On Cyclic Spectrum Estimation with Estimated Cycle Frequency

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***Abstract*—The problem of cyclic spectrum estimation for almost-cyclostationary processes with unknown cycle frequencies is addressed. This problem arises in spectrum sensing and source location algorithms in the presence of relative motion between transmitter and receiver. Sufficient conditions on the process and the cycle frequency estimator are derived such that frequency-smoothed cyclic periodograms with estimated cycle frequencies are mean-square consistent and asymptotically jointly complex normal. Under the same conditions, the asymptotic complex normal law is shown to coincide with the normal law of the case of known cycle frequencies. Monte Carlo simulations corroborate the effectiveness of the theoretical results.**

***Index Terms*—Cyclostationarity; Asymptotic Normality; Doppler effect**

I. Introduction

Second-order almost-cyclostationary (ACS) processes in the wide sense are an appropriate model for signals encountered in several fields of application including communications, radar, telemetry, acoustics, mechanics, biology, econometrics, climatology, and astronomy. For these processes, underlying periodicities are hidden by the mixture or composition with random phenomena and the autocorrelation function is almost-periodic in time. Its Fourier series expansion has frequencies and coefficients referred to as cycle frequencies and cyclic autocorrelation functions, respectively. The Fourier transform of the cyclic autocorrelation function at a given cycle frequency is called cyclic spectrum. It represents the density of spectral correlation between two spectral components separated by a quantity equal to the cycle frequency [8], [16].

Cyclostationarity-based signal processing algorithms are signal selective since cycle frequencies are related to the frequencies of the underlying periodicities that are characteristic of each ACS process. These algorithms allow to extract characteristics of the signal-of-interest (SOI) even if it is embedded in noise and interference, provided that the observation interval is sufficiently large. In fact, cyclostationarity-based algorithms benefit from the existence of reliable estimators of the cyclic statistical functions, provided that the process fulfills very mild conditions on its finite or practically finite memory and the cycle frequency of interest is exactly known. In particular, the frequency-smoothed cyclic periodogram is a mean-square consistent and asymptotically complex normal estimator of the cyclic spectrum [4], [5].

This work is partially supported by CNIT (AIBER project).

In many cases of interest the underlying periodicities are not exactly known and, consequently, the cycle frequencies of the SOI are known with some degree of uncertainty. In communications, this is the case of non cooperative signal detection and estimation, or in the presence of Doppler effect [15]. In human biology, periods of circadian rhythms and hormonal cycles are not exactly known and can vary from different individuals and, for the same individual, in different time periods.

An error in the knowledge of the cycle frequency puts a limit to the finest cycle frequency resolution of the signal processing algorithm. This resolution is of the order of the reciprocal of the data-record length. Consequently, the cycle frequency uncertainty puts an upper limit to the maximum data-record length and, hence, to the minimum signal-to-noise ratio (SNR) or signal-to-interference ratio (SIR) for which satisfactory performance can be achieved.

In this paper, the problem of cyclic spectral analysis is treated in the case of uncertain cycle frequency. Specifically, the frequency-smoothed cyclic periodogram is considered as estimator of the cyclic spectrum when an estimate of the cycle frequency is adopted in place of its exact value. Sufficient conditions are found on the cycle frequency estimate such that the frequency-smoothed cyclic periodogram with estimated cycle frequency is mean-square consistent and asymptotically complex normal. A joint characterization of frequency-smoothed cyclic periodograms at different cycle frequencies is also provided. Moreover, it is shown that asymptotically, under the derived conditions, frequency-smoothed cyclic periodograms with known cycle frequencies and those with estimated cycle frequencies have the same joint complex normal distribution. The fulfillment of these conditions for existing cycle frequency estimators is discussed.

Results of extensive Monte Carlo simulations show the effectiveness of the derived theoretical results.

A notable application of the derived results consists in extending to the case of unknown cycle frequencies the structures of detectors and signal-parameter estimators based on cyclic spectrum estimates with known cycle frequencies. In particular, detectors [10], [16, Sec. 6], signal classification algorithms [1], [2], [16, Sec. 8], and source location and

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channel identification algorithms [8, Secs. 9, 10] designed for the absence of motion between transmitter and receiver, can be considered in the presence of Doppler effect by replacing, into the cyclic spectrum estimators, the known cycle frequency value with a sufficiently accurate estimate satisfying the conditions derived in the paper.

