WAVELET BASED UNSUPERVISED VARIATIONAL BAYESIAN IMAGE RECONSTRUCTION APPROACH

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ABSTRACT
In this paper, we present a variational Bayesian approach in the wavelet domain for linear image reconstruction problems. This approach is based on a Gaussian Scale Mixture prior and an improved variational Bayesian approximation method. Its main advantages are that it is unsupervised and can be used to solve various linear inverse problems. We show the good performance of our approach through comparisons with state of the art approaches on a deconvolution problem.

Index Terms— unsupervised approach, wavelet transform, variational Bayesian, GSM, Generalized Gaussian

1. INTRODUCTION
Most image reconstruction tasks can be considered as ill-posed linear inverse problems. The resolution of such problems generally relies on the introduction of additional information thanks to regularization terms or prior distributions either in the spatial or in a transform domain (e.g., Fourier, wavelet, ...). It is popular to treat image reconstruction problems in the wavelet domain as it provides sparse representations for a large class of images.

As a result, sparsity prior information on wavelet coefficients can be introduced to regularize ill-posed inverse problems. To do this, \( L^1 \) norm regularization has been used extensively, e.g. [1]. However, a main difficulty encountered is a proper choice of hyperparameters which control the trade-off between data fidelity and regularization terms. Thanks to the work in [2], for denoising problems, hyperparameters can be determined by minimizing the Stein’s unbiased risk estimate (SURE) [1]. SURE has also been generalized (GSURE) [3–6] for general linear inverse problems. Another way to solve this problem is to work in the Bayesian framework where we jointly estimate hyperparameters and unknown coefficients by assigning prior distributions to both of them [7].

In this paper, we choose to work in the Bayesian framework. To obtain a more accurate estimation, rather than using orthogonal wavelet transforms, we consider a dictionary decomposition over an union of wavelet bases [8]. Concerning the prior for unknown coefficients, we consider a Gaussian Scale Mixture (GSM) model [9, 10] which encompasses many heavy tailed priors, e.g., Generalized Gaussian (GG), which has shown considerable success in wavelet domain image reconstructions [11, 12]. Moreover, in order to adapt sparsity degrees and relative importance of prior information to different subbands, we take different hyperparameters (shape and scale parameters) for GSM priors in different subbands.

Nevertheless, due to the introduction of hyperpriors, the involved posterior distribution is too complicated for the computation of classical estimators such as the Maximum A Posteriori (MAP) and the Posterior Mean (PM). To tackle this problem efficiently, rather than using MCMC approaches [12], we resort to variational Bayesian methods [13–15] which provide an approximate posterior distribution of simpler form than the original one. The PM estimator can be more easily derived from this approximate distribution. Furthermore, to get a more efficient method, we adopt here an improved variational Bayesian algorithm recently proposed in [15] which can be well adapted to large dimensional problems.

The rest of this paper is organized as follows: we present the involved Bayesian model in Section 2; Section 3 is devoted to the introduction of our variational Bayesian approach whereas algorithm evaluation through simulation results on a deconvolution problem is given in Section 4; Finally, we draw our conclusions in Section 5.

2. BAYESIAN MODELING
The unknown image \( x \in \mathbb{R}^P \) can be represented through a dictionary expansion as \( x = Du \) where \( D \in \mathbb{R}^{P \times N} \) and \( u \in \mathbb{R}^N \) denotes the associated coefficients. In the case of an overcomplete dictionary, e.g., the union of several orthonormal bases, we have \( N > P \). We consider in the following a linear forward model in the transformed domain:

\[
y = ADu + n,
\]

where \( y \in \mathbb{R}^M \) denotes the data, the operator \( A \in \mathbb{R}^{M \times P} \) is assumed to be known and \( n \) is a Gaussian white noise,
\( z \sim N(0, \gamma_{n}^{-1}I) \), with \( \gamma_{n} \) as the inverse of the noise variance. Therefore \( p(y|u, \gamma_{n}) = N(ADu, \gamma_{n}^{-1}I) \).

