Low-Complexity Weighted Pseudolinear Estimator for TDOA Localization with Systematic Error Correction

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***Abstract*—A closed-form pseudolinear estimation algorithm for time-difference-of-arrival (TDOA) emitter localization was previously proposed by replacing TDOA hyperbolae with their asymptotes, effectively transforming the TDOA localization problem into a bearings-only localization problem. Despite having a good mean-square error (MSE) performance and low computational complexity, this algorithm was observed to suffer from systematic errors arising from the large misalignment between TDOA hyperbolae and asymptotes particularly in the near field and for poor geometries, resulting in increased estimation bias. To address the bias problem, this paper presents a systematic error correction technique based on a two-stage estimation process in which the estimation errors due to asymptote misalignment are computed from an initial location estimate and then subsequently corrected. The superior performance of the new algorithm compared with the maximum likelihood estimator is demonstrated with simulation examples.**

I. Introduction

Time-difference-of-arrival (TDOA) localization is a passive emitter localization method with many applications in multistatic radar, sonar, wireless communications and sensor networks, to name just a few. In the 2D plane, TDOA of the emitter signal at two sensors defines a hyperbola as possible locations of the emitter with two foci placed at the sensor locations. The emitter location is fixed by intersecting multiple TDOA hyperbolae obtained from different sensor pairs. Broadly speaking, the existing estimators for TDOA localization can be classified as (i) maximum likelihood estimator (MLE), which is implemented as a nonlinear least squares estimator for Gaussian noise [1], (ii) "linearized" estimators with parameter constraints due to nuisance parameter (see e.g. [2]–[5]), and (iii) hyperbolic asymptote intersection, in which the TDOA hyperbolae are substituted by linear asymptotes [6], [7]. TDOA localization is closely related to time-of-arrival localization, which has been studied extensively (see e.g. [8]–[11] and the references therein).

The MLE is asymptotically unbiased and efficient. However it does not have a closed-form solution and therefore requires a numerical search method, which can be computationally expensive. Being a nonlinear estimator, it is also subject to the threshold effect [12], causing sudden degradation of performance as noise is increased. The MLE cost function for TDOA localization is nonconvex [3], which means that unless an appropriate initial guess is used, numerical search methods can diverge. Linearized TDOA algorithms suffer from bias problems due to the correlation between data matrix and measurement vector resulting from noise injection into the data matrix, as well as introduction of a nuisance parameter that is dependent on the emitter location to be estimated. Some progress has been made in the development of solutions for the constrained optimization problem, arising from the nuisance parameter, achieving near-MLE performance [4]. However the constrained solutions are computationally expensive requiring generalized eigenvalue computations, as well as numerical solutions for polynomial equation roots. The method of hyperbolic asymptote intersection, on the other hand, results in a weighted pseudolinear estimator (weighted PLE) [6], which has low computational complexity, but is subject to bias problems mainly due to systematic errors caused by the gap between the TDOA hyperbolae and their asymptotes.

The main contribution of this paper is to correct the systematic errors present in the method of hyperbolic asymptote intersection method. The systematic error correction requires a two-stage estimation process consisting of the weighted PLE to generate an initial estimate followed by systematic error estimation and correction. The main advantages of this new two-stage TDOA localization algorithm are its superior estimation performance over the MLE and significantly low computational complexity compared with both the MLE and constrained optimization solutions in [4].

The paper is organized as follows. Section II defines the TDOA localization problem. Section III provides a brief review of the MLE and the Cramér-Rao lower bound (CRLB) for TDOA localization. The hyperbolic asymptotes and associated angle errors are discussed in Section IV. Section V presents the weighted PLE with systematic error correction. Comparative simulation studies are presented in Section VI. The paper concludes in Section VII.

