Penalized Linear Regression for Discrete Ill-posed Problems: A Hybrid Least-Squares and Mean-Squared Error Approach

Mohamed Suliman, Tarig Ballal, Abla Kammoun, and Tareq Y. Al-Naffouri
Computer, Electrical, and Mathematical Sciences and Engineering (CEMSE) Division,
King Abdullah University of Science and Technology (KAUST), Thuwal, Makkah Province, Saudi Arabia.
Emails: {mohamed.suliman, tarig.ahmed, abla.kammoun, tareq.alnaffouri}@kaust.edu.sa

***Abstract*—This paper proposes a new approach to find the regularization parameter for linear least-squares discrete ill-posed problems. In the proposed approach, an artificial perturbation matrix with a bounded norm is forced into the discrete ill-posed model matrix. This perturbation is introduced to enhance the singular-value (SV) structure of the matrix and hence to provide a better solution. The proposed approach is derived to select the regularization parameter in a way that minimizes the mean-squared error (MSE) of the estimator. Numerical results demonstrate that the proposed approach outperforms a set of benchmark methods in most cases when applied to different scenarios of discrete ill-posed problems. Jointly, the proposed approach enjoys the lowest run-time and offers the highest level of robustness amongst all the tested methods.**

***Index Terms*—linear estimation, linear least-squares, ill-posed problem, regularization.**

I. Introduction

The central problem of this paper is to estimate a vector $\mathbf{x}$ from an observation vector $\mathbf{y}$, related through

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{r}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a *known* severely ill-conditioned matrix, and $\mathbf{r}$ is a white Gaussian noise vector with unknown variance $\sigma_{\mathbf{r}}^2$ that is independent of $\mathbf{x}$. We focus on the case where $m \geq n$ and we do not impose assumptions on $\mathbf{x}$. Problems with the formulation as in (1) arise in various areas of engineering [1].

A popular approach to find an estimate of $\mathbf{x}$ is the ordinary *least-squares* (LS) method [2]. The LS enjoys a wide popularity in many areas of application due to its low computational complexity and its ability to evaluate a closed-form expression for the final solution. The LS attempts to solve

$$\min_{\hat{\mathbf{x}}} ||\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}||_2, \tag{2}$$

and its solution is given by

$$\hat{\mathbf{x}}_{\text{LS}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{y}, \tag{3}$$

where $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ is the singular value decomposition (SVD) of $\mathbf{A}$. The difficulty with the LS occurs when the matrix $\mathbf{A}$ is *ill-conditioned*. In such a case, the computed LS solution is potentially very sensitive to perturbations in the data such as from the noise $\mathbf{r}$. Discrete ill-posed problems arise in a variety of applications such as computerized tomography [3], astronomy [4], and image deblurring [5], to mention just few.

To overcome the difficulties associated with the LS, *regularization methods* are frequently used [6]. These methods depend on a parameter called the *regularization parameter*. The most common and well-known form of regularization is the one known as *Tikhonov regularization* [6]. The idea of Tikhonov regularization is to define the regularized solution as the minimizer of a weighted problem. A simplified form of Tikhonov regularization problem is

$$\min_{\hat{\mathbf{x}}} ||\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}||_2^2 + \lambda \; ||\hat{\mathbf{x}}||_2^2, \tag{4}$$

where the regularization parameter λ controls the weight given to the minimization of the side constraint relative to the minimization of the residual norm. It has been proved that the solution to (4) is given by the regularized LS (RLS) estimator

$$\hat{\mathbf{x}}_{\text{RLS}} = (\mathbf{A}^T\mathbf{A} + \gamma\mathbf{I})^{-1}\mathbf{A}^T\mathbf{y}, \tag{5}$$

where $\gamma = \lambda \in \mathbb{R}^+$, $\mathbf{I}$ is the identity matrix, and $(.)^T$ is the matrix transpose operation. Several regularization parameter selection methods have been proposed to find the regularization parameter in the regularization methods. These methods include the *L-curve* [7], the *generalized cross validation* (GCV) [8], and the *quasi-optimal* [9], to name a few.