Concerning the prior for the coefficients \( u \), we consider a GSM. In fact, all the coefficients do not exhibit the same statistical characteristics. To take this fact into account, we divide the coefficients into \( L \) subbands and we assign GSM priors with different parameters to coefficients in different subbands. By using \((I_{l})_{l=1,\ldots,L}\) to denote sets of indices of coefficients in the \(l\)th subband, the prior distribution of \( u \) can be written as

\[
p(u|\gamma_{p}, \tau) = \prod_{l=1}^{L} \prod_{i \in I_{l}} p(u_{i}|\gamma_{p}^{l}, \tau^{l})
\]

\[
= \prod_{l=1}^{L} \prod_{i \in I_{l}} \int_{\mathbb{R}} N(u_{i}|0, (z_{l}^{(l)}\gamma_{p}^{l})^{-1}) p(z_{i}|\tau^{l})dz_{i}
\]

where \( z = (z_{1}, \ldots, z_{N}) \) is the vector of hidden variables whose distributions are given by \( p(z_{i}|\tau^{l}) \) with parameters denoted by \( \tau^{l} \). Note that \( p(z_{i}|\tau^{l}) \) depends on the expression of \( p(u_{i}|\gamma_{p}^{l}, \tau^{l}) \). Moreover, in the above equation, \( \gamma_{p} = (\gamma_{p}^{1}, \ldots, \gamma_{p}^{L}) \) are scale parameters of our GSM prior.

In total, three types of hyperparameters are involved in the above Bayesian formulation: \( \gamma_{n}, \gamma_{p} \), and \( \tau = (\tau^{1}, \ldots, \tau^{L}) \). Generally, \( \tau \) are shape parameters of the GSM prior which determine the type of prior information introduced. As a result, we choose to fix \( \tau \) according to our prior knowledge. However, we estimate \( \gamma_{n} \) and \( \gamma_{p} \) since they determine a compromise between data fidelity and fidelity to the prior information.

For \( \gamma_{n} \) and \( (\gamma_{p}^{l})_{l=1,\ldots,L} \), Jeffreys’ non-informative priors are assigned. Using the Bayes’ rule, we can derive the posterior distribution of the unknown parameters given data

\[
p(u, z, \gamma_{n}, \gamma_{p}|y, \tau) \propto \gamma_{n}^{M/2} \exp \left[ -\frac{\gamma_{n}\|y - ADu\|^{2}}{2} \right]
\times \prod_{l=1}^{L} \prod_{i \in I_{l}} \sqrt{z_{i}^{(l)}\gamma_{p}^{l}} \exp \left[ -\frac{\gamma_{p}^{l}}{2} z_{i} u_{i}^{2} \right] p(z_{i}|\tau^{l})
\times \gamma_{n}^{-1} \prod_{l=1}^{L} (\gamma_{p}^{l})^{-1}.
\]

(2)

### 3. Variational Bayesian Approaches

In the following, we introduce a variable \( \Theta = \{u, z, \gamma_{n}, \gamma_{p}\} \) which includes all the parameters to be estimated. The estimation of these parameters is based on the joint posterior distribution given by (2). However, this distribution is intractable since its partition function is difficult to calculate in practice. To tackle this problem, we resort to variational Bayesian approximations (VBA) which generate a separable approximation \( q_{\Theta} \) of the true posterior distribution \( p(\Theta|y, \tau) \) by minimizing the Kullback-Leibler divergence between them. Assuming that \( q_{\Theta}(\Theta) = \prod_{i} q_{i}(\Theta_{i}) \), classical VBA gives the following analytic solution (see [13] for details)

\[
q_{i}(\Theta_{i}) \propto \exp \left( \langle \log p(y, \Theta)\rangle_{q_{j}(\Theta_{j})} \right),
\]

