A REVISIT OF THE UNIQUE POLAR REPRESENTATION OF THE VECTOR-VALUED HYPERANALYTIC SIGNAL

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ABSTRACT

In this paper, we extend the classic analytic signal to the Vector-valued Hyperanalytic Signal (VHaS) that is denoted to distinguish from the multivariate hypercomplex data. The $2d$-Dimensional ($2d$-D) VHaS, $\mathbf{S}(t) : [0,1] \to \mathbb{C}_{2d}$, is defined by a complexification of two d-D Vector-valued Hypercomplex Signals (VHcS), $\mathbf{S}(t) := \mathbf{G}(t)\mathbf{e}_0 + \mathcal{H}^{\mathbb{C}_{2d}}_{\mathbf{e}_d}[\mathbf{G}](t)\,\mathbf{e}_d$, where $\mathcal{H}^{\mathbb{C}_{2d}}_{\mathbf{e}_d}$ and $\mathbf{e}_i$ represent the Hilbert transform and the ith unit axis, and $\mathbf{G}(t) \in \mathbb{C}_d, \mathbf{e}_i \in \mathbb{C}_{2d}$. Inspired by the unique polar form of a classic analytic signal and the one of a 4-D VHaS proposed in [1], we provide a theoretical explanation of the unique polar representation of a 6-D or 8-D VHaS by replacing the quaternion with octonion, which further implies the possible extension for d-D VHaS with $d > 8$. Moreover, the derived continuous VHcS envelope and phase from the polar form lead to a unified definition of the time-frequency-amplitude spectrum of the given VHcS $\mathbf{G}(t)$.

Index Terms— vector-valued hypercomplex signal, vector-valued hyperanalytic signal, quaternionic signal, octonionic signal, unique polar representation, time-frequency-amplitude spectrum.

1. INTRODUCTION

Given a real bounded signal $g(t) : [0,1] \to \mathbb{R}$, the corresponding 2-Dimensional (2-D) Analytic Signal (AS) $\mathbf{s}(t) : [0,1] \to \mathbb{C}_2$ is defined as a complexification of $g(t)$ and its Hilbert transform $\mathcal{H}[g](t)$, $\mathbf{s}(t) := g(t)\mathbf{e}_0 + \mathcal{H}[g](t)\mathbf{e}_1$, where $\mathbf{e}_0, \mathbf{e}_1 \in \mathbb{C}_2$ are the unit complex axes [2]. It is well known that $\mathbf{s}(t)$ also can be uniquely represented as a polar form $\mathbf{s}(t) := a(t)e^{\phi(t)\mathbf{e}_1}$, in which the nonnegative $a(t)$ and the monotonically nondecreasing $\phi(t)$ are called the instantaneous envelope and the instantaneous phase [3]. In other words, the canonical pair $(a(t), \phi(t))$ is in a one-to-one correspondence with $g(t)$ [4]. The nonnegative instantaneous frequency can be defined as the derivative of the phase $f(t) := \frac{1}{2\pi}\varphi'(t)$, and the Hilbert Time-Frequency-Amplitude (TFA) spectrum is defined as $\{a(t) \text{ on the curve } (t, f(t))\}$ [5]. Since the AS model provides a splendid relationship between the temporal signal and its instantaneous TFA spectrum, the valuable information involved in the signal, e.g the feature, mode, or pattern, that might be invisible to our eyes, can be clearly observed on the instantaneous time-frequency (TF) plane [6]. Therefore, the AS model attracts many researchers' and engineers' interests in both mathematics and signal processing communities.

With the progress of science and technology, sensor array based detection method has been widely used in practically every profession, e.g. the multi-lead electrocardiogram or electroencephalogram recording system in biomedicine, the sensor array based source localization system in industrial, or the vector-sensor array network monitoring the geophysical data. To supply the increasing requirement of the multi-D data analysis, it is pressing and significant to extend the classic AS theory into multi-D space.