This paper proposes a new regularization approach called constrained perturbation regularization approach (COPRA). The new approach is based on adding an artificial perturbation matrix with a bounded norm into $\mathbf{A}$ in (1). This perturbation is introduced to improve the singular-value (SV) structure of $\mathbf{A}$ and hence the estimator solution. The proposed COPRA is developed mainly to provide a solution that minimizes the mean-squared error (MSE) between $\mathbf{x}$ and its estimate $\hat{\mathbf{x}}$. To demonstrate the performance of the proposed COPRA, it is applied to solve a set of 11 real-world discrete ill-posed problems. The results demonstrate that, in most cases, COPRA outperforms a set of benchmark methods in terms of the MSE and also the average run-time.

This paper is organized as follows. In Section II, we derive the regularization parameter by optimizing the MSE. Section III presents the derivation of the proposed COPRA, whereas Section IV demonstrates the performance of COPRA by using numerical simulations. Finally, the conclusion of this paper is given in Section V.

978-0-9928-6265-7/16/$31.00 ©2016 IEEE

II. Minimizing the MSE

The MSE for an estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ can be defined as

$$\text{MSE} = \text{tr}\left\{\mathbb{E}\left((\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^T\right)\right\}, \tag{6}$$

where $\text{tr}(.)$ denotes the trace of the matrix. Substituting the SVD of $\mathbf{A}$ in (5), then plugging the result in (6), we obtain

$$\begin{aligned}\text{MSE} &= \sigma_{\mathbf{r}}^2 \text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-2}\right)\\ &+ \gamma^2 \text{tr}\left(\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-2}\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\right),\end{aligned} \tag{7}$$

where $\mathbf{C}_{\mathbf{xx}} \triangleq \mathbb{E}\left(\mathbf{x}\mathbf{x}^T\right)$ is the covariance matrix of $\mathbf{x}$. For a deterministic $\mathbf{x}$, $\mathbf{C}_{\mathbf{xx}} = \mathbf{x}\mathbf{x}^T$ is used for notational simplicity. Since the MSE in (7) is a convex function in γ, the global minimizer can be obtained by differentiating (7) with respect to γ and equating the result to zero. By carrying out this procedure, and after some manipulations, we obtain

$$\begin{aligned}\frac{\partial\,(\text{MSE})}{\partial\,\gamma} &= -2\sigma_{\mathbf{r}}^2 \text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right)\\ &+ 2\gamma\underbrace{\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\right)}_{R} = 0.\end{aligned} \tag{8}$$

Solving (8) gives the optimal regularizer γ_{o}. However, in the general case, and with the lack of knowledge on $\mathbf{x}$, we cannot obtain a closed-form expression for γ_{o}. To obtain a closed-form expression we need to use approximation.

By using the inequalities in [10] (Equation (5)), we can bound R in (8) as

$$\begin{aligned}&\min\left(\text{diag}\left(\mathbf{C}_{\mathbf{xx}}\right)\right)\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right)\\ &\le \text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\right)\\ &\le \max\left(\text{diag}\left(\mathbf{C}_{\mathbf{xx}}\right)\right)\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right),\end{aligned} \tag{9}$$

where $\text{diag}(.)$ returns a vector that contains the diagonal elements of the matrix.

To obtain a solution that is feasible for the general case, and also sub-optimal in some sense, we consider an *average* value of R based on (9) as our evaluation point. i.e.,

$$R \approx \frac{\text{tr}\left(\mathbf{C}_{\mathbf{xx}}\right)}{n}\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right). \tag{10}$$

Applying (10) to (8) yields

$$\begin{aligned}\frac{\partial\,(\text{MSE})}{\partial\,\gamma} &\approx -2\sigma_{\mathbf{r}}^2 \text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right)\\ &+ 2\gamma\frac{\text{tr}\left(\mathbf{C}_{\mathbf{xx}}\right)}{n}\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2+\gamma\mathbf{I}\right)^{-3}\right) = 0.\end{aligned} \tag{11}$$

Equation (11) can be easily solved to obtain the regularizer that approximately minimizes the MSE in (7) as

$$\gamma_{\text{o}} \approx \frac{n\sigma_{\mathbf{r}}^2}{\text{tr}\left(\mathbf{C}_{\mathbf{xx}}\right)}. \tag{12}$$

The result in (12) shows that there always exists a positive γ_{o}, for $\sigma_{\mathbf{r}}^2 \neq 0$, which approximately minimizes the MSE in (7). But the question is how to find this γ_{o}. If we know $\sigma_{\mathbf{r}}^2$ and $\mathbf{C}_{\mathbf{xx}}$, γ_{o} can be directly obtained. However prior statistics are not always available. In the following section, we show how the proposed COPRA can be used to obtain γ_{o}.

