CLOSED-FORM SOLUTION FOR TDOA-BASED JOINT SOURCE AND SENSOR LOCALIZATION IN TWO-DIMENSIONAL SPACE

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ABSTRACT

In this paper, we propose a closed-form solution for *time-difference-of-arrival* (TDOA) based joint source and sensor localization in two-dimensional space (2D). This closed-form solution is a combination of two closed-form solutions for *time-of-arrival information recovery* and *time-of-arrival (TOA)-based joint source and sensor localization* in 2D. In our previous works, we derived closed-form solutions for TOA-based joint source and sensor localization and near-closed-form solutions for TOA information recovery in three-dimensional space (3D). Since the localization in 2D is simpler than that in 3D, closed-form solutions for both problems in 2D are derived in this paper. The root-mean-square errors (RMSEs) achieved by the proposed closed-form solution are compared with the Cramér-Rao lower bound (CRLB) in synthetic experiments. The results show that the proposed solution works well in both low-noise and noisy cases and with both small and large numbers of sources and sensors.

Index Terms— Time Difference of Arrival, Time of Arrival, Joint Source and Sensor Localization

1. INTRODUCTION

With the rapid development of industrial technology and because ad hoc microphone arrays have a wide range of applications, joint source and sensor localization has recently received significant attention from the scientific community.

Fundamentally, joint source and sensor localization can be classified into three cases: (i) *time-of-arrival (TOA)-based localization*, (ii) *time-difference-of-arrival (TDOA)-based localization*, and (iii) *asynchronous observation-based localization*. To obtain TOA information, the sources and sensors must be synchronous, whereas to obtain TDOA information, synchronization among the sources or among the sensors is only required. If both the sources and the sensors are asynchronous, the problem is more difficult. In this paper, we focus on the second problem, *TDOA-based joint source and sensor localization*. When the sources and sensors are and are not in the same plane, the localization is called localization in 2D and in 3D, respectively.

Since the objective function in the joint localization problem is non linear and has many local minima, closed-form solutions are very important. Recently, some closed-form solutions have been derived. For example, closed-form solutions for TOA-based localization can be derived by the *multidimensional scaling* (MDS) method [1, 2] when for any source there is a sensor very close to it. In this case TOA measurements can provide the distances between any pair of sensors (or sources). The mathematical properties of the localization in this case were studied in [3], and an excellent review of its applications was given in [4]. Crocco et al. [5] proposed an attractive method for deriving a closed-form solution for TOA-based localization with the addition of a weak condition that one source and one sensor must be very close. Kuang and coworkers [6, 7] proposed non-iterative methods, rather than closed-form or near-closed-form solutions, for TOA-based and TDOA-based localizations without adding any conditions. Their results provide a good mathematical perspective for these localization problems. On the basis of their results, in our previous works, we proposed closed-form solutions for TOA-based localization [8, 9] and near-closed-form solutions for TDOA-based localization [10]. An experimental evaluation of TOA-based localization [11] showed that the solutions given in [8, 9] worked very well. All the above methods are for localizations in 3D, and are based on the *low-rank property* of a TOA-distance matrix. However, this property is different in 3D and 2D, hence these methods cannot be used directly in 2D. For example, based on the techniques given in [6], Burgess et al. [12] studied TOA-based localizations when the sources or sensors are in 2D because the methods given in [6] do not work for these cases.

To our knowledge, although localizations in 2D are simpler than those in 3D, closed-form solutions for TDOA-based joint source and sensor localization have not yet been found. Note that, closed-form solutions for TDOA-based source localization are found in [13]. Since localization in 2D has a wide range of applications, it is worth studying. In this paper, closed-form solution for TDOA-based localization in 2D is studied and derived. More precisely, this closed-form solution is the closed-form solution for TOA-distance matrix recovery studied in Subsection 3.1 and the closed-form solution

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for TOA-based localization in 2D studied in Subsection 3.2. The CRLB for TDOA-based localization in 2D is obtained and compared with RMSEs achieved by the proposed method in Section 4. A conclusion is also presented in this section.

