DOA ESTIMATION WITH CO-PRIME ARRAYS IN THE PRESENCE OF MUTUAL COUPLING

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ABSTRACT

In this paper, we present a method for performing direction-of-arrival (DOA) estimation using co-prime arrays in the presence of mutual coupling. The effects of mutual coupling are first examined for extended co-prime arrays configurations using the Receiving-Mutual-Impedance Method (RMIM). DOA estimation is then achieved by performing a joint estimation of the angles of arrival and the mutual coupling matrix, using the mixed-parameter covariance matrix adaptation evolution strategy. Simulation results demonstrating the effectiveness of the proposed method are provided.

***Index Terms*—** Co-prime arrays, DOA estimation, mutual coupling.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation has received considerable research interest due to its applications in radar, sonar, and wireless communications [1-2]. In practical antenna arrays, high-resolution DOA estimation techniques, such as MUSIC [3] and ESPRIT [4], suffer from a model mismatch due to the mutual coupling between the sensors. Sparsity-based DOA estimation techniques, such as $\ell_1 - SVD$ [5], are also affected by mutual coupling due to the mismatch between the assumed dictionary matrix and the actual measurements.

Mutual coupling between the array elements can be captured in a mutual coupling matrix (MCM), which depends on the self and mutual impedances between the array elements. Several methods have been proposed in the literature to model the mutual coupling between the array elements [6-9]. One of the earliest methods to characterize mutual coupling is the open-circuit method [6]. This method is based on circuit theory and models the array as a bilateral terminal network. The antenna terminal voltages are related to the open-circuit voltages through a mutual impedance matrix. The MCM is then computed as the inverse of the mutual impedance matrix. In [7], a modification of the open-circuit method was proposed, wherein two types of mutual impedances, namely, the transmission mutual impedance and the re-radiation mutual impedance, were defined. The receiving-mutual-impedance method (RMIM) was presented in [8], and was suggested for use in receiving antenna arrays. This method considers antenna elements in pairs to compute the receiving mutual impedances. An improvement on this method was presented in [9], where all the antennas were taken into account simultaneously to compute the mutual impedances.

For a known mutual coupling matrix, accurate DOA estimation can be achieved by compensating for the MCM in the estimation algorithm [10]. On the other hand, in case of arrays with unknown mutual coupling, the MCM is modeled and the performance of the DOA estimation is degraded if the computed MCM is not exact. Further, the MCM estimate must be re-calibrated periodically to account for changes in local conditions. Several methods for jointly estimating the MCM and the DOAs have been reported in the literature. An iterative method to estimate the MCM, the DOAs, and the antenna gains was presented in [11]. In [12], a maximum likelihood estimator for DOA estimation under unknown mutual coupling and multipath was proposed. A sparse reconstruction method was discussed in [13] to perform DOA estimation in the presence of mutual coupling. All of these aforementioned methods were developed for uniform linear arrays (ULAs) and take advantage of the special structure of their MCMs. As a result, these methods are not applicable to nonuniform arrays. An iterative method for DOA estimation under mutual coupling using nonuniform arrays was proposed in [14]. However, this method treats the nonuniform array as a subset of a ULA and cannot take full advantage of the increased degrees-of-freedom (DOFs) of the nonuniform array for DOA estimation.

In this paper, we present a method for DOA estimation with unknown mutual coupling using co-prime arrays, which are a new class of nonuniform arrays [15]. More specifically, we consider the extended co-prime array configuration [16], which consists of two ULAs; the first ULA consists of $2M$ elements with Nd_0 inter-element spacing, while the second consists of N elements with Md_0 inter-element spacing (See Fig. 1). M and N are co-prime numbers, and d_0 is the typically chosen as one-half wavelength. The two sub-arrays share the first sensor at the zeroth position. The RMIM is first used to examine the structure of the MCM for the extended co-prime configurations. DOA estimation is then performed by jointly estimating the MCM, the sources powers, and their DOAs using the covariance matrix adaptation evolution strategy (CMA-ES) [17]. In particular, a mixed-parameter version of CMA-ES [18-19] is utilized where the DOAs are as-

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sumed to fall on a discrete grid of angles, and the remaining parameters are assumed to be continuous. Simulation results are provided that not only show the effects of mutual coupling on the DOA estimation performance of co-prime arrays, but also demonstrate the effectiveness of the proposed method.

The remainder of the paper is organized as follows. In Section 2, we review high-resolution DOA estimation using extended co-prime arrays. Section 3 discusses the RMIM method and the structure of the MCM for the extended co-prime configuration. The proposed MCM and DOA estimation method is presented in Section 4. Supporting simulations results are provided in Section 5 and conclusions are drawn in Section 6.