III. Constrained Perturbation Regularization Approach (COPRA)

In this section, we show how the proposed COPRA can be used to estimate γ_{o} without any prior knowledge on the signal or the noise.

To this end, we propose adding an artificial perturbation $\Delta\mathbf{A} \in \mathbb{R}^{m\times n}$ to $\mathbf{A}$. We assume that this perturbation, which replaces $\mathbf{A}$ by $(\mathbf{A}+\Delta\mathbf{A})$ in (1), improves the challenging SV structure of $\mathbf{A}$, and hence is capable of producing a better estimate of $\mathbf{x}$. Finally, in order to provide a balance between improving the SV and maintaining the fidelity of the basic model in (1), we add the constraint $||\Delta\mathbf{A}||_2 \le \eta$, $\eta \in \mathbb{R}^+$. As a result, the model in (1) is modified to

$$\mathbf{y} \approx \left(\mathbf{A}+\Delta\mathbf{A}\right)\mathbf{x}+\mathbf{r};\ \ ||\Delta\mathbf{A}||_2 \le \eta. \tag{13}$$

The question now is what is the best $\Delta\mathbf{A}$ and η. It is clear that these values are critical since they affect the model fidelity and the quality of the estimator. This question is addressed ahead in this section. For now, we assume that η is available to us.

To obtain an estimate of $\mathbf{x}$, we consider minimizing the worst-case residual function of (13) which can be defined as

$$\begin{aligned}\min_{\hat{\mathbf{x}}}\ \max_{\Delta\mathbf{A}}\ &||\mathbf{y}-\left(\mathbf{A}+\Delta\mathbf{A}\right)\hat{\mathbf{x}}||_2\\ &\text{subject to: } ||\Delta\mathbf{A}||_2 \le \eta.\end{aligned} \tag{14}$$

It can be shown that the optimization problem in (14) is equivalent to

$$\min_{\hat{\mathbf{x}}}\ \underbrace{||\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}||_2+\eta\,||\hat{\mathbf{x}}||_2}_{F(\hat{\mathbf{x}})}. \tag{15}$$

Thus, the solution of (14) depends only on the perturbation bound η and is agnostic to the structure of $\Delta\mathbf{A}$. It is easy to check that $F(\hat{\mathbf{x}})$ is a convex continuous function in $\hat{\mathbf{x}}$ and its gradient can be obtained as

$$\begin{aligned}\nabla_{\hat{\mathbf{x}}}F(\hat{\mathbf{x}}) &= \frac{1}{||\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}||_2}\mathbf{A}^T\left(\mathbf{A}\hat{\mathbf{x}}-\mathbf{y}\right)+\frac{\eta\,\hat{\mathbf{x}}}{||\hat{\mathbf{x}}||_2}\\ &= \frac{1}{||\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}||_2}\left(\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}}+\frac{\eta\,||\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}||_2\,\hat{\mathbf{x}}}{||\hat{\mathbf{x}}||_2}-\mathbf{A}^T\mathbf{y}\right).\end{aligned} \tag{16}$$

Now, define

$$\gamma = \frac{\eta\,||\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}||_2}{||\hat{\mathbf{x}}||_2}. \tag{17}$$

Substituting (17) in (16), then solving for $\nabla_{\hat{\mathbf{x}}}F(\hat{\mathbf{x}}) = 0$, yields to the RLS equation in (5).