2. TDOA-BASED LOCALIZATION IN 2D

2.1. Problem formulation

TDOA-based localization in 2D is stated simply as follows: given a real matrix $\mathbf{\Gamma} = (\tau_{mn})_{M\times(N-1)}$, find M points $\{(u_m, v_m)\}_{m=1}^M$ and N points $\{(h_n, k_n)\}_{n=1}^N$ in $\mathbb{R}^2$ such that for all m and n

$$\left\| \begin{pmatrix} u_m \\ v_m \end{pmatrix} - \begin{pmatrix} h_{n+1} \\ k_{n+1} \end{pmatrix} \right\|_2 - \left\| \begin{pmatrix} u_m \\ v_m \end{pmatrix} - \begin{pmatrix} h_1 \\ k_1 \end{pmatrix} \right\|_2 = \tau_{mn}, \quad (1)$$

where $\|\cdot\|_2$ denotes the Euclidean distance. If (1) is satisfied, $\mathbf{\Gamma}$ is called a *TDOA-distance matrix*, and $\{(u_m, v_m)\}_{m=1}^M$ and $\{(h_n, k_n)\}_{n=1}^N$ are called *TDOA-based solutions* of $\mathbf{\Gamma}$.

Let $\mathbf{D} = (d_{mn})_{M\times N}$ be a nonnegative matrix where

$$d_{mn} = \left\| \begin{pmatrix} u_m \\ v_m \end{pmatrix} - \begin{pmatrix} h_n \\ k_n \end{pmatrix} \right\|_2 = \sqrt{(u_m - h_n)^2 + (v_m - k_n)^2}. \quad (2)$$

$\mathbf{D}$ is called a *TOA-distance matrix*, and $\{(u_m, v_m)\}_{m=1}^M$ and $\{(h_n, k_n)\}_{n=1}^N$ are also called *TOA-based solutions* of $\mathbf{D}$.

Equations (1) and (2) imply that for all m, n,

$$\tau_{mn} = d_{m,n+1} - d_{m1}. \quad (3)$$

Thus, if d_{m1} is given for all m, the TOA-distance matrix $\mathbf{D}$ can be recovered from the TDOA-distance matrix $\mathbf{\Gamma}$. On the basis of this idea, in this paper we solve TDOA-based localization in 2D by solving the following two problems: *(i) recovering* $\mathbf{D}$ *from* $\mathbf{\Gamma}$ *(recovering the TOA-distance matrix)* and *(ii) solving TOA-based localization in 2D*. The main contribution of this paper is to solve these two problems as closed-form solutions, and the key to obtaining these solutions is the *low-rank property* of a TOA-distance matrix, which is discussed in the following subsection (see [9, 14]).

2.2. Low-rank property of a TOA-distance matrix

Equation (2) implies that

$$\begin{aligned} &d_{mn}^2 - d_{m1}^2 - d_{1n}^2 + d_{11}^2 \\ &\quad = -2(u_m - u_1)(h_n - h_1) - 2(v_m - v_1)(k_n - k_1). \end{aligned} \quad (4)$$

Let us denote

$$\mathbf{X} = \begin{pmatrix} u_2 - u_1 & v_2 - v_1 \\ \vdots & \vdots \\ u_M - u_1 & v_M - v_1 \end{pmatrix}^T, \ \mathbf{Y} = -2\begin{pmatrix} h_2 - h_1 & k_2 - k_1 \\ \vdots & \vdots \\ h_N - h_1 & k_N - k_1 \end{pmatrix}^T \quad (5)$$

and $\mathbf{\Delta_D} = (\delta_{mn})_{(M-1)\times(N-1)}$, where

$$\delta_{mn} = d_{m+1,n+1}^2 - d_{m+1,1}^2 - d_{1,n+1}^2 + d_{11}^2. \quad (6)$$

Equation (4) implies that $\mathbf{\Delta_D} = \mathbf{X}^T\mathbf{Y}$ and also $rank(\mathbf{\Delta_D}) \leqslant 2$. This property is called the *low-rank* property of the TOA-distance matrix $\mathbf{D}$.

3. DERIVATION OF CLOSED-FORM SOLUTIONS

3.1. Recovery of TOA-distance matrix

In this subsection, we study the problem of recovering the TOA-distance matrix, which is stated as follows: given a TDOA-distance matrix $\mathbf{\Gamma} = (\tau_{mn})_{M\times(N-1)}$, determine a matrix $\mathbf{D} = (d_{mn})_{M\times N}$ such that $\tau_{mn} = d_{m,n+1} - d_{m,1}$ for all m, n, where $\mathbf{D}$ is a TOA-distance matrix.