2. DOA ESTIMATION USING CO-PRIME ARRAYS

Consider the extended co-prime configuration, shown in Fig. 1. The corresponding difference coarray (the set of all spatial lags generated by the physical array), shown in Fig. 2, can be expressed as

$$S_0 = \pm\{Mnd_0 - Nmd_0\}, \tag{2}$$

where $0 \leq n \leq N-1$ and $0 \leq m \leq 2M-1$, and has contiguous elements between $-(MN+M-1)d_0$ and $(MN+M-1)d_0$.

Assuming that D narrowband sources with powers $[\sigma_1^2\ \sigma_2^2\ \dots\ \sigma_D^2]$ impinge on the array from directions $[\theta_1\ \theta_2 \dots \theta_D]$, the received data vector at the co-prime array can be expressed as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{3}$$

where $\mathbf{s}(t) = [s_1(t)\ s_2(t)\ \dots\ s_D(t)]^T$ is the source signal vector at snapshot t, and $\mathbf{n}(t)$ is the noise vector. The matrix $\mathbf{A} = [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2) \dots \mathbf{a}(\theta_D)]$ is the array manifold, whose columns are steering vectors of the co-prime array corresponding to the source directions. The steering vector corresponding to θ_d is given by

$$\mathbf{a}(\theta_d) = \left[\, e^{jk_0 x_1 \sin(\theta_d)}, \dots,\ e^{jk_0 x_{2M+N-1} \sin(\theta_d)} \right]^T, \tag{4}$$

where $[x_1,\ x_2, \dots,\ x_{2M+N-1}]$ are the positions of the co-prime array elements, and k_0 is the wavenumber at the operating frequency. With the assumption that the sources are uncorrelated and the noise is spatially and temporally white, the autocorrelation matrix is obtained as

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}(t)^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma_n^2\mathbf{I}, \tag{5}$$

where $\mathbf{R}_{ss} = diag([\sigma_1^2\ \sigma_2^2\ \dots\ \sigma_D^2])$ is the source covariance matrix, σ_n^2 is the noise variance, and $\mathbf{I}$ is an identity matrix.

After forming the autocorrelation matrix, we can proceed with DOA estimation using two methods. The first method uses covariance matrix augmentation [20-22] and the second applies spatial smoothing on the vectorized form of the autocorrelation matrix [15-16]. The latter is reviewed below. Following [16], the autocorrelation matrix is vectorized as

$$\mathbf{z} = vec(\mathbf{R}_{xx}) = \widetilde{\mathbf{A}}\mathbf{p} + \sigma_n^2\tilde{\mathbf{i}}, \tag{7}$$

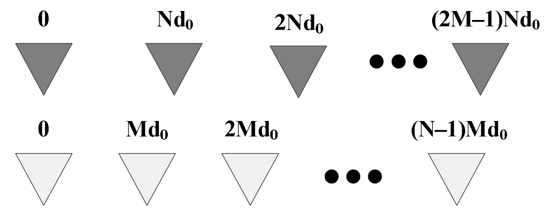

Fig. 1. Extended co-prime array configuration

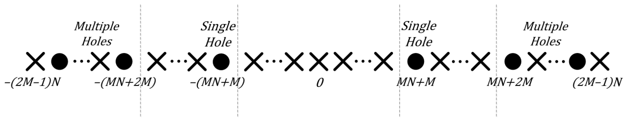

Fig. 2. Extended co-prime array on difference coarray

where $\tilde{\mathbf{i}}$ is the vector form of $\mathbf{I}$, $\widetilde{\mathbf{A}} = [\mathbf{a}(\theta_1) \otimes \mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_D) \otimes \mathbf{a}(\theta_D)]$, the operator '$\otimes$' denotes the Kronecker product, and $\mathbf{p} = [\sigma_1^2\ \sigma_2^2\ \dots\ \sigma_D^2]^T$. The vector $\mathbf{z}$ acts as the received signal vector of a longer array whose elements positions are given by the difference coarray. The sources are replaced by their powers and the noise is deterministic. As the sources now appear as coherent, subspace-based high-resolution methods, such as MUSIC, can no longer be applied directly to perform DOA estimation.

