REPRESENTATION OF SIGNALS BY LOCAL SYMMETRY DECOMPOSITION

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ABSTRACT

In this paper we propose a segmentation of finite support sequences based on the even/odd decomposition of a signal. The objective is to find a more compact representation of information. To this aim, the paper starts to generalize the even/odd decomposition by concentrating the energy on either the even or the odd part by optimally placing the centre of symmetry. Local symmetry intervals are thus located. The sequence segmentation is further processed by applying an iterative growth on the candidate segments to remove any overlapping portions. Experimental results show that the set of segments can be more efficiently compressed with respect to the DCT transformation of the entire sequence, which corresponds to the near optimal KLT transform of the data chosen for the experiment.

Index Terms— Symmetry, 1-D segmentation, signal decomposition, compact representation, compression.

1. INTRODUCTION

Classifying and labeling signals is of utmost importance. Applications range from anomaly detection [1] to event classification [2] and more [3]. Typically, the approach involves steps such as extracting features, measuring stochastic properties, looking for patterns, measuring correlations, computing fractal dimensions... [4] [5] [6]. Traditionally, these approaches only indirectly or do not at all involve exploiting inherent symmetry properties that may be present in many signals, e.g. locally periodic waveforms, luminance variation along a particular direction in a natural image... In addition, exploiting inherent symmetries in signals may also improve the signal coding efficiency.

Fourier or multiresolution representations have been conceived and lead to sparse representation of the information, hence improving compression. This has turned out also useful for tasks such as denoising [7]. Even/odd decomposition is consistent with the simplest decomposition which is proposed for signal manipulation [8]. As a matter of fact, due to the resulting intuitive geometrical interpretation and the parity preservation of the Fourier transform, even/odd decomposition is omnipresent in signal processing.

This paper presents a new methodology to effectively represent a 1-D signal, based on the even/odd decomposition of the signal around its local symmetry points. To do so, a signal is segmented into adjacent, variable length intervals in which strong symmetry characteristics exist. We argue that operating this segmentation leads to a compact representation, and we prove it by showing how individually compressing those intervals achieves better performance than compressing the whole signal using a linear transformation of the data.

The rest of the paper is organized as follows. Section 2 derives a method for finding the globally optimum symmetry point for a 1-D continuous-time, finite support signal, in the process extending the definition of even/odd decomposition to arbitrary points. This analysis is extended to digital sequences in Section 3. Then, Section 4 presents the algorithm that we devised to segment a discrete sequence into multiple near-symmetric sub-sequences, using the theoretic approach described in the previous sections and the application of an iterative process to remove overlapping intervals. Experimental results are obtained showing the effectiveness of this type of representation in terms of compression performance with respect to a traditional decorrelating linear transformation of the data: these are reported in Section 5. Conclusions are finally drawn in Section 6.

2. GLOBALLY OPTIMAL CONTINUOUS SYMMETRY

Let us consider a continuous-time, real-valued, finite-energy signal $x(t)$. For simplicity, we shall assume without loss of generality that the signal has finite support and is time-centered, i.e. with support $[-T, T]$. The well known parity decomposition of a signal states that $x(t)$ can be expressed as the sum of its even and odd parts, respectively $x_e(t)$ and $x_o(t)$, given by:

$$x_e(t) = \frac{x(t) + x(-t)}{2}; \quad x_o(t) = \frac{x(t) - x(-t)}{2} \quad (1)$$

The energy E is defined as the squared Euclidean norm of the signal $x(t)$ and it is easy to see that:

$$E = \int_{-T}^{T} |x(t)|^2 dt = \int_{-T}^{T} |x_e(t) + x_o(t)|^2 dt = \quad (2)$$
$$= \int_{-T}^{T} |x_e(t)|^2 dt + \int_{-T}^{T} |x_o(t)|^2 dt = E_e + E_o$$

since the even and odd parts are orthogonal. An example of such decomposition is shown in Fig. 1.
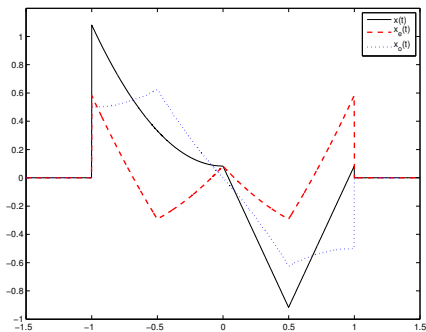
Fig. 1: An example of standard parity decomposition, $T = 1$.