In fact, the multi-D signal consists of multi-variate signal, $s(X) \in \mathbb{R}, X \in [0,1]^d$, and the Vector-valued Signal (VvS), $S(t) \in \mathbb{R}^d, t \in [0,1]$. It should be noted that the VvS sometimes can also be considered as Vector-valued Hypercomplex Signal (VHcS), $\mathbf{S}(t) := \sum_{i=1}^{d} s_i(t)\mathbf{e}_{i-1}$, where $s_i(t)$ is the ith subcomponent of $S(t)$, $\mathbf{e}_0$ is the unit real axis, and $\mathbf{e}_i, i = 1, \dots, d-1$, are other unit imaginary axes. For the multi-variate signal, there are substantial studies covering the basic theory [7, 8], image processing [9–11], and the N-D extension [12]. However, for the VvS, the AS theory in multi-D space has not yet been achieved completely. Inspired by the original study in [13], Sangwine el al. firstly investigated the complexification of two complex signals by introducing quaternion numbers [14], and the polar representation of the Quaternionic Hyperanalytic Signal (QHaS) [15], from which one can derive the instantaneous complex envelope and the instantaneous complex frequency. Huang el al. developed an algorithm to uniquely represent the polar form of a given QHaS [1]. However, the AS theory and the uniqueness of the polar representation of a Vector-valued Hyperanalytic Signal (VHaS) are unknown when the dimension of the VHaS is greater than four. In this paper, we will clarify the associated problems in dimension extension.

On the other hand, we have to point out that the VvS to be analyzed in this paper might not be the raw data obtained from the sensor array based detection system, it should be some component decomposed from the raw data, e.g. the

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one decomposed by using an empirical mode decomposition method [16,17], or by using a wavelet based method [18]. The main reason is because of our original definition of the TFA spectrum, in which the amplitude should be drawn on the corresponding curve $(t, f(t))$ on the TF plane. It clearly implies that the given component should be some simple amplitude-modulated and frequency-modulated (AM-FM) signal.

In the rest of this paper, Section 2 introduces the basic knowledge of quaternion and octonion, the unique polar representation of the complexified QHaS, together with the idea to extend the classic AS theory into multi-D space; Section 3 explains how to obtain unique instantaneous quaternionic envelope, phase and the corresponding TFA spectrum from a complexified VHaS; Section 4 presents representative numerical results that clearly illustrate the performance of the proposed method; and Section 5 summarizes the paper.

2. FROM COMPLEX TO X-ON NUMBER

In mathematics, the quaternion and octonion are typical number systems that extend the complex number. The quaternion was firstly introduced by Hamilton in 1843, and later the extended number systems had been comprehensively studied.

2.1. Basic definition of the hypercomplex number

To introduce the whole number system in a uniform version, we firstly define our number and signal as follows:

Definition 2.1 *The d-D hypercomplex number, $\mathbf{G} \in \mathbb{C}_d, d \geq 2, d \in \mathbb{N}$, is defined as $\mathbf{G} := \sum_{i=1}^{d} g_i \mathbf{e}_{i-1}$, where g_i is the ith subcomponent of the vector $G \in \mathbb{R}^d$, $\mathbf{e}_0$ is the unit real axis, and $\mathbf{e}_i, i = 1, \ldots, d-1$, are other imaginary axes. $\mathbf{G}$ would be called as complex, quaternion, octonion, 16-on, ... , 2^n-on number if we set $d = 2, 4, 8, 16, \ldots, 2^n, n \in \mathbb{N}$, respectively. The d-D Vector-valued Hypercomplex Signal (VHcS) is defined as the function $\mathbf{G}(t) : [0,1] \to \mathbb{C}_d, \mathbf{G}(t) := \sum_{i=1}^{d} g_i(t)\mathbf{e}_{i-1}$, where $g_i(t)$ is the subcomponent of the corresponding Vector-valued Signal (VvS) $G(t) \in \mathbb{R}^d$.*

Given a hypercomplex number $\mathbf{G}$, its real/scalar part and residual/vector part are denoted as $\mathcal{S}[\mathbf{G}] := g_1$ and $\mathcal{V}[\mathbf{G}] := \mathbf{G} - \mathcal{S}[\mathbf{G}]\mathbf{e}_0$ respectively. $\mathbf{G}$ is called a pure hypercomplex number if $g_1 = 0$. Each basis $\mathbf{e}_i$ is considered as the root of -1 or $\mathbf{e}_i^T \mathbf{e}_i = -1$. The multiplication rule among these bases depends on which dimension d we are considering. Table 1 illustrates the computation rules for complex, quaternion and octonion number systems. As one can see, the size of the table will be doubled if we increase the number of element basis to support a larger number system.