Spatial smoothing can be used to decorrelate the sources and restore the rank of the covariance matrix of $\mathbf{z}$ [23]. The elements of $\mathbf{z}$, which correspond to the contiguous co-array elements between $-(MN+M-1)$ and $(MN+M-1)$, are used to form a new vector $\mathbf{z}_f$ expressed as

$$\mathbf{z}_f = \widetilde{\mathbf{A}}_f\mathbf{p} + \sigma_n^2\tilde{\mathbf{i}}_f, \tag{8}$$

where $\widetilde{\mathbf{A}}_f$ is the array manifold matrix corresponding to the filled part of the difference co-array, and $\tilde{\mathbf{i}}_f$ is a $(2MN+2M-1) \times 1$ vector whose $(MN+M)$-th element is equal to one and all its remaining elements are zeros. The filled virtual array is then divided into $(MN+M)$ overlapping subarrays, each having $(MN+M)$ elements. The element positions of the ith subarray are given by

$$\{(n+1-i)d_0, n = 0, 1, \dots, MN+M-1\}. \tag{9}$$

The received signal vector at the ith subarray is denoted by $\mathbf{z}_{fi}$ and its elements consist of the $(MN+M-i+1)$th to $(2MN+2M-i)$th elements of $\mathbf{z}_f$. The autocorrelation matrix of each received signal vector is then formed as

$$\mathbf{R}_{fi} = \mathbf{z}_{fi}\mathbf{z}_{fi}^H. \tag{10}$$

The overall spatially smoothed covariance matrix is finally computed as the average of the autocorrelation matrices of the various subarrays

$$\mathbf{R}_{zz} = \frac{1}{MN+M}\sum_{i=1}^{MN+M}\mathbf{R}_{fi}. \tag{11}$$

It can be shown that the rank of $\mathbf{R}_{zz}$ is equal to $(MN + M)$. This implies that up to $(MN + M - 1)$ sources can be estimated by applying high-resolution subspace techniques, such as MUSIC, on $\mathbf{R}_{zz}$.

3. MUTUAL COUPLING MODELING

When mutual coupling is taken into account, the array output vector takes the form

$$\mathbf{x}(t) = \mathbf{CAs}(t) + \mathbf{n}(t). \tag{12}$$

The only modification from (3) is the addition of the $(2M + N - 1) \times (2M + N - 1)$ mutual coupling matrix, $\mathbf{C}$, which captures the mutual coupling between the different elements in the antenna array. Without compensating for the MCM, the traditional DOA estimation methods fail due to the mismatch between the model (3) and the measurements (12). Below, we characterize the matrix $\mathbf{C}$ using RMIM.

The RMIM can be used to characterize the MCM in receiving antenna arrays [8-9]. This method works under two conditions. First, the antenna elements are assumed to be in the receiving mode, which is valid in DOA estimation applications. Second, the antenna elements are terminated with a known load impedance Z_L.

When an external source excites the array, the voltage across the terminal load of a particular element can be obtained using the superposition principle. This voltage is due to two excitation sources, namely, the incoming source and the induced currents on the other antenna elements. The terminal voltage V_n at the nth element can be expressed as [8-9],

$$V_n = Z_L I_n = U_n + W_n, \tag{13}$$

where I_n is the current induced at the nth antenna element, U_n is the voltage due to the external source, and W_n is the voltage due to mutual coupling from the other elements, and is given by

$$W_n = Z_{n,1} I_1 + \cdots + Z_{n,n-1} I_{n-1} + Z_{n,n+1} I_{n+1} + \cdots + Z_{n,N_t} I_{N_t}. \tag{14}$$

Here, $Z_{m,n}$ is the mutual impedance between the mth and nth elements, and N_t is the total number of elements in the array. The coupled and uncoupled voltages at the antenna terminals are related by the mutual impedance matrix $\mathbf{Z}$ as

$$\mathbf{U} = \mathbf{ZV}, \tag{15}$$

where $\mathbf{U} = [U_1, U_2, \cdots, U_{N_t}]^T$ is the uncoupled voltages vector, $\mathbf{V} = [V_1, V_2, \cdots, V_{N_t}]^T$ contains the coupled voltages, and

$$\mathbf{Z} = \begin{bmatrix} 1 & -\frac{Z_{1,2}}{Z_L} & \cdots & -\frac{Z_{1,N_t-1}}{Z_L} & -\frac{Z_{1,N_t}}{Z_L} \\ -\frac{Z_{2,1}}{Z_L} & 1 & \cdots & -\frac{Z_{2,N_t-1}}{Z_L} & -\frac{Z_{2,N_t}}{Z_L} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{Z_{N_t-1,1}}{Z_L} & -\frac{Z_{N_t-1,2}}{Z_L} & \cdots & 1 & -\frac{Z_{N_t-1,N_t}}{Z_L} \\ -\frac{Z_{N_t,1}}{Z_L} & -\frac{Z_{N_t,2}}{Z_L} & \cdots & -\frac{Z_{N_t,N_t-1}}{Z_L} & 1 \end{bmatrix}. \tag{16}$$

The MCM is then computed as the inverse of the mutual impedance matrix,

$$\mathbf{C} = \mathbf{Z}^{-1}. \tag{17}$$

Mutual coupling between two antenna elements depends on the distance that separates them [11]. A smaller distance results in a larger mutual coupling effect. A distance beyond two or three wavelengths at the operating frequency renders the mutual coupling effect negligible. This results in the banded symmetric Toeplitz structure for the MCM of a ULA [11].