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II. TDOA Localization

The objective of TDOA localization is to estimate the unknown location of an emitter at $\boldsymbol{s} = [x, y]^T$ (where ^{T} denotes matrix transpose) using TDOA measurements obtained from N spatially distributed sensors ($N \geq 3$) at $\boldsymbol{r}_i = [x_i, y_i]^T$, $i = 1, \ldots, N$. The TDOA between sensors i and j is given by

$$\tau_{ij} = \tau_j - \tau_i, \quad i \neq j, \quad i, j \in \{1, \ldots, N\} \tag{1}$$

where

$$\tau_i = \frac{\|\boldsymbol{d}_i\|}{c} \tag{2}$$

is the time it takes for the emitter signal to arrive at sensor i, and c is the speed of propagation for the transmitted signal. Here $\|\cdot\|$ denotes the Euclidean norm, and $\boldsymbol{d}_i$ is the emitter range vector from sensor i:

$$\boldsymbol{d}_i = \boldsymbol{s} - \boldsymbol{r}_i. \tag{3}$$

The range-difference-of-arrival (RDOA), g_{ij}, is related to the TDOA, τ_{ij}, through

$$g_{ij} = \|\boldsymbol{d}_j\| - \|\boldsymbol{d}_i\|, \quad i, j \in \{1, \ldots, N\} \tag{4a}$$
$$= c\tau_{ij}. \tag{4b}$$

Each RDOA defines a hyperbola of possible emitter locations. We assume sensor 1 is the reference sensor for TDOA measurements, following the common practice [2]. Given a sequence of $N-1$ RDOAs, the emitter location $\boldsymbol{s}$ is obtained from the intersection of $N-1$ hyperbolae:

$$\begin{aligned} \|\boldsymbol{s} - \boldsymbol{r}_2\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{12} \\ \|\boldsymbol{s} - \boldsymbol{r}_3\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{13} \\ &\vdots \\ \|\boldsymbol{s} - \boldsymbol{r}_{N-1}\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{1N}. \end{aligned} \tag{5}$$

To solve the above set of nonlinear equations for $\boldsymbol{s}$, a minimum of two equations are required (i.e., $N \geq 3$) since there are two unknowns. However, in practice, four or more sensors may be necessary to avoid "ghost" emitters.

For continuous-wave signals, the RDOAs can be estimated using the method of generalized cross-correlation [13]. The resulting RDOA measurements $\tilde{g}_{1j}$ are

$$\tilde{g}_{1j} = g_{1j} + n_{1j}, \quad j \in \{2, \ldots, N\} \tag{6}$$

where the RDOA noise n_{1j} is assumed to be zero-mean Gaussian. Supposing that the signal received at each sensor is subject to i.i.d. additive Gaussian noise, the covariance matrix of n_{1j} becomes

$$\boldsymbol{\Sigma} = E\left\{ \begin{bmatrix} n_{12} \\ \vdots \\ n_{1N} \end{bmatrix} \begin{bmatrix} n_{12} & \cdots & n_{1N} \end{bmatrix} \right\} \tag{7a}$$
$$= \sigma_n^2 \boldsymbol{T} \tag{7b}$$

where $\sigma_n^2 = E\{n_{1j}^2\}$, $j = 2, \ldots, N$, is the RDOA noise variance and

$$\boldsymbol{T} = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{1}) \tag{8}$$

with $\boldsymbol{I}$ denoting the identity matrix and $\boldsymbol{1}$ a matrix of ones.

III. Maximum Likelihood Estimator and CRLB

For Gaussian noise the MLE is a nonlinear least-squares estimator

$$\hat{\boldsymbol{s}}_{\text{MLE}} = \arg\min_{\boldsymbol{s} \in \mathbb{R}^2} J_{\text{MLE}}(\boldsymbol{s}) \tag{9}$$

where

$$J_{\text{MLE}}(\boldsymbol{s}) = \boldsymbol{e}^T(\boldsymbol{s})\boldsymbol{\Sigma}^{-1}\boldsymbol{e}(\boldsymbol{s}), \quad \boldsymbol{e}(\boldsymbol{s}) = \tilde{\boldsymbol{g}} - \boldsymbol{g}(\boldsymbol{s}) \tag{10}$$

and

$$\tilde{\boldsymbol{g}} = \begin{bmatrix} \tilde{g}_{12} \\ \vdots \\ \tilde{g}_{1N} \end{bmatrix}, \quad \boldsymbol{g}(\boldsymbol{s}) = \begin{bmatrix} \|\boldsymbol{s} - \boldsymbol{r}_2\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| \\ \vdots \\ \|\boldsymbol{s} - \boldsymbol{r}_N\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| \end{bmatrix}.$$