Let us denote d_{m1} by z_{m-1} as a unknown variable. The recovery of the TOA-distance matrix is understood as determining M nonnegative variables $z_0, \ldots, z_{M-1}$ such that

$$\mathbf{D} = \begin{pmatrix} z_0 & z_0 + \tau_{11} & \cdots & z_0 + \tau_{1,N-1} \\ z_1 & z_1 + \tau_{21} & \cdots & z_1 + \tau_{2,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{M-1} & z_{M-1} + \tau_{M1} & \cdots & z_{M-1} + \tau_{M,N-1} \end{pmatrix} \quad (7)$$

is a TOA-distance matrix. Let $a_{mn} = 2\tau_{m+1,n}$, $b_n = -2\tau_{1n}$, and $c_{mn} = \tau_{m+1,n}^2 - \tau_{1n}^2$. Then (6) and (7) imply that

$$\delta_{mn} = a_{mn}z_m + b_n z_0 + c_{mn}. \quad (8)$$

If $\mathbf{D}$ is a TOA-distance matrix, the low rank property confirms that for all $1 \leqslant m_1 < m_2 < m_3 \leqslant M-1$ and $1 \leqslant n_1 < n_2 < n_3 \leqslant N-1$,

$$\det\begin{pmatrix} \delta_{m_1n_1} & \delta_{m_1n_2} & \delta_{m_1n_3} \\ \delta_{m_2n_1} & \delta_{m_2n_2} & \delta_{m_2n_3} \\ \delta_{m_3n_1} & \delta_{m_3n_2} & \delta_{m_3n_3} \end{pmatrix} = 0. \quad (9)$$

(8) and (9) give us a polynomial equation in four variables $z_0, z_{m_1}, z_{m_2}, z_{m_3}$ expressed by the following linear equation:

$$\mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)\mathbf{T}_{m_1m_2m_3}^T = \mathbf{0}, \quad (10)$$

where $\mathbf{T}_{m_1m_2m_3}$ and $\mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)$ are given in Table 1. Note that $\mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)$ is known from $\mathbf{\Gamma}$, and that $\mathbf{T}_{m_1m_2m_3}$ contains four unknown variables $z_0, z_{m_1}, z_{m_2}, z_{m_3}$.

Let $\mathcal{F}$ be a system of all the polynomial equations given in (9) for all $m_1, m_2, m_3, n_1, n_2, n_3$. The number of monomials in $\mathcal{F}$ is $K = \binom{M}{3} + \binom{M}{2} + M + 1$ and they are given by

$$\begin{aligned} \mathbf{T}_{\mathcal{F}} = (&z_0z_1z_2, z_0z_1z_3, \ldots, z_{M-3}z_{M-2}z_{M-1}, \\ &z_0z_1, z_0z_2, \ldots, z_{M-2}z_{M-1}, z_0, z_1, \ldots, z_{M-1}, 1). \end{aligned}$$

$\mathcal{F}$ has $\binom{M-1}{3}\binom{N-1}{3}$ polynomial equations. On the basis of the formulae for $\mathbf{T}_{m_1m_2m_3}$, $\mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)$, and $\mathbf{T}_{\mathcal{F}}$, we generate a coefficient matrix $\mathbf{C}_{\mathcal{F}}$ of $\mathcal{F}$ as follows:

1. $\mathbf{C}_{\mathcal{F}} \leftarrow [\,]$; $K \leftarrow \sum_{k=0}^{3}\binom{M}{k}$;
2. **For each** $m_1, m_2, m_3, n_1, n_2, n_3$
3. $\quad \mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3) \leftarrow$ Table 1; $A \leftarrow$ zeros$(1, K)$;
4. $\quad A([I_1, k_1 + I_2, k_2 + I_3, K]) \leftarrow \mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)$;
5. $\quad \mathbf{C}_{\mathcal{F}} \leftarrow [\mathbf{C}_{\mathcal{F}}\ ;\ A]$
6. **end.**

Table 1. Formulae for $\mathbf{T}_{m_1m_2m_3}$ and $\mathbf{C}_{m_1m_2m_3}(n_1,n_2,n_3)$.

$\mathbf{T}_{m_1m_2m_3}=(z_0z_{m_1}z_{m_2},z_0z_{m_1}z_{m_3},z_0z_{m_2}z_{m_3},z_{m_1}z_{m_2}z_{m_3},z_0z_{m_1},z_0z_{m_2},z_0z_{m_3},z_{m_1}z_{m_2},z_{m_1}z_{m_3},z_{m_2}z_{m_3},z_0,z_{m_1},z_{m_2},z_{m_3},1)$