The paper is organized as follows. In Section II, the statistical characterization of ACS processes and the estimation of the cyclic spectrum with known cycle frequency are briefly reviewed to introduce notation and for subsequent reference. In Section III, the needed assumptions are made and the mean-square consistency and joint asymptotic complex normality of frequency-smoothed cyclic periodograms with estimated cycle frequencies is proved. Numerical results are presented in Section IV. Conclusions are drawn in Section V.

II. Almost-Cyclostationary Processes

A. Statistical Characterization

A second-order complex-valued stochastic process $x(t)$ is said to be almost-cyclostationary in the wide-sense if its first and second-order moments are almost-periodic functions of t [8]. That is,

$$\mathrm{E}\big\{x(t+\tau)\,x^{(*)}(t)\big\} = \sum_{\alpha\in A} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{j2\pi\alpha t} \tag{2.1}$$

where superscript $(*)$ denotes an optional complex conjugation. The function in (2.1) is the autocorrelation function if $(*)$ is present and the conjugate autocorrelation function if $(*)$ is absent. In (2.1), A is the countable set, depending on $(*)$, of the possibly incommensurate (conjugate) cycle frequencies α, and the Fourier coefficients

$$R_{\boldsymbol{x}}^{\alpha}(\tau) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{E}\big\{x(t+\tau)x^{(*)}(t)\big\}e^{-j2\pi\alpha t}\mathrm{d}t \tag{2.2}$$

where subscript $\boldsymbol{x} \triangleq [xx^{(*)}]$, are referred to as *(conjugate) cyclic autocorrelation functions*.

Let

$$X_Z(t_0,f) \triangleq \int_{t_0-Z/2}^{t_0+Z/2} x(t)\, e^{-j2\pi f t}\mathrm{d}t \tag{2.3}$$

be the short-time Fourier transform (STFT) of $x(t)$. The *(conjugate) cyclic spectrum* is defined as

$$S_{\boldsymbol{x}}^{\alpha}(f) \triangleq \lim_{\Delta f\to 0}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\Delta f\,\mathrm{E}\big\{X_{\frac{1}{\Delta f}}(t,f) \cdot X_{\frac{1}{\Delta f}}^{(*)}(t,{}_{(-)}(\alpha-f))\big\}\mathrm{d}t \tag{2.4}$$

where ${}_{(-)}$ is an optional minus sign linked to the optional conjugation $(*)$. It is the Fourier transform of $R_{\boldsymbol{x}}^{\alpha}(\tau)$ and is non zero only for $\alpha\in A$. That is, when $(*)$ is present, spectral components of $x(t)$ separated by quantities equal to a cycle frequency α are correlated.

B. Cyclic Spectrum Estimation

Let $x(t)$ be a zero-mean process satisfying the following assumptions, where z_1 and z_2 denote any of x or x^*.

AS1. All the 2nd- and 4th-order cumulants of z_1 and z_2 are uniformly almost-periodic functions of t for all values of the lag parameters [11, As. 3.1], [12, As. 2.4.2], with summable sequences of the suprema of the Fourier coefficients [11, As. 3.2, 3.3], [12, As. 2.4.3]. □

AS2. For every k, the kth-order absolute cross-moments of $x(t+\tau_1),\dots,x(t+\tau_k)$ are bounded [11, As. 5.2], [12, As. 2.4.16]. □

AS3. For every k, the kth-order cumulants are almost-periodic functions of t, summable with respect to the lag parameters [11, As. 5.1], [12, As. 2.4.15]. □

AS4. There is no cluster (accumulation point) of (conjugate) cycle frequencies except possibly $\pm\infty$ [11, As. 4.4], [12, As. 2.4.10]. □

Assumptions AS1–AS4 are verified by almost all man-made modulated signals adopted in communications, radar, sonar, and telemetry.