It is clear that this idea suffers from the shortcoming that the regularization parameter γ dependents on η as in (17). In practice, we do not know which value of η should be used to obtain a feasible solution. Subsequently, we show how to overcome this issue.

$$\eta_{\mathrm{o}}^2 = \frac{\sigma_{\mathbf{r}}^2 \operatorname{tr}\left(\boldsymbol{\Sigma}^2\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\right) + \operatorname{tr}\left(\boldsymbol{\Sigma}^2\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\boldsymbol{\Sigma}^2\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\right)}{\sigma_{\mathbf{r}}^2 \operatorname{tr}\left(\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\right) + \operatorname{tr}\left(\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\boldsymbol{\Sigma}^2\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\right)}. \tag{21}$$

To proceed further, we apply the SVD of $\mathbf{A}$ to (17), then substitute for $\hat{\mathbf{x}}$ from (5) and manipulate to obtain

$$\mathbf{y}^T\mathbf{U}\left(\boldsymbol{\Sigma}^2-\eta^2\mathbf{I}\right)\left(\boldsymbol{\Sigma}^2+\gamma\mathbf{I}\right)^{-2}\mathbf{U}^T\mathbf{y}=0. \tag{18}$$

As shown, the solution of (15) is the RLS (5) when (17) is satisfied. In the following, we show how we can use the combination of (12) and (18) to obtain the regularization parameter without knowing η explicitly.

Starting from (18), by setting $\gamma=\gamma_{\mathrm{o}}$, $\eta=\eta_{\mathrm{o}}$, solving for η_{o}^2, and finally taking the trace of both sides, we reach

$$\eta_{\mathrm{o}}^2 = \frac{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\mathbf{U}^T\left(\mathbf{y}\mathbf{y}^T\right)\mathbf{U}\right)}{\operatorname{tr}\left(\left(\boldsymbol{\Sigma}^2+\gamma_{\mathrm{o}}\mathbf{I}\right)^{-2}\mathbf{U}^T\left(\mathbf{y}\mathbf{y}^T\right)\mathbf{U}\right)}. \tag{19}$$

To obtain a useful result, let us think of η_{o} as a global value computed over many realizations of the observation vector $\mathbf{y}$. Accordingly, we can replace $\mathbf{y}\mathbf{y}^T$ by its expected value $\mathbb{E}(\mathbf{y}\mathbf{y}^T)$ which can be written using (1) as

$$\mathbb{E}\left(\mathbf{y}\mathbf{y}^T\right) = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^T+\sigma_{\mathbf{r}}^2\mathbf{I}. \tag{20}$$

Substituting (20) in (19) and manipulating, we obtain (21).

In this work, we focus on the case where the SVs of $\mathbf{A}$ decay exponentially [11]. Such a fast decay allows us to divide the SVs into two groups of *significant*, or relatively large, and *trivial*, or nearly zero SVs. As an example, Fig. 1 shows the normalized SV decay for a matrix $\mathbf{A}$ of size 50×50.

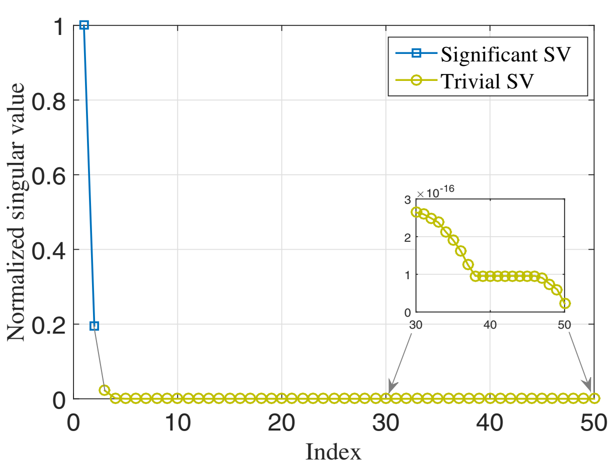


Fig. 1: Example of the singular-value decay for $\mathbf{A}\in\mathbb{R}^{50\times 50}$.

The SVs of the problem are decaying very fast, making it possible to identify two groups of SVs. Based of this, the matrix $\boldsymbol{\Sigma}$ can be divided into two diagonal sub-matrices, $\boldsymbol{\Sigma}_1$, which contains the first (significant) n_1 diagonal entries, and $\boldsymbol{\Sigma}_2$, which contains the last (trivial) $n_2=n-n_1$ diagonal entries. The splitting threshold can be obtained as the mean of the SVs multiplied by a certain constant ρ, where $\rho\in(0,1)$. Similarly, we can write $\mathbf{V}=[\mathbf{V}_1\ \mathbf{V}_2]$, where $\mathbf{V}_1\in\mathbb{R}^{n\times n_1}$, and $\mathbf{V}_2\in\mathbb{R}^{n\times n_2}$.