For extended co-prime arrays, mutual coupling depends on the values of M and N, which define the inter-element spacings of the two constituent ULAs. Two types of mutual coupling can be observed in co-prime arrays. The first one is the intra-mutual coupling which refers to the mutual coupling between the elements of each individual ULA, whereas the second is the inter-mutual coupling which refers to the mutual coupling between elements of the two ULAs. The former can change from one subarray to the other, depending on the value of M and N. A large value of M or N implies that the inter-element spacing is large in one of the subarrays or in both. As a result, the mutual coupling between the elements that belong to those subarrays becomes negligible. The inter-mutual coupling depends on the values of M and N and the locations of the considered elements.

4. DOA ESTIMATION WITH CO-PRIME ARRAYS UNDER UNKNOWN MUTUAL COUPLING

For DOA estimation in the presence of mutual coupling, we consider the received signal vector in (12), with $\mathbf{C}$ defined in (17). In order to take advantage of the increased DOFs offered by co-prime arrays, the autocorrelation matrix must first be computed. The autocorrelation matrix can be expressed as

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}(t)^H] = \mathbf{CAR}_{ss}\mathbf{A}^H\mathbf{C}^H + \sigma_n^2\mathbf{I}, \tag{18}$$

In practice, the autocorrelation matrix is replaced by the following sample average,

$$\widehat{\boldsymbol{R}}_{xx} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}(t)^H, \tag{19}$$

where T is the number of snapshots. Assuming that the number of sources D is known, a joint DOA and mutual coupling estimation requires finding the powers and angles of arrival of the sources, and the mutual coupling coefficients. This problem can be cast as minimization of the objection function

$$FV = \left\|\widehat{\boldsymbol{R}}_{xx} - \mathbf{CAR}_{ss}\mathbf{A}^H\mathbf{C}^H - \sigma_n^2\mathbf{I}\right\|_F, \tag{20}$$

with respect to DOA estimates, sources powers, noise variance, and mutual coupling coefficients. In (20), $\|\cdot\|_F$ denotes the Frobenius norm of the matrix. The total number of optimization parameters is given by $N_{opt} = 2D + 1 + (2M + N - 1)(2M + N)$. The parameters consist of D source powers $[\sigma_1^2\ \sigma_2^2 \ldots \sigma_D^2]$, D source directions $[\theta_1\ \theta_2 \ldots \theta_D]$, the noise variance, and $(2M + N - 1)(2M + N)$ elements in $\mathbf{C}$. Due to symmetry in $\mathbf{C}$, only

$(2M+N-1)(2M+N)/2$ complex values have to be estimated. These complex values require $(2M+N-1)(2M+N)$ parameters.

In this paper, mixed-parameter CMA-ES is used to perform the optimization [17-19]. CMA-ES is chosen since it is one of the most efficient nature-based global optimization techniques and has been shown to provide a better performance than other evolutionary algorithms in many complex engineering problems [19]. This algorithm samples potential solutions from a multivariate Gaussian distribution and adapts its parameters based on the performance of these solutions. The sources directions are picked from a predetermined grid and the remaining parameters are assumed to be continuous.

5. NUMERICAL RESULTS

The effect of mutual coupling on DOA estimation using co-prime arrays is first examined. We consider an extended co-prime array configuration with six physical sensors. The values of M and N are set to 2 and 3, respectively. The first subarray consists of four elements positioned at $[0, 3d_0, 6d_0, 9d_0]$, and the second one has three elements with positions $[0, 2d_0, 4d_0]$, with d_0 equal to one-half wavelength at the operating frequency. Note that the element at 0 is shared between the two ULAs. As a result, the co-prime array consists of six elements positioned at $[0, 2d_0, 3d_0, 4d_0, 6d_0, 9d_0]$. The difference coarray of this configuration has contiguous elements between $-7d_0$ and $7d_0$. As a result, MUSIC with spatial smoothing can resolve up to 7 sources. We consider 7 sources with directions $[-64°, -37°, -17°, 0°, 17°, 37°, 64°]$. A total of 1000 snapshots are used, and the SNR is set to 10 dB for all sources. The estimated spatial spectrum is provided in Fig. 3. The dashed vertical lines indicate the true directions. We can clearly observe that all the sources have been correctly resolved.