(3)

where \( p(y, \Theta) \) is the joint distribution which is explicitly known. As shown by (3), each distribution \( q_{j} \) depends on all the other distributions \( q_{j} \) with \( j \) different from \( i \). In practice, this dependence implies the use of iterative methods such as the Gauss-Seidel one, which are not very efficient to iteratively approximate \( q_{\Theta} \). Recently in a prior work [14], an efficient exponentiated gradient based VBA method has been proposed. This method has been further developed in [15], leading to a more efficient Memory Gradient subspace based variational Bayesian approximation (MG-VBA) method. The MG-VBA integrates the subspace optimization principle and adopts the following updating equation:

\[
q_{i}^{k+1}(\Theta_{i}) = K^{k}(s^{k}) q_{i}^{k}(\Theta_{i}) \left( \frac{\langle \log p(y, \Theta)\rangle_{\prod_{j \neq i} q_{j}^{k}(\Theta_{j})}}{q_{i}^{k}(\Theta_{i})} \right)^{s_{i}^{k}}
\times \left( \frac{q_{i}^{k}(\Theta_{i})}{q_{i}^{k-1}(\Theta_{i})} \right)^{s_{i}^{k}},
\]

(4)

where \( s^{k} = [s_{1}^{k}, s_{2}^{k}] \) is the two-dimensional algorithm step size. In this work, we adopt the approximate optimal step size proposed in [15] thanks to the second order Taylor expansion of the objective criterion. We can see from (4) that \( q_{i}^{k+1} \) does not depend on \( q_{j}^{k+1} \) with \( j \) different from \( i \), but depends on \( \prod_{j \neq i} q_{j}^{k} \) which is known from the 4th iteration. As a result, all the \( (q_{i}^{k+1})_{i=1,\ldots,N} \) are updated in parallel.

Concerning the separability assumption, we consider here a total separability given as follows

\[
q_{\Theta}(\Theta) = \prod_{i=1,\ldots,N} q_{u_{i}}(u_{i}) q_{z_{i}}(z_{i}) q_{\gamma_{n}}(\gamma_{n}) \prod_{l=1}^{L} q_{\gamma_{p}^{l}}(\gamma_{p}^{l}).
\]

In fact, since \( p(z, \gamma_{n}, \gamma_{p}|u, y, \tau) \) is separable, the classical VBA yields directly explicit solutions for \( (q_{z_{i}})_{i=1,\ldots,N}, q_{\gamma_{n}} \) and \( (q_{\gamma_{p}^{l}})_{l=1,\ldots,L} \). Nevertheless, this is not the case for \( (q_{u_{i}})_{i=1,\ldots,N} \). Therefore, we adopt the MG-VBA for the optimization of \( (q_{u_{i}})_{i=1,\ldots,N} \).

#### 3.1. Determination of \( q_{u_{i}} \)

Since a GSM prior is used, the conditional distribution \( p(u_{i}|z_{i}, \gamma_{p}^{l}) \) is a Gaussian one, which is conjugate with the Gaussian likelihood \( p(y|u, \gamma_{n}) \). Therefore, the optimal approximate distributions \( (q_{u_{i}})_{i=1,\ldots,N} \) belong to a Gaussian family. As a result, we take

\[
q_{u_{i}}^{k}(u_{i}) = N(\langle m_{k} \rangle_{i}, \langle \sigma_{k}^{2} \rangle_{i}).
\]

In this case, the optimization of \( (q_{u_{i}})_{i=1,\ldots,N} \) is performed by optimizing their parameters: mean \( m_{k} \) and variance \( \sigma_{k}^{2} \).
Using (4), the following update equations have been obtained (details can be found in [15]):

\[
\sigma_{k+1}^2 = \left[ \frac{1}{\sigma_k^2} + s_1 \left( \frac{1}{\sigma_k^2} - \frac{1}{\sigma_{k-1}^2} \right) + s_2 \left( \frac{1}{\sigma_k^2} - \frac{1}{\sigma_{k-1}^2} \right) \right]^{-1},
\]

\[
m_{k+1} = \sigma_{k+1}^2 \left\{ \frac{m_k}{\sigma_k^2} + s_1 \left( \frac{m_k}{\sigma_k^2} - \frac{m_{k-1}}{\sigma_{k-1}^2} \right) + s_2 \left( \frac{m_k}{\sigma_k^2} - \frac{m_{k-1}}{\sigma_{k-1}^2} \right) \right\}.
\]