Now, substituting the partitioning of $\boldsymbol{\Sigma}$ and $\mathbf{V}$ in the numerator $(\mathcal{N})$ of (21), we obtain

$$\begin{aligned}\mathcal{N} &= \sigma_{\mathbf{r}}^2\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\right)+\sigma_{\mathbf{r}}^2\operatorname{tr}\left(\boldsymbol{\Sigma}_2^2\left(\boldsymbol{\Sigma}_2^2+\gamma_{\mathrm{o}}\mathbf{I}_2\right)^{-2}\right)\\ &+\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\mathbf{V}_1^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}_1\right)\\ &+\operatorname{tr}\left(\boldsymbol{\Sigma}_2^2\left(\boldsymbol{\Sigma}_2^2+\gamma_{\mathrm{o}}\mathbf{I}_2\right)^{-2}\boldsymbol{\Sigma}_2^2\mathbf{V}_2^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}_2\right).\end{aligned} \tag{22}$$

Given how we choose n_1 and n_2, we have $\|\boldsymbol{\Sigma}_2\|\approx 0$. Based on this, we can approximate (22) by

$$\begin{aligned}\mathcal{N} &\approx \sigma_{\mathbf{r}}^2\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\right)\\ &+\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\mathbf{V}_1^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}_1\right).\end{aligned} \tag{23}$$

By applying the same justification used to obtain (10), the second term of (23) can be written as

$$\begin{aligned}&\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\mathbf{V}_1^T\mathbf{C}_{\mathbf{xx}}\mathbf{V}_1\right)\\ &\approx\frac{1}{n_1}\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\right)\operatorname{tr}\left(\mathbf{C}_{\mathbf{xx}}\right).\end{aligned} \tag{24}$$

Substituting (24) in (23) we obtain

$$\begin{aligned}\mathcal{N} &\approx \sigma_{\mathbf{r}}^2\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\right)\\ &+\frac{1}{n_1}\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\right)\operatorname{tr}\left(\mathbf{C}_{\mathbf{xx}}\right).\end{aligned} \tag{25}$$

Similarly, the denominator $(\mathcal{D})$ of (21) can be approximated as

$$\begin{aligned}\mathcal{D} &\approx \sigma_{\mathbf{r}}^2\operatorname{tr}\left(\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\right)+\frac{\sigma_{\mathbf{r}}^2 n_2}{\gamma_{\mathrm{o}}^2}\\ &+\frac{1}{n_1}\operatorname{tr}\left(\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\boldsymbol{\Sigma}_1^2\right)\operatorname{tr}\left(\mathbf{C}_{\mathbf{xx}}\right).\end{aligned} \tag{26}$$