The conjugate of $\mathbf{G}$ is defined as $\bar{\mathbf{G}} := \mathcal{S}[\mathbf{G}]\mathbf{e}_0 - \mathcal{V}[\mathbf{G}]$, the Euclidean norm or the ℓ_2 norm of $\mathbf{G}$ is defined as $\|\mathbf{G}\|_{\ell_2} := \sqrt{\mathbf{G}\bar{\mathbf{G}}} = \sqrt{\sum_{i=1}^{d} g_i^2}$, and the inverse of $\mathbf{G}$ is

$\times$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$

Table 1. Multiplication rule among basis elements of complex, quaternion and octonion number system.

defined as $\mathbf{G}^{-1} := \frac{\bar{\mathbf{G}}}{\|\mathbf{G}\|_{\ell_2}^2}$. By introducing the Cayley-Dickson algebra, any $2d$-D hypercomplex number $\mathbf{G} \in \mathbb{C}_{2d}$ can be represented as a pair of d-D hypercomplex numbers $\mathbf{C}, \mathbf{D} \in \mathbb{C}_d$, e.g. $\mathbf{G} := \mathbf{C} + \mathbf{D}\mathbf{e}_d$.

Definition 2.2 *Given a hypercomplex number $\mathbf{G} \in \mathbb{C}_d$, the exponential and the natural logarithm of $\mathbf{G}$ can be defined by*

$$e^{\mathbf{G}} := e^{\mathcal{S}(\mathbf{G})}\left(\cos(\|\mathcal{V}(\mathbf{G})\|_{\ell_2}) + \frac{\mathcal{V}(\mathbf{G})}{\|\mathcal{V}(\mathbf{G})\|_{\ell_2}}\sin(\|\mathcal{V}(\mathbf{G})\|_{\ell_2})\right) \quad (1)$$

$$\ln(\mathbf{G}) := \ln(\|\mathbf{G}\|_{\ell_2}) + \frac{\mathcal{V}(\mathbf{G})}{\|\mathcal{V}(\mathbf{G})\|_{\ell_2}}\arccos\left(\frac{\mathcal{S}(\mathbf{G})}{\|\mathbf{G}\|_{\ell_2}}\right) \quad (2)$$

2.2. Vector-valued Hyperanalytic Signal (VHaS)

Since the AS theory of the quaternionic signal has been studied in [1, 14], we only present necessary definitions and corollaries here to support our generalized complexification of VHcS for $d \geq 3, d \in \mathbb{N}$.

Corollary 2.3 *Given a VHcS $\mathbf{G}(t) \in \mathbb{C}_d, d = 2^n, n \in \mathbb{N}$, and a unit imaginary axis $\mathbf{e}_d \in \mathbb{C}_{d+1}$, the Right Fourier Transform (RFT) of $\mathbf{G}(t)$ with respect to (w.r.t) the axis $\mathbf{e}_d$ can be sped up by the Fast Fourier Transform (FFT).*

Proof: denotes $\mathcal{F}, \mathcal{F}_{\mathbf{e}_d}^{\mathbb{C}_{2d}}$ as the FFT and the RFT w.r.t $\mathbf{e}_d$ in the hypercomplex space $\mathbb{C}_{2d}$, and sets operators $\mathcal{R}, \mathcal{I}$ to take the real and imaginary part of a complex number respectively,

$$\begin{aligned}
\hat{\mathbf{G}}_{\mathbf{e}_d}(f) &:= \mathcal{F}_{\mathbf{e}_d}^{\mathbb{C}_{2d}}[\mathbf{G}](t) := \int_{[0,1]} \mathbf{G}(t) e^{-2\pi f t \mathbf{e}_d} dt \\
&= \int_{[0,1]} \sum_{i=1}^{d} g_i(t)\mathbf{e}_{i-1} e^{-2\pi f t \mathbf{e}_d} dt \\
&= \sum_{i=1}^{d} \mathbf{e}_{i-1}\left(\mathcal{R}\left(\mathcal{F}[g_i](t)\right) + \mathcal{I}\left(\mathcal{F}[g_i](t)\right)\mathbf{e}_d\right) \\
&= \sum_{i=1}^{d} \left(\mathcal{R}\left(\mathcal{F}[g_i](t)\right)\mathbf{e}_{i-1} + \mathcal{I}\left(\mathcal{F}[g_i](t)\right)\mathbf{e}_{i+d-1}\right)
\end{aligned} \quad (3)$$