AS5. Let $w_T(t) = a(t/T)/T$ be the lag-product tapering window, with $a(t)$ bounded, summable, with unit area, compact support $[-1/2, 1/2]$, and Fourier transform $A(f)$ with rate of decay to zero $\mathcal{O}(|f|^{-r})$, $r \geqslant 1$, as $|f|\to\infty$ [11, As. 3.5], [12, As. 2.4.5]. □

AS6. Let be $q(\tau)\in L^1(\mathbb{R})$ and its Fourier transform $Q(f)\in L^1(\mathbb{R})$ be continuous almost everywhere (a.e.) and such that $q(0)=\int_{\mathbb{R}} Q(f)\,\mathrm{d}f = 1$ [14, As. 3.2]. □

AS7. The (conjugate) cyclic spectra $S_{\boldsymbol{x}}^{\alpha}(f)$ are summable [14, As. 3.3]. □

Let

$$I_{\boldsymbol{x}}^{(T)}(\alpha,f) \triangleq \frac{1}{T}X_T(t_0,f)\,X_T^{(*)}(t_0,{}_{(-)}(\alpha-f)) \tag{2.5}$$

be the *(conjugate) cyclic periodogram* at (conjugate) cycle frequency α, where $X_T(t_0,f)$ is the STFT of $x(t)$ defined according to (2.3).

The frequency-smoothed (conjugate) cyclic periodogram is defined as [7, Chap. 13]

$$S_{\boldsymbol{x}}^{(T,\Delta f)}(\alpha,f) \triangleq I_{\boldsymbol{x}}^{(T)}(\alpha,f) \underset{f}{\otimes} \frac{1}{\Delta f}Q\Big(\frac{f}{\Delta f}\Big) \tag{2.6a}$$

$$= \int_{\mathbb{R}} R_{\boldsymbol{x}}^{(T)}(\alpha,\tau)\, q(\Delta f\,\tau)\, e^{-j2\pi f\tau}\mathrm{d}\tau \tag{2.6b}$$

where $R_{\boldsymbol{x}}^{(T)}(\alpha,\tau)$ is the (conjugate) cyclic correlogram of $x(t)$ [8, eq. (4.1)], [16, eq. (3.1)], $\otimes_f$ denotes convolution with respect to f, and $Q(f/\Delta f)/\Delta f$ is the frequency-smoothing window with bandwidth Δf and inverse Fourier transform $q(\Delta f\,\tau)$.

Theorem 1. Under Assumptions AS1–AS7, the frequency-smoothed (conjugate) cyclic periodogram (2.6a), (2.6b) is a mean-square consistent estimator of the (conjugate) cyclic spectrum as $T \to \infty$ and $\Delta f \to 0$ with $T\Delta f \to \infty$. Moreover, the random variables

$$U_i^{(T,\Delta f)} \triangleq \sqrt{T\Delta f}\big[S_{\boldsymbol{x}}^{(T,\Delta f)}(\alpha_i, f_i) - S_{\boldsymbol{x}}^{\alpha_i}(f_i)\big] \quad i = 1,\dots,M \tag{2.7}$$

are asymptotically jointly complex normal [4], [5], [7, Chap. 15], [12, Sec. 4.7]. □

Cycle frequency estimators based on (conjugate) cyclic correlograms and (conjugate) frequency-smoothed cyclic periodograms are proposed in [3], [6], and [9]. See [16, Sec. 7] for a review.

III. Frequency-Smoothed Cyclic Periodogram with Estimated Cycle Frequency

AS8. Let $\alpha^{(T)}$ be an estimate of the (conjugate) cycle frequency α_0 that converges almost surely (a.s.) to α_0 with rate T^{-b}, with $b \geqslant 0$. □

AS9. The estimation error $\Delta\alpha^{(T)} \triangleq \alpha^{(T)} - \alpha_0$ is zero mean and such that, as $T \to \infty$

$$\left|\mathrm{E}\{(\Delta\alpha^{(T)})^n\}\right| \leqslant \frac{c_n}{T^{n(1+\mu)}} \tag{3.1}$$

with $\mu > 0$ and $c_n > 0$ satisfying

$$\sum_{n=1}^{+\infty} \frac{a^n}{n!}\, c_n < \infty \tag{3.2}$$

for some $a > 0$. □

AS10. Let $\alpha_i^{(T)}$ be estimates of M different (conjugate) cycle frequencies $\alpha_{0,i}$, $i = 1,\dots,M$, such that

$$\lim_{T\to\infty} \alpha_i^{(T)} = \alpha_{0,i} \qquad \text{almost surely} \tag{3.3}$$

where estimates $\alpha_i^{(T)}$ are made starting from (conjugate) cyclic correlograms $R_{\boldsymbol{x}}^{(T)}(\alpha,\tau)$, $\alpha \in D_i$, in non overlapping cycle frequency intervals D_i, that is, $D_i \bigcap D_j = \varnothing$ for $i \neq j$. □

In the following, the main result of the paper is stated.