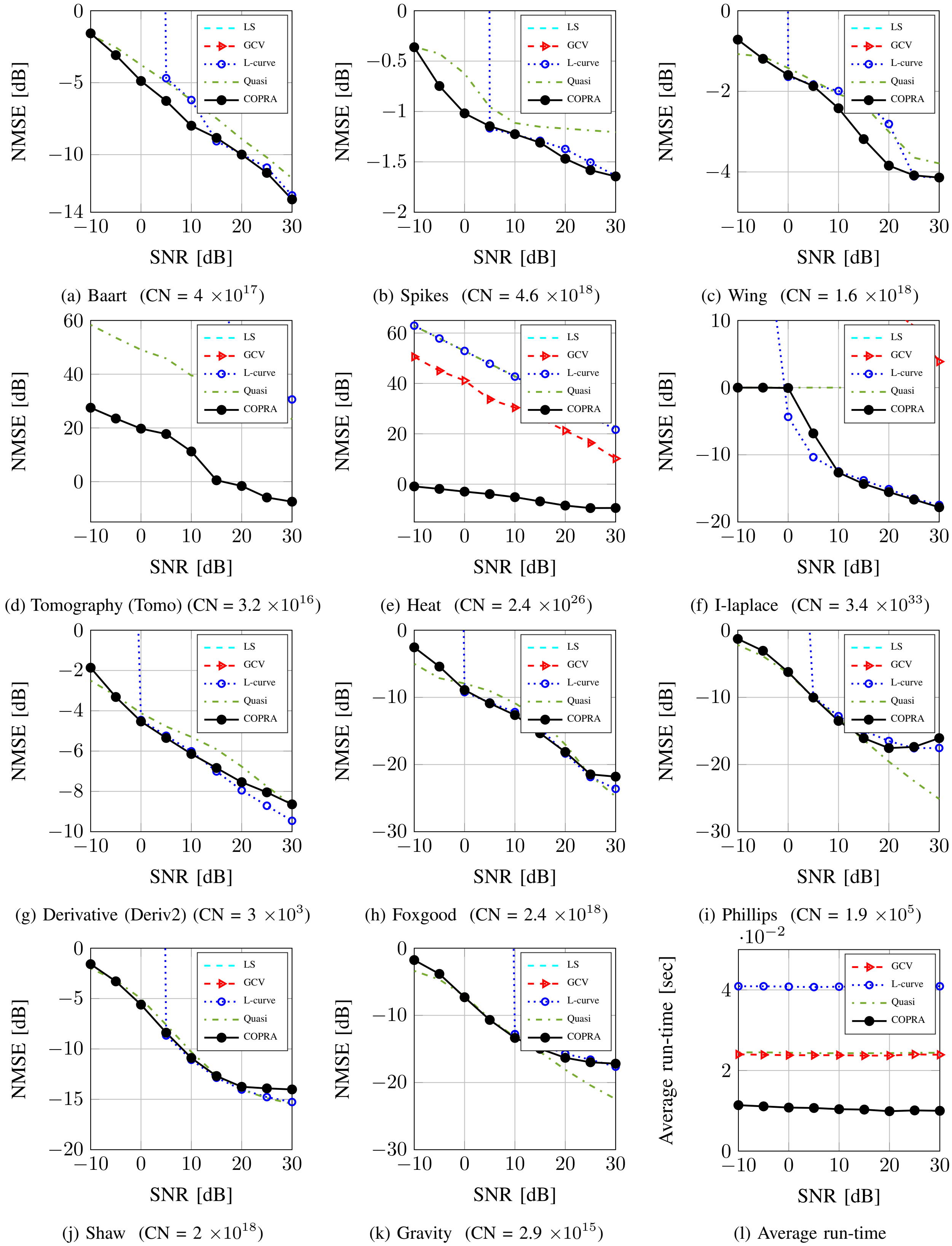
Substituting (25) and (26) in (21), then manipulating yields

$$\eta_{\mathrm{o}}^2 \approx \frac{\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\left(\boldsymbol{\Sigma}_1^2+\frac{n_1\sigma_{\mathbf{r}}^2}{\operatorname{tr}(\mathbf{C}_{\mathbf{xx}})}\mathbf{I}_1\right)\right)}{\operatorname{tr}\left(\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\left(\boldsymbol{\Sigma}_1^2+\frac{n_1\sigma_{\mathbf{r}}^2}{\operatorname{tr}(\mathbf{C}_{\mathbf{xx}})}\mathbf{I}_1\right)\right)+\frac{n_2 n_1\sigma_{\mathbf{r}}^2}{\gamma_{\mathrm{o}}^2\operatorname{tr}(\mathbf{C}_{\mathbf{xx}})}}. \tag{27}$$

Now, based on (12) we can insert $\frac{n_1}{n}\gamma_{\mathrm{o}}$ to replace $\frac{n_1\sigma_{\mathbf{r}}^2}{\operatorname{tr}(\mathbf{C}_{\mathbf{xx}})}$ in (27) and manipulate to obtain

$$\eta_{\mathrm{o}}^2 \approx \frac{\operatorname{tr}\left(\boldsymbol{\Sigma}_1^2\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\left(\frac{n}{n_1}\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)\right)}{\operatorname{tr}\left(\left(\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)^{-2}\left(\frac{n}{n_1}\boldsymbol{\Sigma}_1^2+\gamma_{\mathrm{o}}\mathbf{I}_1\right)\right)+\frac{n_2}{\gamma_{\mathrm{o}}}}. \tag{28}$$

The expression in (28) reveals that any η_{o} satisfying (28)