Equation (9) does not have a closed-form solution. The MLE can be computed using the Gauss-Newton (GN) algorithm:

$$\hat{\boldsymbol{s}}(i+1) = \hat{\boldsymbol{s}}(i) + (\boldsymbol{J}^T(i)\boldsymbol{\Sigma}^{-1}\boldsymbol{J}(i))^{-1}\boldsymbol{J}^T(i)\boldsymbol{\Sigma}^{-1}\boldsymbol{e}(\hat{\boldsymbol{s}}(i)), \quad i = 0, 1, \ldots \tag{11}$$

Here $\boldsymbol{J}(i)$ is the Jacobian matrix of $\boldsymbol{g}(\boldsymbol{s})$ with respect to $\boldsymbol{s}$ evaluated at $\boldsymbol{s} = \hat{\boldsymbol{s}}(i)$:

$$\boldsymbol{J}(i) = \begin{bmatrix} (\boldsymbol{u}_2(\hat{\boldsymbol{s}}(i)) - \boldsymbol{u}_1(\hat{\boldsymbol{s}}(i)))^T \\ (\boldsymbol{u}_3(\hat{\boldsymbol{s}}(i)) - \boldsymbol{u}_1(\hat{\boldsymbol{s}}(i)))^T \\ \vdots \\ (\boldsymbol{u}_N(\hat{\boldsymbol{s}}(i)) - \boldsymbol{u}_1(\hat{\boldsymbol{s}}(i)))^T \end{bmatrix}, \quad \boldsymbol{u}_j(\hat{\boldsymbol{s}}) = \frac{\hat{\boldsymbol{s}} - \boldsymbol{r}_j}{\|\hat{\boldsymbol{s}} - \boldsymbol{r}_j\|} \tag{12}$$

The CRLB for TDOA localization is given by

$$\text{CRLB} = (\boldsymbol{J}_o^T \boldsymbol{\Sigma}^{-1} \boldsymbol{J}_o)^{-1} \tag{13}$$

where $\boldsymbol{J}_o$ is the Jacobian evaluated at the true emitter location.

IV. Hyperbolic Asymptotes for TDOA Localization

For a given RDOA measurement $\tilde{g}_{1j}$ taken at sensors $\boldsymbol{r}_1$ and $\boldsymbol{r}_j$, the hyperbolic asymptote associated with the emitter is a bearing line that emanates from the mid-point of sensors

$$\boldsymbol{m}_{1j} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_j) \tag{14}$$

with a bearing angle

$$\tilde{\theta}_{1j} = \tan^{-1}\left(\frac{y_j - y_1}{x_j - x_1}\right) + \mathcal{I}_{1j}\cos^{-1}\left(-\frac{\tilde{g}_{1j}}{\|\boldsymbol{r}_{1j}\|}\right) \tag{15}$$

where $\mathcal{I}_{1j}$ takes on the values ± 1 depending on which side of the sensor baseline the emitter lies (see Fig. 1). The determination of $\mathcal{I}_{1j}$ can be accomplished by resorting to clustering [6] or prior directional knowledge about the emitter location.

The second term in the right hand side of (15) is approximately given by

$$\cos^{-1}\left(-\frac{\tilde{g}_{1j}}{\|\boldsymbol{r}_{1j}\|}\right) \approx \cos^{-1}\left(-\frac{g_{1j}}{\|\boldsymbol{r}_{1j}\|}\right) + \frac{n_{1j}}{\sqrt{\|\boldsymbol{r}_{1j}\|^2 - g_{1j}^2}} \tag{16}$$

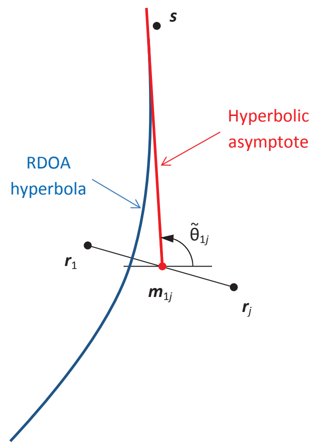

Fig. 1. Hyperbolic asymptote for RDOA measurement $\tilde{g}_{1j}$.