■

It should be noted that the reason that we require $d = 2^n, n \in \mathbb{N}$ in Corollary 2.3 is because we have to figure out the basis multiplication problem. Therefore we can gain the benefit from $\mathbf{e}_{i-1}\mathbf{e}_d = \mathbf{e}_{i+d-1}$ for $i = 1, \ldots, d$ (recall the multiplication rule in Table 1). In practice such requirement also cover the case with dimension $d \neq 2^n$ if we can complement additional components with zeros. For instance, if $d = 3$ and the given VvS is $G(t) = [g_1(t), g_2(t), g_3(t)]^T$, the constructed VHcS would be $\mathbf{G}(t) = g_1(t)\mathbf{e}_0 + g_2(t)\mathbf{e}_1 + g_3(t)\mathbf{e}_2 + 0\mathbf{e}_3$. Now, we can extend the classic Hilbert transform into the hypercomplex space without proof.

Corollary 2.4 *Given a VHcS $\mathbf{G}(t) \in \mathbb{C}_d, d = 2^n, n \in \mathbb{N}$ and a unit hypercomplex axis $\mathbf{e}_d \in \mathbb{C}_{d+1}$, the Vector-valued Hypercomplex Hilbert Transform (VHHT) of $\mathbf{G}(t)$ w.r.t $\mathbf{e}_d$ can be expressed in terms of the classic Hilbert transform*

$$\begin{aligned} \mathcal{H}^{\mathbb{C}_{2d}}_{\mathbf{e}_d}[\mathbf{G}](t) &:= \mathcal{F}^{\mathbb{C}_{2d}\,-1}_{\mathbf{e}_d}\left[-\mathbf{e}_d \operatorname{sgn}(f)\mathcal{F}^{\mathbb{C}_{2d}}_{\mathbf{e}_d}[\mathbf{G}]\right](t), \\ &= \sum_{i=1}^{d} \mathcal{H}[g_i](t)\mathbf{e}_{i-1} \end{aligned} \tag{4}$$

where $\mathcal{F}^{\mathbb{C}_{2d}\,-1}_{\mathbf{e}_d}$ is the inverse RFT, and $\operatorname{sgn}(\cdot)$ is the signum function. Finally, we can construct the VHaS by complexifying the given VHcS and its VHHT

Definition 2.5 *Given a VHcS $\mathbf{G}(t) \in \mathbb{C}_d, d = 2^n, n \in \mathbb{N}$, the VHaS $\mathbf{S}(t) \in \mathbb{C}_{2d}$ w.r.t the $\mathbf{e}_d$ axis is defined by*

$$\mathbf{S}(t) := \mathbf{G}(t) + \mathcal{H}^{\mathbb{C}_{2d}}_{\mathbf{e}_d}[\mathbf{G}](t)\mathbf{e}_d. \tag{5}$$

3. UNIQUE POLAR REPRESENTATION

To keep consistency of the AS theory, we hope the constructed VHaS, $\mathbf{S}(t) \in \mathbb{C}_{2d}, d = 2^n, n \in \mathbb{N}$, can be uniquely represented as a polar form $\mathbf{S}(t) := \mathbf{A}(t)e^{\mathbf{\Phi}(t)\mathbf{e}_d}$, $\mathbf{A}(t), \mathbf{\Phi}(t) \in \mathbb{C}_d$, where $\mathbf{A}(t)$ and $\mathbf{\Phi}(t)$ denote the instantaneous hypercomplex envelope and phase respectively. As [1] has already investigated the unique polar representation of a VHaS for $n = 1$, in the following, we study the dimension extended case for $n = 2$. However, we should bear in mind that the dimension of the VvS $G(t)$ can be 3 or 4. And when $G(t) \in \mathbb{R}^3$, we need to supplement one more component of zeros such that the constructed VHcS $\mathbf{G}(t) \in \mathbb{C}_4$ or VHaS $\mathbf{S}(t) \in \mathbb{C}_8$. Considering that the X-on number system will be more complicated when $n > 2$, but the unique polar representation methods are similar, we only consider octonion number based unique polar representation in this section.