$\mathbf{C}_{m_1m_2m_3}(n_1,n_2,n_3)=A_{123}+A_{312}+A_{231}-A_{321}-A_{213}-A_{132}$

where $A_{i_1i_2i_3}=(\hat{a}_1\hat{a}_2\hat{b}_3\ ,\ \hat{a}_1\hat{b}_2\hat{a}_3\ ,\ \hat{b}_1\hat{a}_2\hat{a}_3\ ,\ \hat{a}_1\hat{a}_2\hat{a}_3\ ,\ \hat{a}_1\hat{b}_2\hat{c}_3+\hat{a}_1\hat{c}_2\hat{b}_3\ ,\ \hat{b}_1\hat{a}_2\hat{c}_3+\hat{c}_1\hat{a}_2\hat{b}_3\ ,\ \hat{b}_1\hat{c}_2\hat{a}_3+\hat{c}_1\hat{b}_2\hat{a}_3\ ,\ \hat{a}_1\hat{a}_2\hat{c}_3\ ,\ \hat{a}_1\hat{c}_2\hat{a}_3\ ,$
$\hat{c}_1\hat{a}_2\hat{a}_3\ ,\ \hat{b}_1\hat{c}_2\hat{c}_3+\hat{c}_1\hat{b}_2\hat{c}_3+\hat{c}_1\hat{c}_2\hat{b}_3\ ,\ \hat{a}_1\hat{c}_2\hat{c}_3\ ,\ \hat{c}_1\hat{a}_2\hat{c}_3\ ,\ \hat{c}_1\hat{c}_2\hat{a}_3\ ,\ \hat{c}_1\hat{c}_2\hat{c}_3)$

and $\hat{a}_k=a_{m_kn_{i_k}},\hat{b}_k=b_{n_{i_k}},\hat{c}_k=c_{m_kn_{i_k}}$.

Here,
$I_1=(p_{0m_1m_2},p_{0m_1m_3},p_{0m_1m_3},p_{m_1m_2m_3})$,
$I_2=(p_{0m_1},p_{0m_2},p_{0m_3},p_{m_1m_2},p_{m_1m_3},p_{m_2m_3})$,
$I_3=(1,m_1\!+\!1,m_2\!+\!1,m_3\!+\!1)$, $k_1=\binom{M}{3}$, $k_2=k_1+\binom{M}{2}$,
$p_{m_1m_2}=m_1M-\frac{1}{2}(m_1+2)(m_1+1)+m_2+1$,
$p_{m_1m_2m_3}=\frac{1}{2}m_1M^2-\frac{1}{2}[(m_1+2)^2-2(m_2+1)]M$
$\quad+\frac{1}{6}(m_1^3+6m_1^2+11m_1-3m_2^2-9m_2+6m_3+6)$.

Note that $p_{m_1m_2m_3}$ and $p_{m_1m_2}+k_1$ are the orders of the monomials $z_{m_1}z_{m_2}z_{m_3}$ and $z_{m_1}z_{m_2}$ in $\mathbf{T}_{\mathcal{F}}$, respectively. Equation (10) implies that

$$\mathbf{C}_{\mathcal{F}}\mathbf{T}_{\mathcal{F}}^T=\mathbf{0}. \tag{11}$$

Since $\mathbf{T}_{\mathcal{F}}$ contains all variables $z_0,\ldots,z_{M-1}$, the TOA-distance matrix is recovered if the linear equations (11) are solved. The solvability of (11) depends on the rank of $\mathbf{C}_{\mathcal{F}}$, which is given by the following proposition.

Proposition 1. *Considering the cases* $(M\geqslant 4,N\geqslant 7)$, $(M\geqslant 5,N\geqslant 6)$, *and* $(M\geqslant 7,N\geqslant 5)$, *if the points* $\{(u_m,v_m)\}_{m=1}^M$ *do not lie on the same line, and a similar condition applies for* $\{(h_n,k_n)\}_{n=1}^N$, *we certainly have*

$$rank(\mathbf{C}_{\mathcal{F}})=\sum_{k=0}^{3}\binom{M}{k}=K-1. \tag{12}$$

A mathematical proof of this proposition has not yet been found. However, we checked the rank of $\mathbf{C}_{\mathcal{F}}$ in many simulated experiments and confirmed that it is correct. From (11), it is obvious that

$$\left[\mathbf{C}_{\mathcal{F}}^T\mathbf{C}_{\mathcal{F}}\right]\mathbf{T}_{\mathcal{F}}^T=\mathbf{0}. \tag{13}$$

Let $\mathbf{E}_{\min}$ be the eigenvector corresponding to the smallest eigenvalue of the matrix $\mathbf{C}_{\mathcal{F}}^T\mathbf{C}_{\mathcal{F}}$ and $E_{\min,i}$ be the ith element of $\mathbf{E}_{\min}$. The closed-form solutions for $z_0,\ldots,z_{M-1}$ are given by the following proposition.