In the above equations, we omit all the indications of vector component (\(i\)) for the sake of clarity. Moreover, \(\sigma_k^2\) and \(m_k\) are two intermediate variables updated using the following equations:

\[
(\sigma_k^2)_{i} = \left( (\gamma_n)^{k}(D^TA^T AD)_{(i,i)} + (\gamma_p)^{k}(z_i) \right)^{-1},
\]

\[
(m_k)_{i} = \left( \sigma_k^2 \right)_{i} (\gamma_n)^{k} [D^TA^T y - D^TA^T AD m_k] + \text{diag}(D^TA^T AD) \circ m_k,
\]

where \((w)^k = E_\theta(w), \text{diag}(M)\) is a vector containing the diagonal elements of \(M\) and \(\odot\) denotes the Hadamard product between two vectors. Actually, the variables \((\sigma_k^2)_{i}\) and \((m_k)_{i}\) are determined by the auxiliary function \(q_i'(\gamma_i) \propto \exp \left( (\log p(y, \Theta))_{i} q_i(\Theta_i) \right)\).

### 3.2. Determination of \(q_{z_i}\)

For the hidden variables \((z_i)_{i=1,\ldots,N}\), using (3), we can obtain

\[
q_{z_i}^{k+1}(z_i) \propto \exp \left( \frac{1}{2} \ln(z_i) - \frac{(\gamma_n)^{k}}{2} z_i (u_i^2)^k + \ln p(z_i | \tau^f) \right) \propto \sqrt{z_i} p(z_i | \tau^f) \exp \left( -\frac{(\gamma_n)^{k}}{2} (u_i^2)^k z_i \right)
\]

\[
= p(z_i | (u_i^2)^k, (\gamma_n)^{k}, \tau^f).
\]  

(7)

We can see that \(q_{z_i}\) depends on \(p(z_i | \tau^f)\). However, for most distributions in the GSM family, we do not know the explicit expression of \(p(z_i | \tau^f)\). As a result, the explicit expression of \(q_{z_i}\) is not known either. Nevertheless, our objective is not to obtain \(q_{z_i}\), but \(q_{u_i}\). As noted in [16] and also shown by (5), to determine \(q_{u_i}\), it is enough to know the expectation of \(q_{z_i}\).

In this case, the main challenge is to determine the expectation of \(q_{z_i}\) without knowing its explicit expression. Since \(p(u_i | \gamma_p^i, \tau^f)\) belongs to the GSM family, we can obtain (see [16] for details)

\[
p'(u_i | \gamma_p^i, \tau^f) = \frac{\partial}{\partial u_i} \int_0^\infty p(u_i | z_i, \gamma_p^i) p(z_i | \tau^f) dz_i = -\gamma_p^i u_i p(u_i | \gamma_p^i, \tau^f) \mathbb{E}_{p(z_i | u_i, \gamma_p^i, \tau^f)}[z_i],
\]

which allows us to get

\[
\mathbb{E}_{p(z_i | u_i, \gamma_p^i, \tau^f)}[z_i] = -\frac{p'(u_i | \gamma_p^i, \tau^f)}{\gamma_p^i u_i p(u_i | \gamma_p^i, \tau^f)}.
\]  

(8)

Combining (7) and (8), we obtain the expectation

\[
(z_i)^{k+1} = -\frac{p'(u_i | \gamma_p^i, \tau^f)}{\gamma_p^i u_i p(u_i | \gamma_p^i, \tau^f)} | u_i = \sqrt{\langle \theta_i^2 \rangle}, \gamma_n = \langle \gamma_i^n \rangle_k.
\]

(9)

In this work, we consider also one special case of the GSM family: the Generalized Gaussian (GG) distribution whose density is given by:

\[
GG(u_i | \gamma_p^i, \tau^f) = \frac{\sqrt{\tau^f}}{2\Gamma(1/\tau^f)} e^{-|\sqrt{\tau^f} u_i|^\tau^f}
\]

(10)

where \(\tau^f > 0\) is the shape parameter of the GG distribution.