3.1. Sign ambiguity in the polar form

To let the discussion easier, we start from the hypercomplex number but not the signal. Suppose the given hyperanalytic number is $\mathbf{S} := \sum_{i=1}^{8} s_i\mathbf{e}_{i-1}, s_i \in \mathbb{R}$. Its Cayley-Dickson form can be set as $\mathbf{S} := \mathbf{C} + \mathbf{D}\mathbf{e}_4$, where $\mathbf{C} := \sum_{i=1}^{4} s_i\mathbf{e}_{i-1}$ and $\mathbf{D} := \sum_{i=1}^{4} s_{i+4}\mathbf{e}_{i-1}$ are quaternions. The corresponding polar form is $\mathbf{S} := \mathbf{A}e^{\mathbf{\Phi}\mathbf{e}_4}$, where

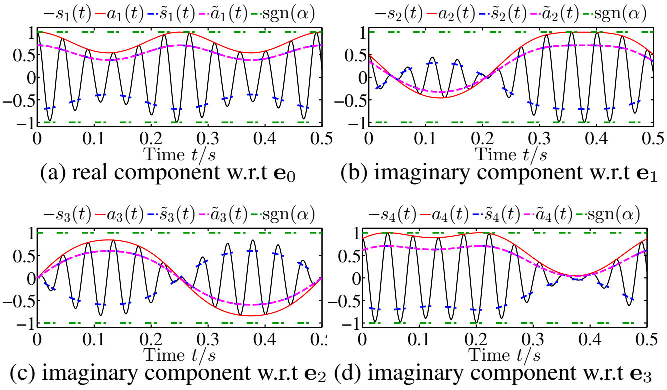


Fig. 1. An example of the sign ambiguity problem. Here $\mathbf{C}(t) := \sum_{i=1}^{4} s_i\mathbf{e}_{i-1}$ is given; $\operatorname{sgn}(\alpha)$ is the unknown sign signal; $\mathbf{A}(t) := \sum_{i=1}^{4} a_i(t)\mathbf{e}_{i-1}$ is the envelope that we want to retrieve; $\mathcal{A}[\mathbf{C}](t) := \sum_{i=1}^{4} \tilde{s}_i\mathbf{e}_{i-1}$ and $\mathcal{A}[\mathbf{A}](t) := \sum_{i=1}^{4} \tilde{a}_i(t)\mathbf{e}_{i-1}$ are the axes of $\mathbf{C}(t)$ and $\mathbf{A}(t)$ respectively (Color online).

$\mathbf{A} := \sum_{i=1}^{4} a_i\mathbf{e}_{i-1}$, $\mathbf{\Phi} := \sum_{i=1}^{4} \phi_i\mathbf{e}_{i-1}$ are quaternions, and all coefficients a_i, ϕ_i are real. As $\mathbf{\Phi}\mathbf{e}_4$ is a pure octonion that does not contain real part, according to (1), the exponential of $\mathbf{\Phi}\mathbf{e}_4$ can be expressed as

$$\begin{aligned} e^{\mathbf{\Phi}\mathbf{e}_4} &:= \alpha\mathbf{e}_0 + \beta\mathbf{e}_4 + \gamma\mathbf{e}_5 + \eta\mathbf{e}_6 + \xi\mathbf{e}_7 \\ &:= \cos(\|\mathbf{\Phi}\|_{\ell_2})\mathbf{e}_0 + \sum_{i=1}^{4} \frac{\phi_i}{\|\mathbf{\Phi}\|_{\ell_2}} \sin(\|\mathbf{\Phi}\|_{\ell_2})\mathbf{e}_{i+3} \end{aligned} \tag{6}$$

where $\|\mathbf{\Phi}\|_{\ell_2} = \sqrt{\sum_{i=1}^{4} \phi_i^2}$. Then, we obtain