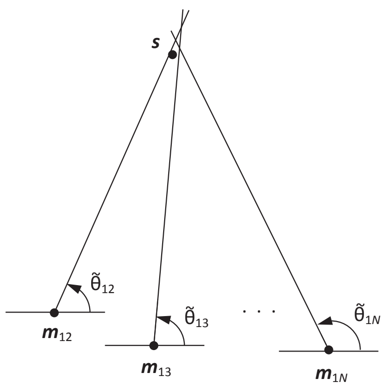

Fig. 2. Equivalent bearings-only localization problem.

whence the noise for the asymptote bearing angle $\tilde{\theta}_{1j}$ is obtained as

$$\epsilon_{1j} \approx \mathcal{I}_{1j} \frac{n_{1j}}{\sqrt{\|\boldsymbol{r}_{1j}\|^2 - g_{1j}^2}}. \tag{17}$$

V. Weighted Pseudolinear Estimator with Systematic Error Correction

A. Weighted Pseudolinear Estimator for TDOA Localization

The TDOA localization problem can be transformed into a bearings-only localization problem as shown in Fig. 2, enabling the bearings-only PLE to be used as a "linear" estimator for TDOA localization:

$$\hat{\boldsymbol{s}}_{\text{WPLE}} = (\boldsymbol{A}^T \boldsymbol{W}^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{W}^{-1} \boldsymbol{b} \tag{18}$$

which we will refer to as the *weighted PLE* (WPLE) for TDOA localization. Here

$$\boldsymbol{A} = \begin{bmatrix} \sin\tilde{\theta}_{12} & -\cos\tilde{\theta}_{12} \\ \vdots & \vdots \\ \sin\tilde{\theta}_{1N} & -\cos\tilde{\theta}_{1N} \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} [\sin\tilde{\theta}_{12}, \ -\cos\tilde{\theta}_{12}]\boldsymbol{m}_{12} \\ \vdots \\ [\sin\tilde{\theta}_{1N}, \ -\cos\tilde{\theta}_{1N}]\boldsymbol{m}_{1N} \end{bmatrix}.$$

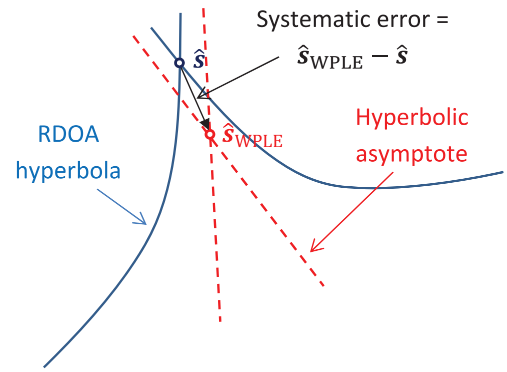

Fig. 3. Illustration of systematic error due to misalignment between asymptotes and hyperbolae.

Assuming small RDOA noise and large range-to-baseline ratio, the weighting matrix is given by

$$\boldsymbol{W} = \boldsymbol{D}\boldsymbol{T}\boldsymbol{D} \tag{19}$$

where $\boldsymbol{T}$ is the Toeplitz matrix in (8) and $\boldsymbol{D}$ is a diagonal matrix defined by

$$\boldsymbol{D} = \text{diag}(d_1, \dots, d_{N-1}) \tag{20}$$

with

$$d_i = \frac{\mathcal{I}_{1,i+1}}{\sqrt{\|\boldsymbol{r}_{1,i+1}\|^2 - \tilde{g}_{1,i+1}^2}}. \tag{21}$$

In matrix $\boldsymbol{D}$, true RDOAs g_{1j} are replaced by RDOA measurements $\tilde{g}_{1j}$.

B. Systematic Errors

The WPLE defined in (18) is a biased estimator for two main reasons:

- The hyperbolic asymptotes do not overlap with the TDOA hyperbolae in the near-field emitters and in poor geometries that are characterized by near collinearity between asymptotes and corresponding sensor pairs (systematic errors)
- The linearization process that leads to the WPLE causes injection of measurement noise into the data matrix $\boldsymbol{A}$, thereby creating correlation between $\boldsymbol{A}$ and the measurement noise vector.