Proposition 2. *If* $rank(\mathbf{C}_{\mathcal{F}})=K-1$, (11) *has a unique solution given by, for* $m=0,1,\ldots,M-1$,

$$z_m=\frac{E_{\min,K-M+m}}{E_{\min,K}}. \tag{14}$$

Proof. The size of $\mathbf{C}_{\mathcal{F}}^T\mathbf{C}_{\mathcal{F}}$ is $K\times K$ and $rank(\mathbf{C}_{\mathcal{F}}^T\mathbf{C}_{\mathcal{F}})=rank(\mathbf{C}_{\mathcal{F}})$. Thus, if $rank(\mathbf{C}_{\mathcal{F}})=K-1$, $\mathbf{E}_{\min}$ is a unique eigenvector corresponding to the zero eigenvalue. The uniqueness of the zero eigenvalue of $\mathbf{C}_{\mathcal{F}}^T\mathbf{C}_{\mathcal{F}}$ and (13) imply that $\mathbf{T}_{\mathcal{F}}$ and $\mathbf{E}_{\min}$ are linearly independent. Hence (14) is satisfied. □

3.2. TOA-based localization in 2D

We assume that a candidate $\mathbf{D}=(d_{mn})_{M\times N}$ of the TOA-distance matrix is determined on the basis of the TDOA-distance matrix $\mathbf{\Gamma}$, the closed-form solutions for $z_0,\ldots,z_{M-1}$ given in Proposition 2, and (7). In this section, we consider TOA-based localization in 2D, which is stated as follows: find $\{(u_m,v_m)\}_{m=1}^M$ and $\{(h_n,k_n)\}_{n=1}^N$ in $\mathbb{R}^2$ such that

$$\sqrt{(u_m-h_n)^2+(v_m-k_n)^2}=d_{mn}\quad\forall\ m,n. \tag{15}$$

Without loss of generality, since the roles of the points $\{(u_m,v_m)\}_{m=1}^M$ and $\{(h_n,k_n)\}_{n=1}^N$ are symmetric in TOA-based localization, we assume that $N\geqslant M$. Moreover, since the TOA-distance matrix is invariant under reflection, translation, and rotation, we assume that $(u_1,v_1)\equiv(0,0)$, $(u_2,v_2)\equiv(0,\alpha)$, $(\alpha>0)$, and $h_1\geqslant 0$. From (15), we first have formulae for k_n based on α as follows:

$$\begin{cases}d_{1n}^2=h_n^2+k_n^2\\ d_{2n}^2=h_n^2+(k_n-\alpha)^2\end{cases}\Rightarrow k_n=\frac{1}{2\alpha}\left(d_{1n}^2+\alpha^2-d_{2n}^2\right). \tag{16}$$

Furthermore, let a matrix $\mathbf{\Delta_D}$ be given by (6). Since $rank(\mathbf{\Delta_D})\leqslant 2$, by factorizing $\mathbf{\Delta_D}$, there exist two matrices $\widehat{\mathbf{X}}\in\mathbb{R}^{2\times(M-1)}$ and $\widehat{\mathbf{Y}}\in\mathbb{R}^{2\times(N-1)}$ such that $\mathbf{\Delta_D}=\widehat{\mathbf{X}}^T\widehat{\mathbf{Y}}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be given by (5). Since $\mathbf{X}^T\mathbf{Y}=\mathbf{\Delta_D}=\widehat{\mathbf{X}}^T\widehat{\mathbf{Y}}$, $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$ are used to determine $\mathbf{X}$ and $\mathbf{Y}$, in the sense that there exists an invertible (2×2) matrix $\begin{pmatrix}l_1&l_2\\l_3&l_4\end{pmatrix}$ such that

$$\mathbf{X}=\begin{pmatrix}l_1&l_2\\l_3&l_4\end{pmatrix}^{-T}\widehat{\mathbf{X}}\quad\text{and}\quad\mathbf{Y}=\begin{pmatrix}l_1&l_2\\l_3&l_4\end{pmatrix}\widehat{\mathbf{Y}}. \tag{17}$$