$$\begin{aligned} \mathbf{S} = \sum_{i=1}^{8} s_i\mathbf{e}_{i-1} &= \mathbf{C} + \mathbf{D}\mathbf{e}_4 = \mathbf{A}e^{\mathbf{\Phi}\mathbf{e}_4} \\ &= \alpha\mathbf{A} + (a_1\beta - a_2\gamma - a_3\eta - a_4\xi)\mathbf{e}_4 \\ &\quad + (a_1\gamma + a_2\beta - a_3\xi + a_4\eta)\mathbf{e}_5 \\ &\quad + (a_1\eta + a_2\xi + a_3\beta - a_4\gamma)\mathbf{e}_6 \\ &\quad + (a_1\xi - a_2\eta + a_3\gamma + a_4\beta)\mathbf{e}_7 \end{aligned} \tag{7}$$

from the equality $\mathbf{C} = \alpha\mathbf{A}$, we can expect that the axis of the left hand side quaternion should be equal to the one of the right hand side quaternion. In other words, by introducing an axis operator defined by $\mathcal{A}[\mathbf{G}] := \frac{\mathbf{G}}{\|\mathbf{G}\|_{\ell_2}}$, $\mathbf{G} \in \mathbb{C}_d$, we have

$$\mathcal{A}[\mathbf{A}] = \frac{\mathcal{A}[\mathbf{C}]}{\operatorname{sgn}(\alpha)}, \tag{8}$$

Therefore, we meet a sign ambiguity problem because of the unknown $\operatorname{sgn}(\alpha)$ for retrieving $\mathcal{A}[\mathbf{A}]$. Figure 1 illustrates this phenomenon clearly.

3.2. Unique polar representation

Since it is reasonable to assume a continuous envelope $a_i(t)$, we can easily retrieve the envelope component by component using a piecewise recovery algorithm proposed in [1]. After we obtain the axis $\mathcal{A}[\mathbf{A}]$, we can directly retrieve the quaternionic envelope together with the unknown α

$$\mathbf{A} = \|\mathbf{A}\|_{\ell_2}\mathcal{A}(\mathbf{A}) = \|\mathbf{S}\|_{\ell_2}\mathcal{A}(\mathbf{A}); \tag{9}$$

$$\alpha = \frac{\mathbf{C}}{\mathbf{A}}. \tag{10}$$

Connecting with (7), we can obtain the other unknowns β, γ, η, and ξ by solving the following system of equations

$$\begin{cases} a_1\beta - a_2\gamma - a_3\eta - a_4\xi = s_5 \\ a_1\gamma + a_2\beta - a_3\xi + a_4\eta = s_6 \\ a_1\eta + a_2\xi + a_3\beta - a_4\gamma = s_7 \\ a_1\xi - a_2\eta + a_3\gamma + a_4\beta = s_8 \end{cases}. \tag{11}$$

Or we can obtain $e^{\mathbf{\Phi}\mathbf{e}_4}$ in an alternative way

$$e^{\mathbf{\Phi}\mathbf{e}_4} := \frac{\bar{\mathbf{A}}\mathbf{S}}{\|\mathbf{S}\|_{\ell_2}^2}. \tag{12}$$

Now, recalling the natural logarithm definition in (2), we can retrieve the quaternionic phase by

$$\mathbf{\Phi} = \mathcal{A}[\beta\mathbf{e}_0 + \gamma\mathbf{e}_1 + \eta\mathbf{e}_2 + \xi\mathbf{e}_3]\arccos(\alpha). \tag{13}$$

In practice, when we consider signals, each component of the octonionic phase signal $\mathbf{\Phi}(t)$ should be monotonically non-decreasing. Fortunately, this requirement can be satisfied by using a simple unwrapping method that introduced in [1]. Afterwards, we can modify the theorem in [1] as