(a) Baart (CN = 4 $\times 10^{17}$)
(b) Spikes (CN = 4.6 $\times 10^{18}$)
(c) Wing (CN = 1.6 $\times 10^{18}$)
(d) Tomography (Tomo) (CN = 3.2 $\times 10^{16}$)
(e) Heat (CN = 2.4 $\times 10^{26}$)
(f) I-laplace (CN = 3.4 $\times 10^{33}$)
(g) Derivative (Deriv2) (CN = 3 $\times 10^{3}$)
(h) Foxgood (CN = 2.4 $\times 10^{18}$)
(i) Phillips (CN = 1.9 $\times 10^{5}$)
(j) Shaw (CN = 2 $\times 10^{18}$)
(k) Gravity (CN = 2.9 $\times 10^{15}$)
(l) Average run-time

Fig. 2: (a)-(k) Normalized mean-squared error (NMSE) [dB] vs SNR [dB].
(1) Average run-time [sec] vs SNR [dB].

minimizes the MSE approximately. Now, to eliminate the dependency on η in (17) we return to equation (18) and substitute $\gamma = \gamma_o$, $\eta = \eta_o$ in this equation. Then we substitute for η_o by using (28). These substitutions result in the *characteristic equation* for COPRA

$$S(\gamma_o) = \text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2 + \gamma_o\mathbf{I}\right)^{-2}\mathbf{b}\mathbf{b}^T\right)\text{tr}\Big\{\left(\mathbf{\Sigma}_1^2 + \gamma_o\mathbf{I}_1\right)^{-2} \times \left(\beta\mathbf{\Sigma}_1^2 + \gamma_o\mathbf{I}_1\right)\Big\} + \frac{n_2}{\gamma_o}\text{tr}\left(\mathbf{\Sigma}^2\left(\mathbf{\Sigma}^2 + \gamma_o\mathbf{I}\right)^{-2}\mathbf{b}\mathbf{b}^T\right) - \text{tr}\left(\left(\mathbf{\Sigma}^2 + \gamma_o\mathbf{I}\right)^{-2}\mathbf{b}\mathbf{b}^T\right)\text{tr}\Big\{\mathbf{\Sigma}_1^2\left(\mathbf{\Sigma}_1^2 + \gamma_o\mathbf{I}_1\right)^{-2} \times \left(\beta\mathbf{\Sigma}_1^2 + \gamma_o\mathbf{I}_1\right)\Big\} = 0, \quad (29)$$

where $\mathbf{b} \triangleq \mathbf{U}^T\mathbf{y}$, and $\beta \triangleq \frac{n}{n_1}$. Equation (29) is solved to obtain the regularization parameter γ_o. In this paper, Newton's method [12], initialized using *small* positive value, is exclusively used to carry out this task. An extension to this work will show that (29) has a unique positive root, and that Newton's method always converges to the desired solution.

IV. Results

In order to examine the performance of the proposed COPRA, we perform extensive testing. The Matlab regularization toolbox [13] is used to generate pairs of $\mathbf{A}$ and $\mathbf{x}$. Noise is added to $\mathbf{Ax}$ according to a certain *signal-to-noise-ratio* (SNR) in dB (SNR (dB) = $10\log_{10}(\text{SNR})$) to generate $\mathbf{y}$. The toolbox provides real-world problems that can be used for testing regularization methods in discrete ill-posed problems. The performance is evaluated using the *normalized* MSE (NMSE) (MSE normalized by $||\mathbf{x}||_2^2$) obtained over 10^4 different noise realizations at each SNR value. To facilitate results visualization, we choose different upper and lower thresholds for the vertical axis in the results sub-figures. Fig. 2 shows the results for a selected 11 problems in addition to the average run-time. Each sub-figure presents the performance and quotes the condition number (CN) of the problem's matrix. These results are obtained for a matrix dimension of 50×50, except for *Tomo* (49×49) and *Phillips* (52×52) which have special dimensionality requirements.

We compare the NMSE of COPRA versus those of L-curve, GCV, quasi-optimal, and LS. If the NMSE curve for any method or part of it does not appear at any SNR in Fig. 2, it means that the performance of the method at this SNR exceeds the *upper threshold* of the vertical axis.