The new representation does not increase the temporal support needed to reconstruct the original signal. In fact, the parity property of the even and odd parts implies that even if they both have the same support of the original signal, only e.g. their causal part is informative and thus sufficient to describe the entire signal. If the original signal $x(t)$ is of a inherently even (resp. odd) shape, almost all of its energy will be carried by its even (resp. odd) part (in Fig. 1 the former case applies). It is also advisable to cancel the input mean, since all the DC energy is carried just by the even part.

There are some cases where the parity decomposition as defined in Eq. (1) has no effect even if the original signal has simple parity characteristics because its center of symmetry is not in the origin. The reason is that the parity decomposition as defined above only considers the parity characteristics with respect to the time origin $t = 0$.

So, let us extend the definition of parity decomposition to allow an arbitrary symmetry point $t = t_0$. In this case, we have:

$$
\begin{aligned}
x_e(t; t_0) &= \frac{x(t) + x(2t_0 - t)}{2} \\
x_o(t; t_0) &= \frac{x(t) - x(2t_0 - t)}{2} \\
x(t) &= x_e(t; t_0) + x_o(t; t_0)
\end{aligned} \tag{3}
$$

and we fall back to the standard case by setting $t_0 = 0$. Still assuming that the original signal $x(t)$ has support $[-T, T]$, after even/odd decomposition with respect to t_0 the (non-informative) support of both $x_e(t)$ and $x_o(t)$ is no more T but $T + 2|t_0|$. However, the informative support is still T since the added support simply mirrors part of the signal confined in the informative support. Thus, the price to pay for searching for the best possible decoupling of the energies E_e and E_o is just increased complexity using the same support T. Note that Eq. (2) for the computation of energies is still valid, provided that the extremes of integration are changed to reflect the increased (non-informative) support of the even and odd signals.

We want to find the global optimal symmetry point t_0, that represents the time instant for which there is a maximum

energy decoupling between the even and odd parts. Since the energies E_e and E_o are now function of the parameter t_0, but still constrained by Eq. (2) to have sum E, both possess the same extrema points. The globally optimal symmetry point t_0' corresponds to the minimum of either E_o or E_e (i.e. the maximum of E_e or E_o).

To find t_0', let us concentrate on the extrema points of the energy of the even part. Its derivative w.r.t. t_0 is:

$$
\frac{dE_e}{dt_0} = \frac{d}{dt_0} \int_{-\infty}^{+\infty} |x_e(t; t_0)|^2 \, dt = \tag{4}
$$
$$
= \frac{d}{dt_0} \int_{-\infty}^{+\infty} \left| \frac{x(t) + x(2t_0 - t)}{2} \right|^2 dt =
$$
$$
= \frac{1}{4} \frac{d}{dt_0} \int_{-\infty}^{+\infty} [\, |x(t)|^2 + |x(2t_0 - t)|^2 +
$$
$$
+ 2x(t)x(2t_0 - t)] dt
$$

where we extended the integral on the whole real axis for simplicity of notation. The first two terms in the last integral give E as a result, since reversing the time axis and the origin do not influence the energy value, thus they are both independent from t_0. Thus:

$$
\frac{dE_e}{dt_0} = \frac{1}{2} \frac{d}{dt_0} \int_{-\infty}^{+\infty} x(t)x(2t_0 - t) dt \tag{5}
$$

Exchanging the order of derivation and integration and then using the usual definition for the linear convolution of two energy signals we obtain:

$$
\frac{dE_e}{dt_0} = (x * x')(2t_0) = -\frac{dE_o}{dt_0} \tag{6}
$$

So, candidate extreme points can be determined by convolving the signal with the derivative of its conjugate, finding the zero-crossing points and then dividing by 2. A local minimum for E_e corresponds to a local maximum for E_o and vice-versa. To find the global minimum between E_e and E_o it is not sufficient to look for the global minimum for E_e but one must consider as well its global maximum (i.e. the global minimum of E_o). t_0' is then declared to correspond to the location that leads to the global minimum between E_e and E_o.