Next, mutual coupling between the antenna elements is incorporated. The RMIM is used to calculate the mutual coupling matrix for the considered co-prime array. Eq. (12) is then used to model the received data. MUSIC with spatial smoothing is then applied to the resulting data without compensating for the mutual coupling effects. The obtained spectrum is shown in Fig. 4. Clearly, the DOA estimation performance has degraded due to the presence of mutual coupling. More specifically, not all sources are resolved. In fact, two sources are completely missed, while the remaining DOA estimates are biased.

Finally, the proposed method is used to compensate for the mutual coupling effects. The MCM is assumed to be unknown, and a joint estimation of the MCM and the DOAs is performed using mixed-parameter CMA-ES [18-19]. The total number of optimization parameters is equal to $N_{opt} = 57$. The bounds for the MCM terms are set according to the distance between the elements. For the main diagonal terms corresponding to self-coupling, the lower bound of the magnitude is set to 0.8. For the remaining off-diagonal terms, the upper bound of the magnitude is varied between 0.1 and 0.4 in accordance to the distance between the considered elements. The number of sources is assumed to be known, and the DOAs are assumed to fall on a grid with a 1° step size. The source powers and noise variance are assumed to be positive values. For CMA-ES, a population size of 1000 is used for 1000 generations. Fig. 5 shows the estimated spectrum using the proposed method. It can be seen that all the sources are correctly estimated.

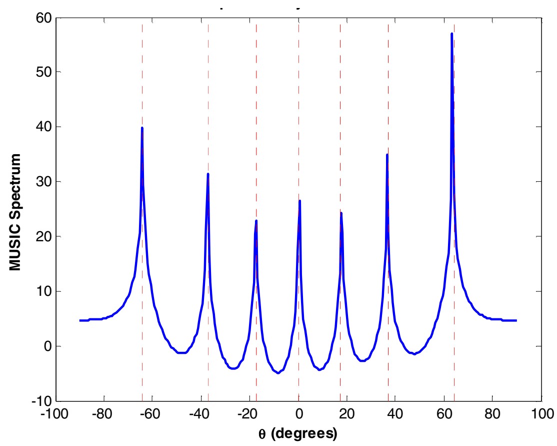

Fig. 3. MUSIC spectrum in dB without mutual coupling effect

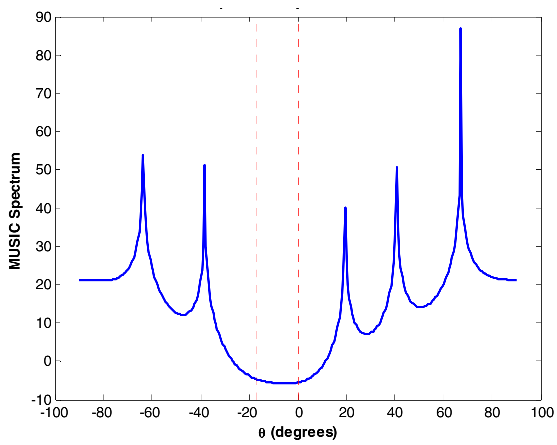

Fig. 4. MUSIC spectrum in dB with mutual coupling effect

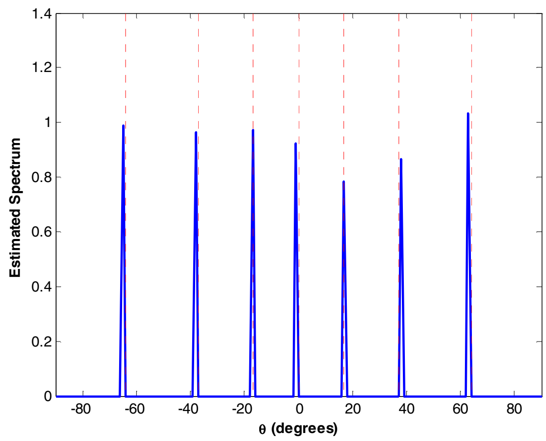

Fig. 5. Estimated spectrum using the proposed method

6. CONCLUSION

A new method has been proposed for performing direction finding with co-prime arrays under unknown mutual coupling. The mutual coupling matrix was characterized using the receiving-mutual-impedance method. DOA estimation was posed as a minimization problem where the sources DOAs, the sources powers, and the mutual coupling matrix were jointly estimated using the mixed-parameter CMA-ES. Simulation results validated the performance of the proposed method.

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