Theorem 2. Under Assumptions AS1–AS7, AS9 (with $\mu > 1/3$) for every $\Delta\alpha_i^{(T)} = \alpha_i^{(T)} - \alpha_{0,i}$, and AS10, the random variables

$$U_i^{(T,\Delta f)} \triangleq \sqrt{T\Delta f}\Big[S_{\boldsymbol{x}}^{(T,\Delta f)}(\alpha_i^{(T)}, f_i) - S_{\boldsymbol{x}}^{\alpha_{0,i}}(f_i)\Big] \quad i = 1,\dots,M \tag{3.4}$$

are asymptotically ($T \to \infty$ and $\Delta f \to 0$ with $T\Delta f \to \infty$) zero-mean jointly complex normal with asymptotic covariance matrix with entries $\overline{\Sigma}_{ij}$ and asymptotic conjugate covariance matrix with entries $\overline{\Sigma}_{ij}^{(c)}$, having the same expressions (see [12, eq. (4.143)] for the covariance) as the case of known (conjugate) cycle frequencies (Theorem 1).

Proof: The proof is only sketched here due to lack of space. See [14] for a complete proof.

Starting from the mean-square consistency of the cyclic correlogram [13, Theorem 3.3] and using the Fourier transform (2.6b), as $T \to \infty$ and $\Delta f \to 0$ with $T\Delta f \to \infty$, we have

1) $\mathrm{E}\{U_i^{(T,\Delta f)}\} \to 0$;

2a) $\mathrm{cov}\{U_i^{(T,\Delta f)}, U_j^{(T,\Delta f)}\} \to \overline{\Sigma}_{ij}$ finite;

2b) $\mathrm{cov}\{U_i^{(T,\Delta f)}, U_j^{(T,\Delta f)*}\} \to \overline{\Sigma}_{ij}^{(c)}$ finite.

In 2a) and 2b), the proof of cases $i = j$ and $i \neq j$ need to be treated separately. In fact, for $i \neq j$, according to Assumption AS10, the asymptotic independence of $\alpha_i^{(T)}$ and $\alpha_j^{(T)}$ is exploited.

Then, using the asymptotic result [11, eq. (C2)], [12, eq. (3.126)]

$$\mathrm{cum}\{\sqrt{T} R_{\boldsymbol{x}}^{(T)}(\alpha_i, \tau_i),\ i = 1,\dots,k\} = \mathcal{O}(T^{-k/2+1}) \tag{3.5}$$

and (2.6b) we have

3) $\mathrm{cum}\{U_1^{(T,\Delta f)[*]_1}, \dots, U_k^{(T,\Delta f)[*]_k}\} \to 0$ for $k \geqslant 3$

where superscript $[*]_i$ denotes ith optional complex conjugation. Define $\boldsymbol{U}^{(T,\Delta f)} \triangleq [U_1^{(T,\Delta f)[*]_1}, \dots, U_k^{(T,\Delta f)[*]_k}]$ and $\boldsymbol{\omega} \triangleq [\omega_1, \dots, \omega_k]$. The cumulant of the complex random variables $U_i^{(T,\Delta f)[*]_i}$, $i = 1,\dots,k$, defined according to [11, App. E], [12, Sec. 1.4.2], [18, App. A] is

$$\begin{aligned}\mathrm{cum}&\Big\{U_1^{(T,\Delta f)[*]_1}, \dots, U_k^{(T,\Delta f)[*]_k}\Big\} \\ &\triangleq (-j)^k \frac{\partial^k}{\partial\omega_1 \dots \partial\omega_k} \log \mathrm{E}\Big\{e^{j\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{U}^{(T,\Delta f)}}\Big\}.\end{aligned} \tag{3.6}$$

Thus, accounting for items 1–3, asymptotically, the logarithm of the joint characteristic function of $U_i^{(T,\Delta f)[*]_i}$, $i = 1,\dots,k$ is a quadratic form in the ω_k's. That is, the $U_i^{(T,\Delta f)}$ are jointly complex normal. □

As a corollary, from item 1 we have that the frequency-smoothed (conjugate) cyclic periodogram with estimated (conjugate) cycle frequency is asymptotically unbiased with rate of decay to zero of the bias faster than $(T\Delta f)^{-1/2}$ and from item 2 we have that its variance is $\mathcal{O}((T\Delta f)^{-1})$. Therefore, it is a mean-square consistent estimator of the (conjugate) cyclic spectrum. Note that the only mean-square consistence can be proved under the less restrictive Assumptions AS1–AS7 and AS8 with $b = 1$ [14].