Generally speaking, It can be said that an estimator offering NMSE above 0 dB is not robust and is worthless. From Fig. 2, it is clear that the proposed COPRA offers the highest level of robustness amongst all the methods. This can be seen from the fact that COPRA is the only approach whose NMSE remains below 0 dB in almost all cases. Comparing the NMSE over the range of the SNR values in each problem, we find *on average* that COPRA exhibits the lowest NMSE amongst all the methods in 8 problems (the first 8 sub-figures). Considering all the problems, the closest contender to COPRA is the quasi-optimal method. However, this method and the remaining benchmark methods show lack of robustness in certain situations. This appears in the form of overly high NMSE in certain cases. For example, in *Heat* and *Tomo* problems, only the proposed COPRA is offering reliable performance. Another obvious observation is that the NMSE curve of LS does not appear in the 11 plots. This is due to the fact that it is providing a NMSE above the upper cut-off NMSE. This clearly emphasizes why LS algorithm cannot be used with ill-posed problems.

Finally, Fig. 2(l) plots the average run-time for each method against the SNR as calculated in Matlab. The figure is a good representation for the run-times of all the 11 problems (no significant run-time variation between problems has been seen). From Fig. 2(l), it can be seen that COPRA has the lowest run-time compared to all benchmark methods. The closet to COPRA are the GCV and quasi-optimal methods which have approximately double the run-time of COPRA.

V. Conclusions

A new regularization approach for linear discrete ill-posed problems has been presented. The proposed constrained perturbation regularization approach (COPRA) combines the simplicity of the least-squares criterion with the robustness of the mean-squared error (MSE) based estimation. The singular-value (SV) structure of the linear operator is utilized to simplify the formulation and obtain the regularization parameter as a solution of a non-linear equation in one variable. Numerical results demonstrate that the proposed COPRA outperforms a set of benchmark methods in most cases while enjoying the lowest run-time and offers the highest level of robustness.

Acknowledgment

This work was supported by the King Abdulaziz City of Science and Technology (KACST) under Grant AT-34-345.

References

[1] Thomas Kailath, Ali H Sayed, and Babak Hassibi, *Linear estimation*, vol. 1, Prentice Hall Upper Saddle River, NJ, 2000.
[2] Steven M Kay, *Fundamentals of Statistical Signal Processing: Practical Algorithm Development*, vol. 3, Pearson Education, 2013.
[3] Gabor T Herman, *Fundamentals of computerized tomography: image reconstruction from projections*, Springer Science & Business Media, 2009.
[4] Ian Craig and John Brown, "Inverse problems in astronomy," 1986.
[5] Per Christian Hansen, James G Nagy, and Dianne P O'leary, *Deblurring images: matrices, spectra, and filtering*, vol. 3, Siam, 2006.
[6] Andrei Nikolaevich Tikhonov, AV Goncharsky, VV Stepanov, and Anatoly G Yagola, *Numerical methods for the solution of ill-posed problems*, vol. 328, Springer Science & Business Media, 2013.
[7] Per Christian Hansen and Dianne Prost O'Leary, "The use of the l-curve in the regularization of discrete ill-posed problems," *SIAM Journal on Scientific Computing*, vol. 14, no. 6, pp. 1487–1503, 1993.
[8] Grace Wahba, *Spline models for observational data*, vol. 59, Siam, 1990.
[9] Vladimir Alekseevich Morozov, *Methods for solving incorrectly posed problems*, Springer Science & Business Media, 2012.
[10] Sheng-de Wang, Te-Son Kuo, and Chen-Fa Hsu, "Trace bounds on the solution of the algebraic matrix riccati and lyapunov equation," *IEEE Transactions on Automatic Control*, vol. 31, no. 7, pp. 654–656, 1986.
[11] Dilip N Ghosh Roy and LS Couchman, *Inverse problems and inverse scattering of plane waves*, Academic Press, 2001.
[12] Christopher J Zarowski, *An introduction to numerical analysis for electrical and computer engineers*, John Wiley & Sons, 2004.
[13] P. C. Hansen, *Regularization Tools: A Matlab Package for Analysis and Solution of Discrete Ill-Posed Problems, Version 4.1*, Technical University of Denmark, Lyngby, Denmark, 2008.