With a GG prior, the expectation given by (9) becomes

\[
(z_i)^{k+1} = \tau^f [\langle \gamma_i^n \rangle_k (\langle m_{k+1} \rangle_k^2 + (\sigma_k^2)_{k+1})]^{\frac{1}{\tau^f} - 1}.
\]

### 3.3. Determination of \(q_{\gamma_n}\) and \(q_{\gamma_p}\)

Thanks to the conjugate priors for the hyperparameters, the optimal approximate distributions \(q_{\gamma_n}\) and \(q_{\gamma_p}\) are Gamma ones. As a result, we take

\[
q_{\gamma_n}^{k+1} = \mathcal{G}(\beta_k, \xi_k),
\]

\[
q_{\gamma_p}^{k+1} = \mathcal{G}(\eta_k, \zeta_k).
\]

Therefore, the optimization of \(q_{\gamma_n}\) and \(q_{\gamma_p}\) can be performed by updating their parameters. Using (3), we can obtain the following update equations for parameters of \(q_{\gamma_n}\) and \(q_{\gamma_p}\):

\[
\beta_k^{k+1} = \frac{M}{2} = \beta
\]

\[
\xi_k^{k+1} = \frac{1}{2} ||y - AD m_{k+1}||^2 + \frac{1}{2} \sum_{i=1}^{N} (D^TA^T AD)_{i,i} (\sigma_k^2)_{k+1}^i
\]

\[
\eta_k^{k+1} = \frac{\text{card}(I_k)}{2} = \eta_k
\]

\[
\zeta_k^{k+1} = \frac{1}{2} \sum_{i \in I_k} (z_i)^{k+1} [(m_{k+1})^2_i + (\sigma_k^2)_{k+1}]_i,
\]

where \(\text{card}(I_k)\) is the number of elements in the set \(I_k\).

The PM estimator is used for each parameter. However, we reconstruct the dictionary coefficients \(\hat{u}\) instead of the unknown image \(\hat{x}\). As a result, we perform the following reconstruction operation to get an estimation of the unknown image: \(\hat{x} = \hat{D} \hat{m}\).

### 4. EXPERIMENTAL RESULTS

The proposed approach is evaluated through an application to a deconvolution problem which is covered by the linear forward model (1). In deconvolution problems, \(A\) corresponds
to a convolution operator. In the following, we present simulation results obtained by the proposed approach and compare the results with two existing approaches: a SURE-LET approach in wavelet domain [4], and a supervised total variation (TV) regularized least-squares deconvolution approach in the image domain which computes the MAP estimate thanks to a primal-dual algorithm [17]. For the TV based approach, the hyperparameter is manually tuned to obtain the best result which has the highest PSNR.

In our simulations, we used Symlet-8 wavelets over three decomposition levels, leading to 10 subbands. Moreover, we used a frame constructed by the union of nine translated wavelet bases which allows reducing blocky artifacts caused by dyadic shifts underlying the orthogonal wavelet transform.

We show in Fig. 1 (a) the original Lena image and in Fig. 1 (b) its blurred version. The reconstructions obtained by SURE-LET [4], the TV based approach and the proposed one are shown in Fig. 1 (c), (d) and (e), respectively. We can see that the result of the proposed approach (Fig. 1 (e)) is of slightly better quality than that of SURE-LET (Fig. 1 (c)). Moreover, by comparing Fig. 1 (d) and (e), we can see that details of Lena are better reconstructed by the proposed approach than the TV based one, e.g. the brim of the hat, textures and the feather on the hat (upper left corner of Lena image) of Fig. 1 (e) are sharper than those in Fig. 1 (d).