IV. Numerical Results

In this section, numerical results are reported aimed at corroborating the theoretical results (Theorem 2) on the mean-square consistency and asymptotic complex normality of the frequency-smoothed cyclic periodogram with estimated cycle frequency. A pulse-amplitude modulated (PAM) signal $x(t)$ with binary stationary white modulating sequence, raised cosine pulse with excess bandwidth $\eta = 0.35$, and bit period $T_p = 8T_s$, with $T_s = 1/f_s$ the sampling period, is considered.

By denoting by $\widehat{\mathrm{E}}\{U\}$ the sample mean of the random variable U and by $U^{(T,\Delta f)}$ any of the random variables $U_i^{(T,\Delta f)}$ in (3.4), the sample variance

$$\sigma^{2(T,\Delta f)}(\alpha, f) \triangleq \widehat{\mathrm{E}}\Big\{\Big|U^{(T,\Delta f)}(\alpha, f) - \widehat{\mathrm{E}}\big[U^{(T,\Delta f)}(\alpha, f)\big]\Big|^2\Big\} \tag{4.1}$$

and the sample second-order moment

$$\gamma^{(T,\Delta f)}(\alpha, f) \triangleq \widehat{\mathrm{E}}\Big\{\Big(U^{(T,\Delta f)}(\alpha, f) - \widehat{\mathrm{E}}\big[U^{(T,\Delta f)}(\alpha, f)\big]\Big)^2\Big\} \tag{4.2}$$

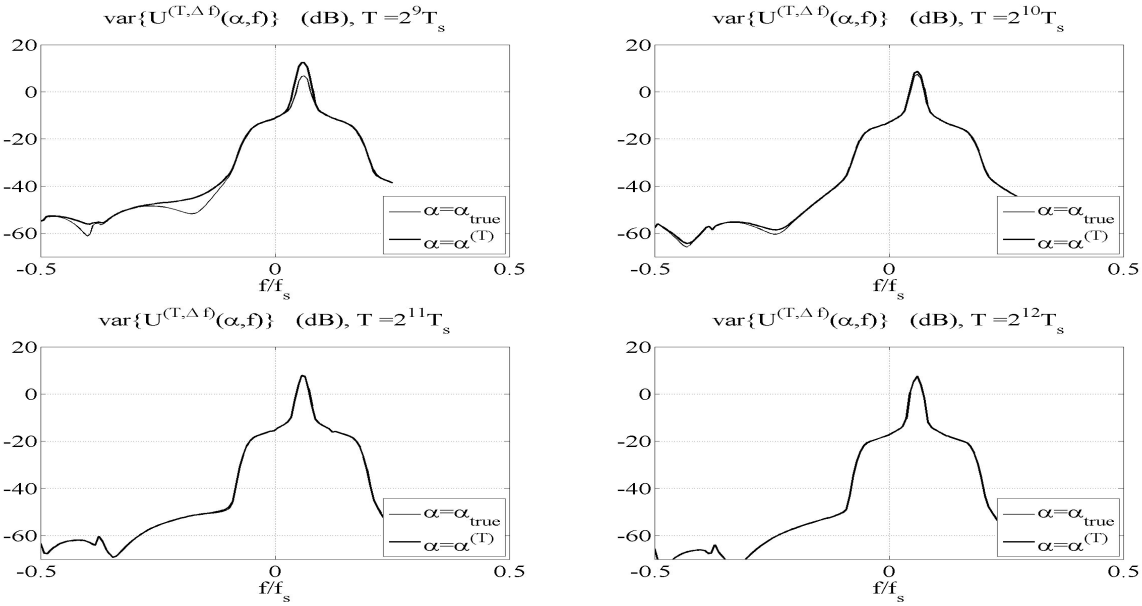

Fig. 1. Sample variance as function of f for different values of T: (Thin line) $U^{(T,\Delta f)}(\alpha_{\text{true}}, f)$ and (thick line) $U^{(T,\Delta f)}(\alpha^{(T)}, f)$.