Combining (5) and (17) with $h_n^2=d_{1n}^2-k_n^2$ and (16), we have

$$4(h_n^2-h_1^2)=\begin{pmatrix}\hat{y}_{1,n-1}^2\\\hat{y}_{2,n-1}^2\\2\hat{y}_{1,n-1}\hat{y}_{2,n-1}\\-4\hat{y}_{1,n-1}\\-4\hat{y}_{2,n-1}\end{pmatrix}^T\times\begin{pmatrix}l_1^2\\l_2^2\\l_1l_2\\h_1l_1\\h_1l_2\end{pmatrix} \tag{18}$$
$$=2(d_{1n}^2+d_{2n}^2-d_{11}^2-d_{21}^2)+\frac{1}{\alpha^2}\left[(d_{21}^2-d_{11}^2)^2-(d_{2n}^2-d_{1n}^2)^2\right],$$

where $\hat{y}_{ij}$ denotes the (i,j)th element of $\widehat{\mathbf{Y}}$. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be $1\times(N-1)$ vectors whose elements are $2(d_{1n}^2+d_{2n}^2-d_{11}^2-d_{21}^2)$ and $(d_{21}^2-d_{11}^2)^2-(d_{2n}^2-d_{1n}^2)^2$ for $n=2,\ldots,N$, respectively, and let

$$\mathbf{S} = \begin{pmatrix} \hat{y}_{11}^2 & \cdots & \hat{y}_{1,N-1}^2 \\ \hat{y}_{21}^2 & \cdots & \hat{y}_{2,N-1}^2 \\ 2\hat{y}_{11}\hat{y}_{21} & \cdots & 2\hat{y}_{1,N-1}\hat{y}_{2,N-1} \\ -4\hat{y}_{11} & \cdots & -4\hat{y}v_{1,N-1} \\ -4\hat{y}_{21} & \cdots & -4\hat{y}_{2,N-1} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} l_1^2 \\ l_2^2 \\ l_1 l_2 \\ h_1 l_1 \\ h_1 l_2 \end{pmatrix}^T. \quad (19)$$

Equation (18) implies that

$$\mathbf{S}^T \mathbf{L} = \mathbf{d}_1 + \frac{1}{\alpha^2}\mathbf{d}_2. \quad (20)$$

Because of limited space in this paper, we give the following result without proof: *if* $N \geqslant 6$, $rank(\mathbf{S}^T) = 5$ *almost surely*. Note that the value of N in this subsection is $\max\{M, N\}$, where M, N are given in the section on TDOA-based localization. Thus, M and N given in Proposition 1 that we study in this paper satisfy $\max\{M, N\} \geqslant 6$, and then we almost surely have $rank(\mathbf{S}) = 5$. Let $\mathbf{S}^\dagger$ denote the *Moore-Penrose pseudoinverse* of $\mathbf{S}^T$, i.e., $\mathbf{S}^\dagger \mathbf{S}^T = \mathbf{I}_5$, and let $\mathbf{a} = \mathbf{S}^\dagger \mathbf{d}_1$, $\mathbf{b} = \mathbf{S}^\dagger \mathbf{d}_2$. Equation (20) implies that

$$\begin{aligned} l_1^2 = a_1 + \frac{1}{\alpha^2}b_1 \ , \ l_2^2 = a_2 + \frac{1}{\alpha^2}b_2 \ , \ l_1 l_2 = a_3 + \frac{1}{\alpha^2}b_3 \\ h_1 l_1 = a_4 + \frac{1}{\alpha^2}b_4 \ , \ h_1 l_2 = a_5 + \frac{1}{\alpha^2}b_5, \end{aligned} \quad (21)$$

where a_i, b_i denote the ith elements of $\mathbf{a}$ and $\mathbf{b}$, respectively. From the first three equations of (21), the closed-form solution for α is given by the following quadratic equation:

$$c_1 + c_2\frac{1}{\alpha^2} + c_3\frac{1}{\alpha^4} = 0, \quad (22)$$

where $c_1 = a_1 a_2 - a_3^2$, $c_2 = a_1 b_2 + a_2 b_1 - 2a_3 b_3$, and $c_3 = b_1 b_2 - b_3^2$. For each candidate α from (22), the closed-form solutions for k_n are given by (16), and the closed-form solutions for h_1, l_1, l_2, l_3, and l_4 are given by

$$\begin{aligned} &(l_3, l_4) = -2(k_2 - k_1, \ldots, k_N - k_1)\widehat{\mathbf{Y}}^\dagger \\ &h_1 = \left|a_4 + \frac{1}{\alpha^2}b_4\right| \Big/ \sqrt{a_1 + \frac{1}{\alpha^2}b_1} \\ &l_1 = \left(a_4 + \frac{1}{\alpha^2}b_4\right)/h_1 \quad \text{and} \quad l_2 = \left(a_5 + \frac{1}{\alpha^2}b_5\right)/h_2, \end{aligned} \quad (23)$$

where $\widehat{\mathbf{Y}}^\dagger$ denotes the *Moore-Penrose pseudoinverse* of $\widehat{\mathbf{Y}}$. Finally, the closed-form solutions for the remaining of unknown parameters of TOA-based localization in 2D are given by (17) and TOA-based localization in 2D is solved.