3. GLOBALLY OPTIMAL DISCRETE SYMMETRY

In the 1-D discrete case, Eq. (3) becomes:

$$
\begin{aligned}
x_e[n; n_0] &= \frac{x[n] + x[2n_0 - n]}{2} \\
x_o[n; n_0] &= \frac{x[n] - x[2n_0 - n]}{2} \\
x[n] &= x_e[n; n_0] + x_o[n; n_0]
\end{aligned} \tag{7}
$$

The symmetry point n_0 cannot be arbitrary but has to correspond to either a sample (i.e. integer) or to a half-sample
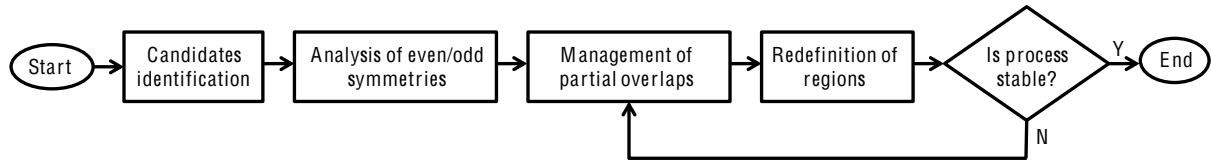
Fig. 2: Block diagram describing the whole process.

position. First, let us suppose that the input sequence $x[n]$ has finite duration, that is without loss of generality $x[n]$, $n = 1, \ldots, L$. Then $n_0 = \frac{k}{2}$, $k \in \{2, \ldots, 2L\}$. When k is odd (resp. even) then the folding point corresponds to a half-sample (resp. integer) position. It is clear that the odd and even parts may not have the same length, that depend on the associated n_0. Eq. (2) is still valid, provided the integration operator is substituted by the summation one.

Again, searching for the optimal symmetry point consists in choosing the position that lets either E_e or E_o be the global minimum. For the discrete case, it is better to evaluate the energy of the even sequence rather than its derivative. Thus:

$$
\begin{aligned}
E_e &= \sum_{-\infty}^{+\infty} |x_e[n; n_0]|^2 = \sum_{-\infty}^{+\infty} \left| \frac{x[n] + x[2n_0 - n]}{2} \right|^2 = \\
&= \frac{1}{4} \sum_{n=-\infty}^{+\infty} |x[n]|^2 + |x[2n_0 - n]|^2 + 2x[n]x[2n_0 - n] = \\
&= \frac{1}{2}E + \frac{1}{2} \sum_{1}^{2n_0} x[n]x[2n_0 - n]
\end{aligned}
\tag{8}
$$

where $[1, 2n_0]$ is the non-null support of the product in the summation. The energy of the odd part E_o has the same expression provided the sign of the summation in the last row is changed. Using the definition of convolution for energy sequences yields:

$$
\begin{aligned}
E_e &= \frac{1}{2}E + (x * x)[2n_0] \\
E_o &= \frac{1}{2}E - (x * x)[2n_0]
\end{aligned}
\tag{9}
$$

so in the end we need to compute the auto-convolution of $x[n]$. The optimal candidate locations can thus be derived from:

$$
2n_0 = \arg\max_m |(x * x)[m]|
\tag{10}
$$

## 4. LOCALLY OPTIMAL DISCRETE SYMMETRY

In this section we use the approach proposed in Section 3 to identify, for an arbitrary sequence $x[n]$, maximal support segments that exhibit near perfect even or odd local symmetries. The objective is to find non-overlapping segments displaying strong symmetry traits that cover at least a good portion of the sequence, so that their identification provides for a significant

information reduction. The procedure is derived from the theory outlined previously. It takes four steps, as shown by the flow diagram of Fig. 2. First, in order to locate possible even or odd symmetry points, we compute the auto-convolution of $x[n]$. All local extrema become candidate points (Fig. 3), whose positions are stored in a list $\vec{P}$.

In the second stage of Fig. 2, we determine the extent of the symmetry degree around each candidate location, say the i-th location P_i. Two parameters must be set for this process: the minimum symmetry support m_l and the even/odd energy ratio threshold p_t, with $0.5 < p_t \leq 1$. At this point, a tentative segment consisting of a sub-sequence $y[n]$ with support m_l centered around P_i is selected. Then, the even/odd decomposition of $y[n]$ with respect to P_i is carried out, including mean removal, computing the E_y, E_{y_e} and E_{y_o}, which represent the energies of $y[n]$ and of its even and odd decomposition respectively. To classify $y[n]$ into an even or odd symmetric segment, we check whether the energy ratio $E_{y_e}/E_y \geq p_t$ or $E_{y_o}/E_y \geq p_t$. If neither condition is verified, P_i is removed from the list of the candidate locations. Alternatively, we gradually increase the support of the sub-sequence $y[n]$ and repeat the process as long as the energy ratio remains larger than p_t. The maximum support for which the latter condition applies is stored as M_i. This operation allows to track the exact extent of the local symmetry around the candidate point P_i, by that identifying a candidate segment. Thus, the support of the considered candidate segment can assume a value anywhere from the minimum support m_l to M_i.