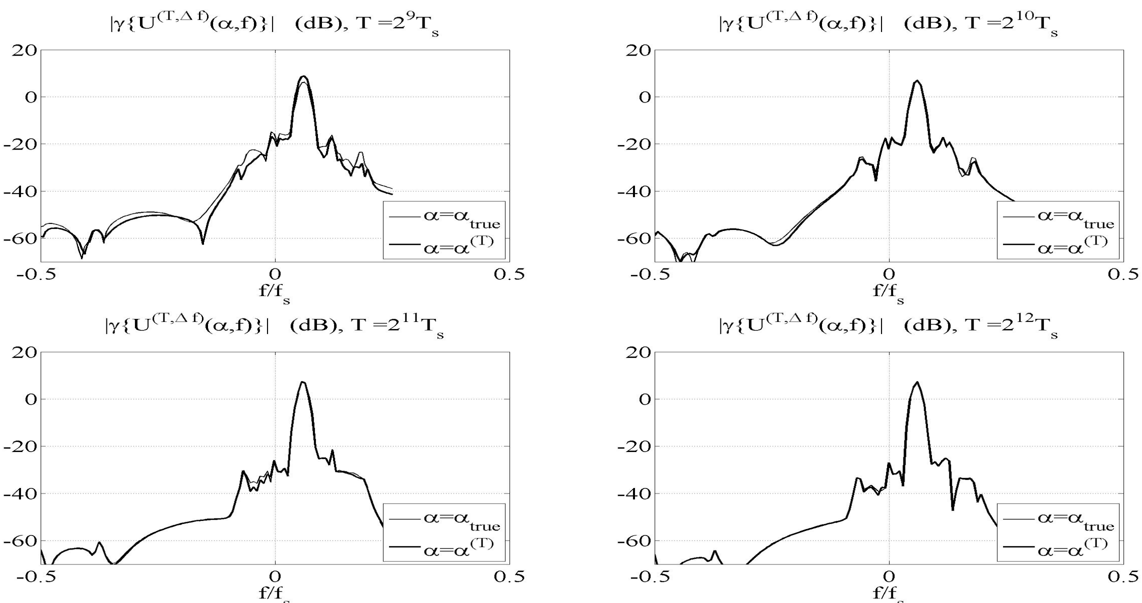

Fig. 2. Magnitude of the sample second-order moment as function of f for different values of T: (Thin line) $U^{(T,\Delta f)}(\alpha_{\text{true}}, f)$ and (thick line) $U^{(T,\Delta f)}(\alpha^{(T)}, f)$.

are evaluated as functions of f by 10^4 Monte Carlo trials for data-record lengths $T = 2^n T_s$ and frequency-smoothing window widths $\Delta f = 2^{-(1+n/2)} f_s$, with $n = 9, \ldots, 12$. Both $\sigma^{2(T,\Delta f)}$ and $\gamma^{(T,\Delta f)}$ are needed to characterize a non circular complex normal distribution [17]. Accordingly with the results of Theorem 2, when ideally $n \to \infty$, we have that $T \to \infty$, $\Delta f \to 0$ and $T\Delta f = 2^{n/2-1} \to \infty$.

The two cases $\alpha = \alpha_{\text{true}} = 1/T_p$ and $\alpha = \alpha^{(T)}$with $\alpha^{(T)}$ the estimate of [3] are compared. The estimate of [3] is asymptotically normal with error variance (and moments) satisfying Assumption AS9 with $\mu = 1/2$ and $\forall a \in \mathbb{R}$ [14]. According to the results of Theorem 2, we have that the functions $\sigma^{2(T,\Delta f)}(\alpha, f)$ (Fig. 1) and $\gamma^{(T,\Delta f)}(\alpha, f)$ (Fig. 2) with $\alpha = \alpha^{(T)}$ become closer and closer to the analogous functions with $\alpha = \alpha_{\text{true}}$.