Theorem 3.1 *Given a quaternionic signal* $\mathbf{G}(t) : [0,1] \to \mathbb{C}_4$*, the VHaS can be constructed by* $\mathbf{S}(t) := \mathbf{G}(t) + \mathcal{H}_{\mathbf{e}_4}^{\mathbb{C}_8}[\mathbf{G}(t)]\mathbf{e}_4$*,* $\mathbf{S}(t) \in \mathbb{C}_8$*, which has a unique polar form* $\mathbf{S}(t) = \mathbf{A}(t)e^{\mathbf{\Phi}(t)\mathbf{e}_4}$*,* $\mathbf{A}(t), \mathbf{\Phi}(t) \in \mathbb{C}_4$*, if* $(\mathbf{A}(t), \mathbf{\Phi}(t))$ *is the canonical quaternion pair where* $\mathbf{A}(t)$ *has a unique polar form, and* $\mathbf{\Phi}(t) := \sum_{i=1}^{4}\phi_i(t)\mathbf{e}_{i-1}$*,* $\phi_i(t) \geq 0$*,* $\|\mathbf{\Phi}(0)\|_{\ell_2} \in [0, 2\pi)$*.*

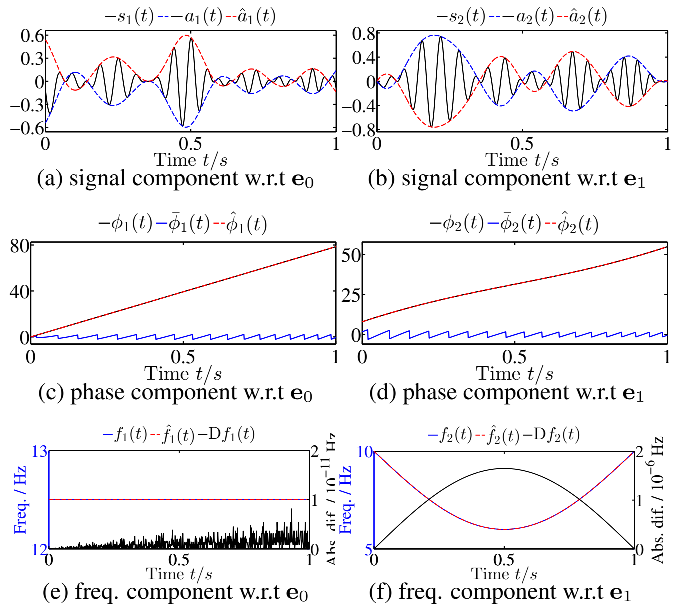
3.3. Time-frequency-amplitude (TFA) spectrum

After we obtain the unique polar form of the given hypercomplex signal, we can derive the corresponding intantaneous hypercomplex frequency together with the TFA spectrum.

Definition 3.2 *Given a quaternionic signal* $\mathbf{G}(t) : [0,1] \to \mathbb{C}_4$ *and its VHaS in a polar form* $\mathbf{S}(t) := \mathbf{A}(t)e^{\mathbf{\Phi}(t)\mathbf{e}_4} := \|\mathbf{S}(t)\|_{\ell_2}\mathcal{A}[\mathbf{A}](t)e^{\sum_{i=1}^{4}\phi_i(t)\mathbf{e}_{i+3}}, \phi_i(t) \geq 0$*. The instantaneous quaternionic frequency of* $\mathbf{G}(t)$ *is defined by*

$$\mathbf{F}_{\mathbf{G}}(t) := \sum_{i=1}^{4} f_i(t)\mathbf{e}_{i-1} = \frac{1}{2\pi}\sum_{i=1}^{4}\frac{d(\phi_i(t))}{dt}\mathbf{e}_{i-1}. \tag{14}$$

The TFA spectrum can be defined by $\{\|\mathbf{A}(t)\|_{\ell_2}$ *on the curve* $(t, f_1(t), f_2(t), f_3(t), f_4(t)), t \in [0,1]\}$*.*



(a) signal component w.r.t $\mathbf{e}_0$ (b) signal component w.r.t $\mathbf{e}_1$

(c) phase component w.r.t $\mathbf{e}_0$ (d) phase component w.r.t $\mathbf{e}_1$

(e) freq. component w.r.t $\mathbf{e}_0$ (f) freq. component w.r.t $\mathbf{e}_1$