Intuitively, we should consider good segments for the sequence $x[n]$ (that is, those with the most comprehensive local symmetries) as the candidate segments around every surviving positions in $\vec{P}$ with the largest possible supports M_i. A complete overlap, i.e. a candidate segment is entirely included in another candidate segment, can be retained since it can offer a relevant hierarchy of the symmetric nature of the data. For example, in images an eye falls within a larger symmetric portion of the face (see also Fig. 5). However, a partial overlap between candidate segments is to be avoided and therefore we need to crop such overlapping areas.

The next step shown in Fig. 2 concerns which segments to keep and how to choose their support, removing their overlaps. All symmetry locations are examined, from left to right, and each candidate segment that overlaps with one or more neighboring segment is analyzed in turn. In particular, it is necessary to define a priority criterion to help choosing which
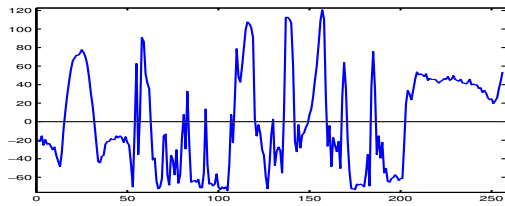
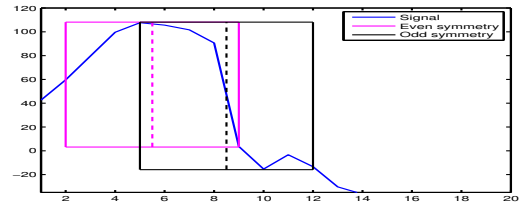(a) Pixel luminance values of a row of the *Lena* image.
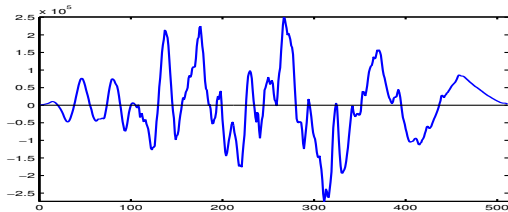
(b) Auto-convolution of (a).

Fig. 3: In (b) it is shown the auto-convolution of a row of *Lena*, depicted in (a). Relative maxima and minima identify candidates for resp. even and odd symmetry locations.
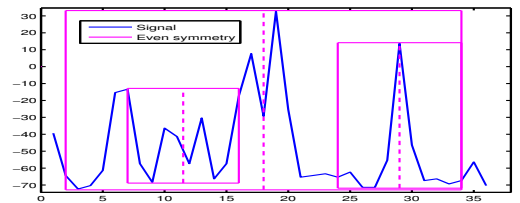

(a) Partial overlap.

(b) Total overlap.

Fig. 4: Two cases: (a) shows a partial overlap between candidate segments, that is to be removed. In (b) there is a multiple total overlap of a segment's support w.r.t. another one, which is instead useful to construct a hierarchy of symmetries.

candidate segments need to be shortened or completely removed to avoid partial overlapping. Possible criterions can be (i) longest segment support (M_i), (ii) highest energy ratio or (iii) highest energy ratio on same-support segments.

According to the first criterion, if the considered candidate segment presents the largest support in a set of overlapping regions, the latter have their support decreased until they are no longer overlapping with the segment of interest; otherwise, the considered segment is the one that is to be shortened. The same happens with the second criterion, however the deciding factor in this case is the highest energy ratio. Of course, if the support of a given segment goes under m_l, it is altogether removed (although possibly just temporarily, see below). The third criterion is a bit more complicated, but it helps preserving the best local symmetry of the data. Among the overlapping candidate segments, the largest common support is chosen and applied to the whole set (thereby shortening the other segments to the same support). Then, the second criterion based on the highest energy ratio is applied to these equally long segments.

After the third stage of Fig. 2, no overlapping segments are present. One last operation remains. In fact, during the removal of overlaps, a segment A can be deleted in favor of segment B, and segment B, in turn, can be deleted in favor of segment C. If in the end there is no overlap between A and C, we must reactivate segment A because it was erroneously canceled by a previous segment that has since been removed. Since this procedure can recreate some overlaps, the overlap removal algorithm must be iterated. The process ends when the segment list converges to a stable state (i.e. when no more changes occur during an iteration), that is guaranteed to hap-

pen by the algorithm structure, in general within 10 iterations.