While these problems with the WPLE were discussed in [6], no solution was offered. In this subsection we address the systematic errors and develop a simple and effective correction method.

Fig. 3 illustrates the systematic error arising from asymptote-hyperbola misalignment for the case of two TDOA measurements obtained from 3 sensors. This error can be significant resulting in nonvanishing estimation bias. Exact determination of the systematic error is not possible without solving the nonlinear hyperbolic fixing problem in the first place, e.g., using the MLE. However this will defeat the purpose of developing an alternative WPLE solution to the localization problem at hand. The systematic error can be

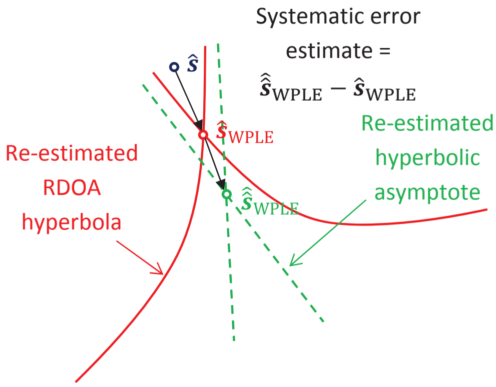

Fig. 4. Estimation of systematic error using initial emitter location estimate. The vector $\hat{\hat{\boldsymbol{s}}}_{\text{WPLE}} - \hat{\boldsymbol{s}}_{\text{WPLE}}$ approximates the true systematic error $\hat{\boldsymbol{s}}_{\text{WPLE}} - \hat{\boldsymbol{s}}$, which is not readily available.

approximated by employing an initial estimate for the emitter location as a substitute for the hyperbolic fixing solution and then finding how much systematic error would have resulted from it if the triangulation of asymptotes were used instead. This is depicted in Fig. 4.

Combining the systematic error approximation method in Fig. 4 with error compensation leads to the following procedure for systematic error correction:

1) Estimate the emitter location $\hat{\boldsymbol{s}}_{\text{WPLE}}$ using the WPLE in (18).
2) Re-estimate the RDOAs from $\hat{\boldsymbol{s}}_{\text{WPLE}}$ using

$$\hat{g}_{1j} = \|\hat{\boldsymbol{s}}_{\text{WPLE}} - \boldsymbol{r}_j\| - \|\hat{\boldsymbol{s}}_{\text{WPLE}} - \boldsymbol{r}_1\|, \quad j = 2, 3, \dots, N. \tag{22}$$

Substituting the re-estimated RDOAs into (5) will give an emitter location estimate exactly at $\hat{\boldsymbol{s}}_{\text{WPLE}}$ as the re-estimated RDOA hyperbolae will intersect uniquely, implying zero "measurement" noise.

3) Next we determine how much error the WPLE introduces using the re-estimated RDOAs. To do this, substitute the re-estimated range differences $\hat{g}_{1j}$ for the measured range differences $\tilde{g}_{1j}$ to re-estimate the bearing angles:

$$\hat{\theta}_{1j} = \tan^{-1}\left(\frac{y_j - y_1}{x_j - x_1}\right) + \mathcal{I}_{1j} \cos^{-1}\left(-\frac{\hat{g}_{1j}}{\|\boldsymbol{r}_{1j}\|}\right). \tag{23}$$

4) Construct $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{D}$ using the re-estimated RDOAs and bearing angles:

$$\begin{aligned}\hat{\boldsymbol{A}} &= \boldsymbol{A}|_{\tilde{\theta}_{12}=\hat{\theta}_{12},\cdots,\tilde{\theta}_{1N}=\hat{\theta}_{1N}}\\ \hat{\boldsymbol{b}} &= \boldsymbol{b}|_{\tilde{\theta}_{12}=\hat{\theta}_{12},\cdots,\tilde{\theta}_{1N}=\hat{\theta}_{1N}}\\ \hat{\boldsymbol{W}} &= \hat{\boldsymbol{D}}\boldsymbol{T}\hat{\boldsymbol{D}}\end{aligned}$$

where $\hat{\boldsymbol{D}} = \text{diag}(\hat{d}_1, \dots, \hat{d}_{N-1})$ with

$$\hat{d}_i = \frac{\mathcal{I}_{1,i+1}}{\sqrt{\|\boldsymbol{r}_{1,i+1}\|^2 - \hat{g}_{1,i+1}^2}}.$$