4. EXPERIMENTS AND CONCLUSION

4.1. Synthetic experiments

To evaluate the proposed closed-form solution for TDOA-based localization in 2D, we perform synthetic experiments, which are introduced as follows. Let $\{(u_m, v_m)\}_{m=1}^M$ and $\{(h_n, k_n)\}_{n=1}^N$ be uniformly distributed and independent points in a virtual square of size 5×5 m. The TDOA-distance matrix $\mathbf{\Gamma}$ is computed from these given positions and the Euclidean distance. Then the TDOA-distance matrix is corrupted by adding i.i.d. Gaussian noises to its elements, i.e., $\mathbf{\Gamma}_\sigma \leftarrow \mathbf{\Gamma} + \mathcal{N}(0, \sigma^2\mathbf{I}_{M(N-1)})$, where $\mathbf{I}_{M(N-1)}$ is the identity matrix of size $M(N-1)$ and $\mathcal{N}(0, \sigma^2\mathbf{I}_{M(N-1)})$ denotes an $(M, N-1)$ Gaussian matrix with zero mean and covariance $\sigma^2\mathbf{I}_{M(N-1)}$. The new positions $\{(\hat{u}_m, \hat{v}_m)\}_{m=1}^M$ and $\{(\hat{h}_n, \hat{k}_n)\}_{n=1}^N$ are estimated by the proposed method and the corrupted TDOA-distance matrix $\mathbf{\Gamma}_\sigma$. The RMSEs of the new positions are used to evaluate the proposed method.

4.2. Cramér-Rao lower bound

Let $\mathbf{Z}$ be an $M(N-1)$ vector containing the observed TDOA-distance matrix $\mathbf{\Gamma}_\sigma$ and let

$$\begin{aligned} \Phi &= (u_1,\ v_1,\ \ldots,u_M,\ v_M,\ h_1,\ k_1,\ \ldots,h_N,\ k_N), \\ \widehat{\Phi} &= (\hat{u}_1,\ \hat{v}_1,\ \ldots,\hat{u}_M,\ \hat{v}_M,\ \hat{h}_1,\ \hat{k}_1,\ \ldots,\hat{h}_N,\ \hat{k}_N). \end{aligned}$$

If our estimator is unbiased, i.e., $\mathbb{E}(\widehat{\Phi}) = \Phi$, the *Cramér-Rao lower bound* (CRLB) states that $Cov(\widehat{\Phi}) \geqslant F(\Phi)^{-1}$, where $F(\Phi)$ is the *Fisher information* of Φ, i.e.,

$$F(\Phi) = \mathbb{E}_\Phi\Big\{\big[\nabla_\Phi \ln p(\mathbf{Z}|\Phi)\big]^2\Big\}, \quad (24)$$

where $p(\mathbf{Z}|\Phi)$ is the conditional probability density function of $\mathbf{Z}$ given Φ, and ∇_Φ is a partial derivative operator with respect to Φ. Let $f(\Phi)$ be an $M(N-1)$ vector containing the TDOA-distance matrix generated by Φ. We have

$$\begin{aligned} Z_{i_{mn}} =& \sqrt{(u_m - h_{n+1})^2 + (v_m - k_{n+1})^2} \\ &- \sqrt{(u_m - h_1)^2 + (v_m - k_1)^2} + \eta_{mn}, \end{aligned} \quad (25)$$

where Z_i denotes the ith element of $\mathbf{Z}$, $i_{mn} = (m-1)(N-1) + n$, and η_{mn} are i.i.d. Gaussian variables with zero mean and standard deviation σ. Thus,

$$\begin{aligned} \ln p(\mathbf{Z}|\Phi) = &-\frac{M(N-1)}{2}\ln(2\pi) - \ln\sigma \\ &- \frac{1}{2\sigma^2}\big[\mathbf{Z} - f(\Phi)\big]^T\big[\mathbf{Z} - f(\Phi)\big]. \end{aligned} \quad (26)$$