5. EXPERIMENTAL RESULTS

First, we report in Fig. 5 the result of the algorithm for the search of locally optimal symmetries applied to the sequence shown in Fig. 3a. In particular we highlight the hierarchy of local even and odd symmetries reflected by smaller segments contained into longer segments.

Next, we perform a preliminary evaluation on how segmentation of the sequence driven by local symmetries can support the design of compact transformations for certain types of data, which might help compacting the information. Accordingly, we compare the traditional 1-D DCT applied on length 8 sequences (representing rows of natural images) with the 1-D DCT applied on the variable length segments centered around the local symmetries. We used the length 8 DCT since it is a well-known decorrelating transform for natural images.

The following steps have been applied to the rows of the *Lena* image. First, we perform the search for local even/odd symmetries, extracting locations and corresponding supports (adopting one of the three priority criterions). Then, the even/odd decomposition is applied and the first half of the samples resulting from it are stored. Last, the 1-D DCT is computed on all halved length even and odd sequences present in the analyzed row. The process has been applied to all the rows of the image to be statistically representative.

Experimental results show that, for each row, all symmetric segments cover almost entirely any given row. It is a rare
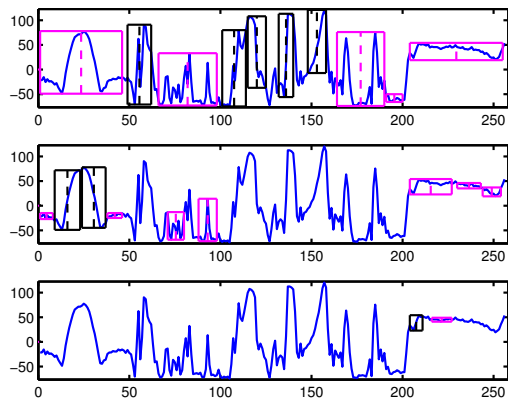
Fig. 5: Three hierarchical levels of symmetry. Here, the third criterion has been used using $m_l = 8$ and $p_t = 0.95$. The legend of Fig. 4 applies here as well.

occurrence that isolated samples are not part of any segment. In addition, when the first priority criterion described in Section 4 is employed, even less isolated samples occur as a result of adopting the highest possible segment support as the deciding factor.

In Fig. 6 we show the performance of the described procedure compared to a classical 1-D DCT applied on length 8 sequences. The PSNR curve has been produced by adjusting a uniform quantizer step size applied to the DCT coefficients. This curve is almost identical when the first or the third criterion of Section 4 are employed while the performance of the second one is slightly lower, indicating how the energy ratio alone is probably inappropriate to finalize segmentation. Rate values take into account the encoding of the even or odd type of symmetry of each segment and its location, including the encoding of its symmetry point and its support.

6. CONCLUSIONS

In this paper we presented a method to identify segments in a 1-D sequence connoted by inherent local symmetry. First, local extrema of its auto-convolution are identified and then the presence of symmetry around them is verified applying one of three possible criterions. A segmentation is obtained by performing an iterative procedure to remove any overlapping between candidate segments. The experimental results show that this representation achieves good compression for the whole sequence.

Current work focuses on studying a transform for the segments to replace the DCT as a second stage, tailored in particular to deal with variable length sub-sequences and their specific correlation. In addition, the extension of the algorithm to the 2-D domain, with the added complexity it introduces, is the subject of currently undergoing studies.
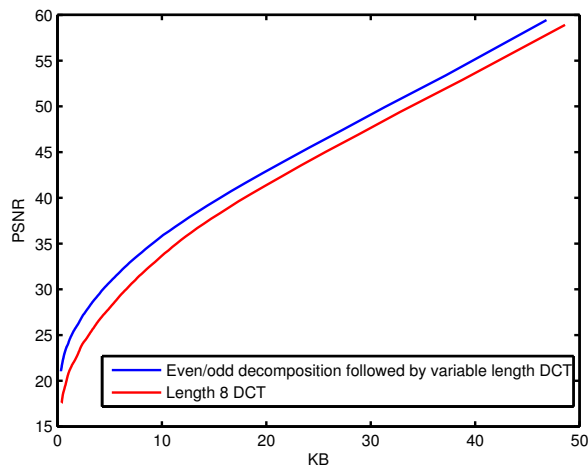

Fig. 6: Mean rate-distortion for the rows of *Lena*. Applying 1-D DCT on the even/odd decompositions provides more effective encoding if compared to a traditional 1-D DCT.

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