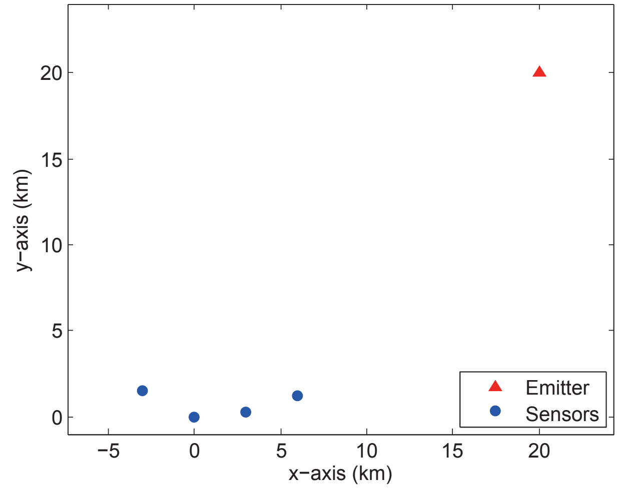

Fig. 5. Simulated TDOA localization geometry. This is a poor geometry as a result of approximate collinearity between some sensor pairs and corresponding RDOA hyperbolae.

5) Re-estimate the emitter location using the WPLE:

$$\hat{\hat{\boldsymbol{s}}}_{\text{WPLE}} = (\hat{\boldsymbol{A}}^T \hat{\boldsymbol{W}}^{-1} \hat{\boldsymbol{A}})^{-1} \hat{\boldsymbol{A}}^T \hat{\boldsymbol{W}}^{-1} \hat{\boldsymbol{b}}. \tag{24}$$

This is the location estimate the WPLE would produce if the true emitter location were at $\hat{\boldsymbol{s}}_{\text{WPLE}}$ and RDOA measurements were available noise-free as given by $\hat{g}_{1j}$.

6) If the WPLE introduced no systematic error, we would have $\hat{\hat{\boldsymbol{s}}}_{\text{WPLE}} = \hat{\boldsymbol{s}}_{\text{WPLE}}$. The systematic error due to approximation of hyperbolae by hyperbolic asymptotes is approximately given by

$$\boldsymbol{\delta} = \hat{\hat{\boldsymbol{s}}}_{\text{WPLE}} - \hat{\boldsymbol{s}}_{\text{WPLE}}. \tag{25}$$

Subtracting this error vector from the original WPLE gives the *WPLE with systematic error correction* (SEC-WPLE):

$$\begin{aligned}\hat{\boldsymbol{s}}_{\text{SEC}} &= \hat{\boldsymbol{s}}_{\text{WPLE}} - \boldsymbol{\delta}\\ &= 2\hat{\boldsymbol{s}}_{\text{WPLE}} - \hat{\hat{\boldsymbol{s}}}_{\text{WPLE}}.\end{aligned} \tag{26}$$

VI. Simulation Studies

In this section we demonstrate the effectiveness of the proposed systematic error correction method by way of simulation examples. The simulated TDOA localization geometry is depicted in Fig. 5. The signal transmitted by an emitter positioned at $\boldsymbol{s} = [20, 20]^T$ km is intercepted by $N = 4$ sensors at $\boldsymbol{r}_1 = [-3, 1.5]^T$ km, $\boldsymbol{r}_2 = [0, 0]^T$ km, $\boldsymbol{r}_3 = [3, 0.3]^T$ km and $\boldsymbol{r}_4 = [6, 1.2]^T$ km. The bias norm and root MSE (RMSE) for the WPLE, SEC-WPLE and MLE were simulated using 5,000 Monte Carlo runs. Note that this is a particularly poor localization geometry as a result of approximate collinearity between some sensor pairs and corresponding RDOA asymptotes. This will cause two problems as will be seen in the simulations: aggravated systematic error and bias problems for the WPLE, and early onset of the threshold effect for the MLE at relatively small noise levels.