Then $\nabla_\Phi \ln p(\mathbf{Z}|\Phi) = \frac{1}{\sigma^2}\mathbf{K}^T\big[\mathbf{Z} - f(\Phi)\big]$, where $\mathbf{K}$ is the *Jacobian matrix* of $f(\Phi)$ given by

$$K_{ij} = \begin{cases} \frac{u_{m_i} - h_{n_i+1}}{\|\mathbf{x}_{m_i} - \mathbf{y}_{n_i+1}\|_2} - \frac{u_{m_i} - h_1}{\|\mathbf{x}_{m_i} - \mathbf{y}_1\|_2} & \text{if } j = 2(m_i - 1) + 1 \\ \frac{h_{n_i+1} - u_{m_i}}{\|\mathbf{x}_{m_i} - \mathbf{y}_{n_i+1}\|_2} & \text{if } j = 2(M + n_i) + 1 \\ \frac{v_{m_i} - k_{n_i+1}}{\|\mathbf{x}_{m_i} - \mathbf{y}_{n_i+1}\|_2} - \frac{v_{m_i} - k_1}{\|\mathbf{x}_{m_i} - \mathbf{y}_1\|_2} & \text{if } j = 2(m_i - 1) + 2 \\ \frac{k_{n_i+1} - v_{m_i}}{\|\mathbf{x}_{m_i} - \mathbf{y}_{n_i+1}\|_2} & \text{if } j = 2(M + n_i) + 2 \\ \frac{u_{m_i} - h_1}{\|\mathbf{x}_{m_i} - \mathbf{y}_1\|_2} & \text{if } j = 2M + 1 \\ \frac{v_{m_i} - k_1}{\|\mathbf{x}_{m_i} - \mathbf{y}_1\|_2} & \text{if } j = 2M + 2 \\ 0 & \text{otherwise,} \end{cases} \quad (27)$$

where ϕ_j denotes the jth element of Φ, $\mathbf{x}_m \equiv (u_m, v_m)$, $\mathbf{y}_n \equiv (h_n, k_n)$, $m_i = \lfloor (i-1)/(N-1) \rfloor + 1$, $n_i = i - (N-1)(m_i - 1)$, and $\lfloor a \rfloor$ is the largest integer not greater than a.

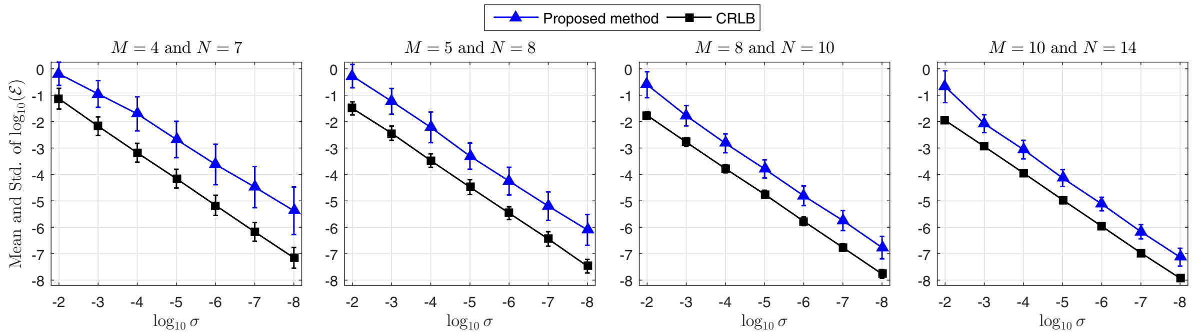

Fig. 1. Comparisons of the root-mean-square error for the proposed method with the CRLB.

Since $\mathbb{E}_\Phi\{[\mathbf{Z}-f(\Phi)][\mathbf{Z}-f(\Phi)]^T\} = \sigma^2\mathbf{I}_{M(N-1)}$, the CRLB for TDOA-based localization in 2D is

$$Cov(\widehat{\Phi}) \geqslant \sigma^2[\mathbf{K}^T\mathbf{K}]^{-1}. \quad (28)$$

4.3. Evaluation and conclusion

Figure 1 presents the *means* and *standard deviations* of RMSEs obtained in 200 synthetic experiments. These values are compared with the means and standard deviations of $\log_{10}\big[\frac{1}{\sqrt{M+N}}\big(\sum_i \text{CRLB}_{ii}\big)^{1/2}\big]$. The symbol $\mathcal{E}$ in the figure is used for both the RMSE and the CRLB. For comparison, we first applied the TDOA-based localization in 3D given in [9, 10] by adding zeros to the third coordinates. However, for almost all values of σ, the RMSEs of these 3D experiments were larger than 1 m (not shown in Figure 1). This means that the 3D algorithm cannot be used directly in 2D.

On the other hand, the simulated results in Figure 1 show that estimations by the proposed method are reliable in the cases of small M and N, i.e., $(M = 4, N = 7)$ and $(M = 5, N = 8)$, and more accurate and stable in the cases of large M and N, i.e., $(M = 8, N = 10)$ and $(M = 10, N = 14)$. Since the proposed method estimates source and sensor positions from the closed-form solution, it is easy to perform and should have a wide range of application.

5. ACKNOWLEDGEMENT

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