Fig. 2. Paritial numerical results for the VHaS $\mathbf{S}(t) \in \mathbb{C}_8$. (a) & (b): the subcomponent $s_1(t), s_2(t)$ of the given signal $\mathbf{G}(t) \in \mathbb{C}_4$ satisfying $\mathbf{S}(t) = \mathbf{G}(t) + \mathcal{H}_{\mathbf{e}_4}^{\mathbb{C}_8}[\mathbf{G}](t)$; negating the theoretical envelope $-a_1(t), -a_2(t)$, and the recovered ones $\hat{a}_1(t), \hat{a}_2(t)$; (c) & (d) the theoretical phase component $\phi_1(t), \phi_2(t)$, the recovered ones $\bar{\phi}_1(t), \bar{\phi}_2(t)$ without unwrapping, and the recovered ones with unwrapping $\hat{\phi}_1(t), \hat{\phi}_2(t)$; (e) & (f): the theoretical instantaneous frequency $f_1(t), f_2(t)$, the calculated ones based on the proposed method $\hat{f}_1(t), \hat{f}_2(t)$, and the absolute difference between them, $\mathrm{D}f_1(t) := |f_1(t) - \hat{f}_1(t)|$, $\mathrm{D}f_2(t) := |f_2(t) - \hat{f}_2(t)|$ (Color online).

Remark 3.3 *Althought we only introduced the unique polar representation of a given VvS* $G(t) \in \mathbb{R}^4$ *or a quaternionic signal* $\mathbf{G}(t) \in \mathbb{C}_4$*, the VvS* $G(t) \in \mathbb{R}^3$ *can also be analyzed by supplement one more component of zeros, e.g. setting* $g_4(t) = 0$*. Afterwards, we can omit the last component of the obtained envelope, phase and instantaneous frequency, and then reduce the dimension to 3.*

4. NUMERICAL STUDY

To test the proposed method for time-frequency analysis of a VHcS signal, we design a representative VHaS signal model

$$\begin{aligned} \mathbf{S}(t) &:= \mathbf{A}(t)e^{\mathbf{\Phi}(t)\mathbf{e}_4}, \qquad t \in [0,1], \\ \mathbf{A}(t) &:= e^{-t}e^{2\sin(2\pi t)\mathbf{e}_1}e^{(4\pi t\mathbf{e}_0 + (2\pi t + \cos(2\pi t))\mathbf{e}_1)\mathbf{e}_2}, \\ \mathbf{\Phi}(t) &:= 25\pi t\mathbf{e}_0 + (20\pi t + 8\cos(\pi t))\mathbf{e}_1 \\ &\quad + (15\pi t - 8\sin(\pi t))\mathbf{e}_2 + (10\pi t + e^{3t})\mathbf{e}_3, \end{aligned} \tag{15}$$

in which $\mathbf{A}(t)$ is a quaternionic signal that can be presented in a polar form. From the model we can derive the instantaneous quaternionic frequency as

$$\begin{aligned}\mathbf{F}(t) = &\frac{25}{2}\mathbf{e}_0 + (10 - 4\sin(\pi t))\mathbf{e}_1 \\ &+ (\frac{15}{2} - 4\cos(\pi t))\mathbf{e}_2 + (5 + \frac{3}{2\pi}e^{3t})\mathbf{e}_3.\end{aligned} \tag{16}$$

To save the space, Fig. 2 illustrates part of the numerical results by using the proposed method. In sub-figures (a) and (b), the recovered envelope $\hat{a}_i(t)$ coincides strongly with the ideal one $a_i(t)$. Sub-figures (c) and (d) illustrate the importance of the phase unwrapping, while sub-figures (e) and (f) imply the efficiency of the instantaneous frequency calculation. Since the first component $f_1(t)$ is a constant, the absolute difference between it and the calculated one $\hat{f}_1(t)$ is around machine accuracy. The absolute difference between $f_2(t)$ and $\hat{f}_2(t)$ is larger since $f_2(t)$ is nonlinear and thus the corresponding accuracy is corrupted by the discrete derivative computation at different time positions.

5. CONCLUSION

We successfully extend the VHaS theory into multi-D space, and illustrate the corresponding unique polar representation. Based on the proposed VHaS model, we can obtain a canonical pair of continuously instantaneous envelope and phase, in which the phase consists of monotonically non-decreasing sub-components that leads to a natural definition of the instantaneous frequency. Although we only studied the VHaS model with the dimension $d \leq 8$, we have to mention that the model with $d > 8$ can be also analyzed by introducing larger X-on hypercomplex number. Moreover, the way to define hyperanalytic signal together with its unique polar representation will be similar to the method proposed in this paper.

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