Unbiased CRB denoted UCRB which is defined by $\mathbf{UCRB} = \mathbb{E}^{-1}_{g_{\mathbf{y};\boldsymbol{\theta}_0}}\left[\frac{\partial \ln g_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}{\partial\boldsymbol{\theta}} \frac{\partial \ln g_{\mathbf{y};\boldsymbol{\theta}}(\mathbf{y})}{\partial\boldsymbol{\theta}^T}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right]$ and with the biased CRB denoted CRB which takes account the bias of the MLE. We can remark that the MCRB is the tightest lower bound achievable by the MLE of MSE under misspecification model in both Fig.(1) and Fig.(2). The UCRB is the lowest bound in both figures which means that, obviously, if the model is correctly specified, estimators can provide better performance in term of the MSE than in the misspecified case. Note that, the CRB which takes into account the bias of the MLE and the MCRB exhibit a threshold effect [14–16]. The origin of this phenomenom is only due to the bias of the estimators as we can see the difference between the UCRB and the CRB in both figures.

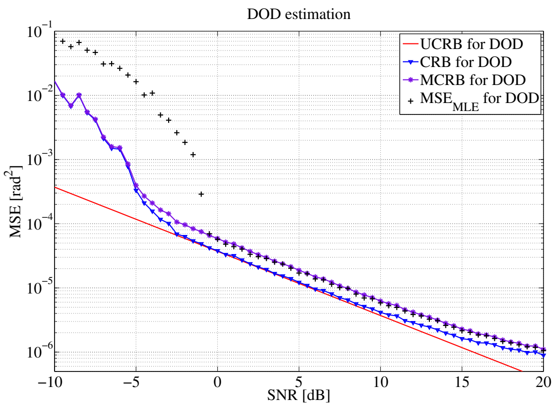

Fig. 1. Comparison of MLE's MSE and lower bounds for the DOD estimation versus SNR

Finally, note that the proposed mismatch Barankin bound in [11] is no longer helpful in this context to predict the threshold effect. Due to lack of space, we do not report the calculus of this bound here, but the explanation comes from the fact that the misspecified Barankin bound is derived in [11] by using asymptotics considerations. Consequently, this bound will be useless in the context of low SNR or low

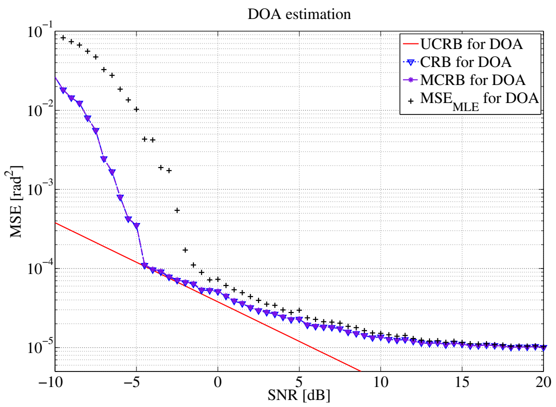

Fig. 2. Comparison of MLE's MSE and lower bounds for the DOA estimation versus SNR

number of observations. However, the misspecified Barankin bound will still be an interesting tool to analyse asymptotic performance of the mismatch MLE, when the regularity conditions of the MCRB will not be satisfied (*i.e.* [17]).

5. CONCLUSION

In this paper, the MCRB is applied to the DOA and DOD estimation in the MIMO radar context with error model on the transmitters and receivers array element positions. Simulations results show that, under misspecfication model, the MCRB is asymptotically achievable by the MSE of the MLE and provides better prediction of the MSE behavior than the classical CRBs.

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