Specifically, let $\theta^{(T,\Delta f)}(\alpha, f)$ denote any of the functions $\sigma^{2(T,\Delta f)}(\alpha, f)$ or $\gamma^{(T,\Delta f)}(\alpha, f)$, and let us define the normalized root mean-squared error (NRMSE)

$$\text{NRMSE}_\theta \triangleq \left[\int_{\mathcal{B}} \left| \theta^{(T,\Delta f)}(\alpha^{(T)}, f) - \theta^{(T,\Delta f)}(\alpha_{\text{true}}, f) \right|^2 \mathrm{d}f \right]^{\frac{1}{2}}$$

$$\cdot \left[\int_{\mathcal{B}} \left| \theta^{(T,\Delta f)}(\alpha_{\mathrm{true}}, f) \right|^2 \mathrm{d}f \right]^{-\frac{1}{2}} \tag{4.3}$$

with $\mathcal{B} = [-f_s/2, f_s/2]$. In Table I, the NRMSE for $\sigma^{2(T,\Delta f)}$ and $\gamma^{(T,\Delta f)}$ is shown to be decreasing for increasing values of the data-record length T and decreasing values of Δf. The results corroborate the convergence properties proved in Theorem 2.

T/T_s	$\Delta f/f_s$	$\mathrm{NRMSE}_{\sigma^2}$	NRMSE_{γ}
2^9	$2^{-5.5}$	2.361	0.801
2^{10}	2^{-6}	0.330	0.094
2^{11}	$2^{-6.5}$	0.069	0.078
2^{12}	2^{-7}	0.027	0.033

TABLE I
NRMSE FOR $\sigma^{2(T,\Delta f)}$ AND $\gamma^{(T,\Delta f)}$ FOR INCREASING VALUES OF THE DATA-RECORD LENGTH T AND DECREASING VALUES OF THE FREQUENCY-SMOOTHING WINDOW-WIDTH Δf.

Finally, in Fig. 3, the marginal empirical CDF of the imaginary part of $U^{(T,\Delta f)}$, with $f/f_s = 0.075$ and $T = 2^{10}T_s$, normalized to its sample variance $\sigma_{\mathrm{I}}^{(T,\Delta f)}$, is reported for known cycle frequency $\alpha = \alpha_{\mathrm{true}}$ (dotted line) and estimated cycle frequency $\alpha = \alpha^{(T)}$ (solid thick line) and is compared with the CDF of a standard normal random variable (solid thin line). The CDF for $\alpha = \alpha^{(T)}$ closely follows the CDF for $\alpha = \alpha_{\mathrm{true}}$ and the standard normal distribution.

Simulation results not reported here show that satisfactory performance can be obtained by exploiting the CDP [9] cycle frequency estimator.

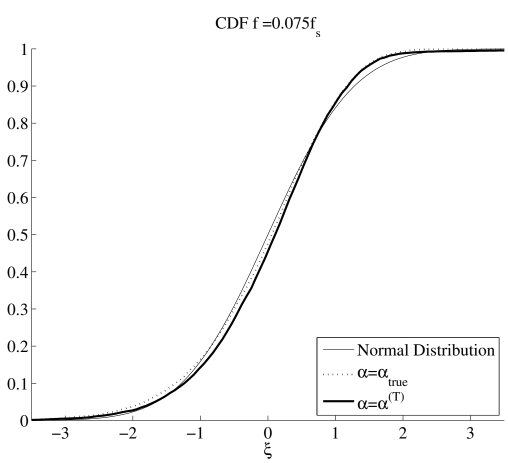

Fig. 3. Marginal empirical CDF of the imaginary part of $U^{(T,\Delta f)}$, with $f/f_s = 0.075$, normalized to its sample variance $\sigma_{\mathrm{I}}^{(T,\Delta f)}$, for known cycle frequency $\alpha = \alpha_{\mathrm{true}}$ (dotted line) and estimated cycle frequency $\alpha = \alpha^{(T)}$ (solid thick line) compared with the CDF of a standard normal random variable (solid thin line).

V. CONCLUSION

Sufficient conditions are derived for almost-cyclostationary processes and for cycle frequency estimators such that frequency-smoothed cyclic periodograms with estimated cycle frequencies are mean-square consistent and asymptotically jointly complex normal. If the nth-order moment of the cycle frequency estimation error is $\mathcal{O}(T^{-n(1+\mu)})$ with $\mu > 1/3$, then frequency-smoothed cyclic periodograms with estimated and known cycle frequencies have the same complex normal distribution. The derived results allow one to extend to the case of relative motion between transmitter and receiver most of the signal processing algorithms based on cyclic spectrum measurements and designed for the case of no motion between transmitter and receiver.

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