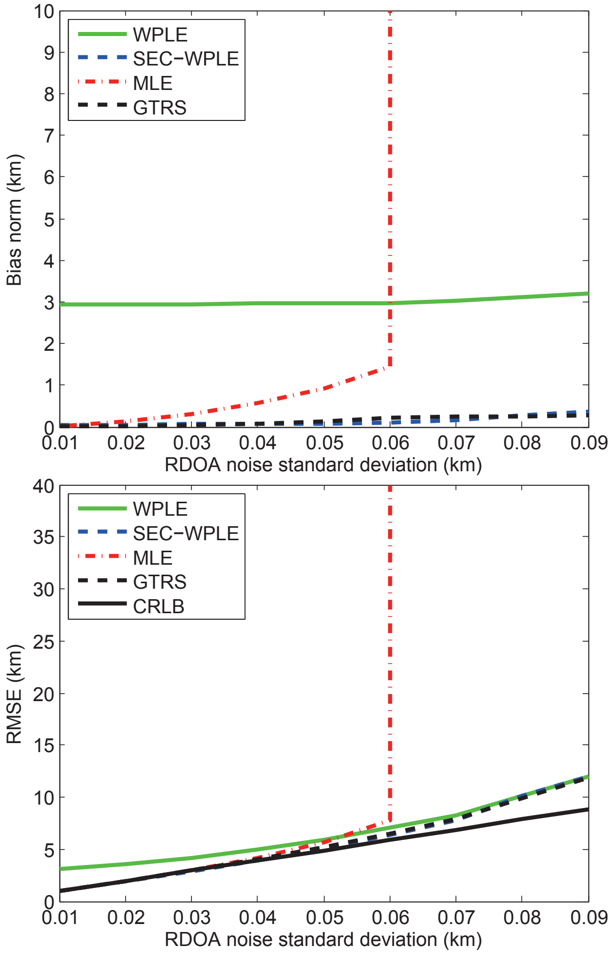

Fig. 6. Comparison of bias and RMSE performance. The proposed SEC-WPLE successfully compensates the systematic errors present in the WPLE and achieves an RMSE performance much better than the MLE and almost identical to the computationally demanding GTRS solution.

We have compared the estimation performance of the WPLE, SEC-WPLE, MLE (using GN iterations) and the constrained optimization solution employing generalized trust region subproblem (GTRS) in [4]. Fig. 6 shows the bias norm and RMSE versus RDOA noise standard deviation for the simulated estimators. The square root of CRLB trace is also included for benchmarking purposes. We observe that the WPLE and SEC-WPLE both perform well compared with the MLE. As the RDOA noise is increased in excess of 70 m, the MLE performance rapidly deteriorates due to the threshold effect. The systematic errors cause the WPLE to have a large nonvanishing bias as is evident from Fig. 6. The SEC-WPLE on the other hand exhibits almost no bias thanks to systematic error compensation. For small RDOA noise, the SEC-WPLE, MLE and GTRS have almost identical estimation performance. At large noise levels, the SEC-WPLE and GTRS perform similarly. However, it should be noted that the GTRS is a very complex solution with extremely high computational complexity compared with the SEC-WPLE. To compare the computational complexities, we measured the average execution times for each algorithm in MATLAB, which produced 1.0431×10^{-4} s (WPLE), 2.111×10^{-4} s (SEC-WPLE), 0.0036 s (MLE), and 0.0033 s (GTRS).

VII. Conclusion

While pseudolinear estimation techniques enjoy low computational complexity, they can be plagued by severe bias problems for a number of reasons. In this paper we presented an effective method to ameliorate the bias problem for the WPLE proposed in [6]. The WPLE is an attractive low-complexity solution for TDOA emitter localization. However it suffers from the so-called systematic errors caused by misalignment between RDOA asymptotes and hyperbolae. These errors become particularly acute in the near field and for poor localization geometries due to non-negligible misalignment between the asymptotes and hyperbolae leading to nonvanishing bias even as the noise tends to zero as observed in Section VI. The proposed systematic error correction method overcomes this problem in a two-stage error estimation process. The efficacy of the SEC-WPLE was demonstrated by way of simulation examples. The SEC-WPLE was observed to outperform the MLE at noise levels near the threshold region and to perform on par with the constrained optimization solution GTRS [4] at a significantly reduced computational complexity.

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