The application was based on synthetic data generated from two images, Lena and Cameraman. A 9 × 9 uniform blur was applied to original images and Gaussian noises were added to the blurred ones resulting in a SNR equal to 40 dB.

The proposed approach was implemented with the following initializations: the wavelet transform of the observed data as the mean and 100 as the variance of dictionary coefficients. From these initial values, we compute the initialization of \( \gamma_n \) using updating equations given in Section 3.3 and \((\gamma_p^i)_{i=1,\ldots,10}\) are initialized by the same value. Concerning the shape parameters of the GG priors, 2, which gives a Gaussian distribution, was assigned to the GG prior for coarse approximation coefficients since they are generally not sparse and [1 0.9 0.8] were assigned to the GG priors for detail coefficients from the coarsest to the finest level to enforce higher degrees of sparsity for coefficients at finer scales.

We show in Table 1 PSNR of reconstructions obtained by SURE-LET, TV based approach and the proposed approach.

<table>
<thead>
<tr>
<th>Image</th>
<th>SURE-LET</th>
<th>TV</th>
<th>Proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lena</td>
<td>29.06</td>
<td>28.19</td>
<td>29.75</td>
</tr>
<tr>
<td>Cameraman</td>
<td>26.56</td>
<td>27.41</td>
<td>28.86</td>
</tr>
</tbody>
</table>

We show in Table 1 PSNR of reconstructions obtained by the three approaches. The highest PSNR in each case is highlighted in bold. We can see that in both cases, it is the proposed approach that gives the best PSNR. For Lena, the proposed approach gives 29.75 dB which is 0.69 dB higher than SURE-LET and 1.56 dB higher than TV. For Cameraman, our approach gives 28.86 dB which is 2.3 dB larger than SURE-LET and 1.45 dB higher than TV. However, concerning the execution time, the proposed approach is slower than SURE-LET. For Lena, SURE-LET takes 95 seconds whereas the proposed approach takes 273 seconds. Moreover, for Cameraman, SURE-LET takes 44 seconds whereas the proposed approach takes 136 seconds. Our approach jointly estimate parameters and hyperparameters in an iterative way which leads to better results but requires more computation time.

In our approach, hyperparameters were determined automatically. For the inverse noise variance \( \gamma_n \), our approach gives \( 2.14 \times 10^5 \) for Lena which is close to the true value: \( 2.26 \times 10^5 \), and gives \( 1.08 \times 10^5 \) for Cameraman, which is also close to the true value: \( 1.09 \times 10^5 \). We show also in Table 2 converged values of the scale parameters \((\gamma_p^i)_{i=1,\ldots,10}\) of the GG prior for Lena. From (2) we can see that larger values of \( \gamma_p^i \) lead to greater importance of the sparse prior information. In Table 2, we can see that the value of \( \gamma_p^1 \) is larger for coefficients at finer scales: the \( \gamma_p^i \) of scale 1 is larger than that of scale 2.
scale 2 which is larger than that of scale 3, which means that we give much more importance on the sparsity prior information for finer scale coefficients. This result is coherent with the fact that coefficients of finer scales are sparser than those of coarser scales. Furthermore, $\gamma_p$ for coarse approximation coefficients is small, which means that the coarse approximation coefficients are mainly determined by the data. For 
Camерамan, we do not show the estimation of $\gamma_p^p$ since it exhibits a similar performance to Lena.

We need to note that another advantage of the proposed approach is that it can be easily used to treat other linear inverse problems, e.g. tomographic reconstruction problems.

5. CONCLUSION

In this paper, by using variational Bayesian approximations, we proposed an unsupervised Bayesian approach based on a prior distribution of the GSM family in a transform domain for linear inverse problems. The first main advantage of this approach is that a large number of hyperparameters can be estimated automatically. The second advantage is that it can be used to solve various linear inverse problems. Experimental results showed that the proposed approach can well estimate the hyperparameters and gives better reconstructions than classical approaches. Furthermore, the proposed approach can be easily applied to large dimensional problems.

REFERENCES