Estimation of DOA and Phase Error Using a Partly Calibrated Sensor Array with Arbitrary Geometry

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***Abstract*—A new method is presented to effectively estimate the direction-of-arrival (DOA) of a source signal and the phase errors of a sensor array with arbitrary geometry. Assuming that one sensor (except the reference one) has been calibrated, the proposed method appropriately reconstruct the data matrix and establish a series of linear equations with respect to the unknown parameters through eigenvalue decomposition (EVD). We build a LS problem with a quadratic constraint and solve it by two approaches. Unlike the conventional methods which are limited to specific array geometries, the proposed can be applied to arbitrary arrays. Moreover, it only requires one calibrated sensor, which may not be consecutively spaced to the reference one. The effectiveness of the proposed method is validated by simulation results.**

I. Introduction

The problem of direction-of-arrival (DOA) estimation using sensor arrays plays an important role in various areas such as wireless communication, radar and radio astronomy [1]-[6]. In general, an accurate knowledge of the array characteristics is required to determine the unknown DOA of the incoming signal. However, the array systems in practical applications usually suffer from various kinds of imperfections and hence, the array manifold is only imprecisely known. In this situation, the performance of direction finding techniques may be significantly degraded due to the mismatch between the actual and nominal array manifolds.

During the past few decades, the problems of array calibration and DOA estimation in the presence of array uncertainties have received extensive attentions [7]-[14]. Assuming that a series of calibration sources are located with exactly known DOAs, the array can be effectively calibrated. In practice, however, the calibration sources are not always available. In order to deal with this problem, some methods are proposed to calibrate arrays in the absence of the exact knowledge of DOAs [9]-[11]. In particular, Weiss and Friedlander proposed an alternative iterative method (named as WF method), which can estimate the DOAs and gain-phase error of each sensor element simultaneously. However, this method may be considerably deteriorated in the presence of relatively large phase uncertainties due to the ambiguity in estimating the phase uncertainties and DOAs. The eigenstructure based methods in [10] and [11] can work well when the phase error is large. Nevertheless, both of these two methods suffer from heavy computational load.

Recently, partly calibrated arrays have received great research interests [12]-[14]. It has been shown in [13] and [14] that if each subarray is calibrated, a spectral rank-reduction algorithm [14] or ESPRIT-like algorithm [13] can be utilized to determine the DOAs. For a partly calibrated uniform linear array (ULA) where some sensors have been calibrated, i.e. the gains/phases of these sensors are known as prior, the shift-invariant property can be employed to estimate the DOAs as well as the gains/phases. It should be noted that the approaches in [12]-[14] require at least one pair of consecutive calibrated sensors.

In this paper, the problem of DOA estimation and phase error calibration in a sensor array with arbitrary geometry is addressed. We develop a new method to estimate DOA of a single signal and phase errors of array provided that one sensor, which is different from the reference one, is calibrated. The proposed method in this paper constructs a series of data matrices and estimates the unknown DOA together with phase errors by LS minimization. For arbitrary array, we propose two approaches (generalized singular value decomposition (BSVD) [15] and semidefinite relaxation (SDR) [16]) to solve the LS problem. Simulation results demonstrate superiority of the proposed method.

II. Problem Formulation

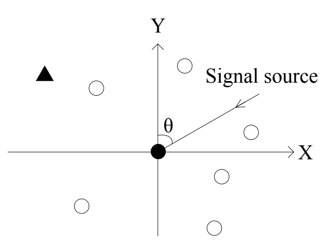

Fig 1: A planar array with one calibrated sensor(● denotes the reference sensor, ▲ denotes calibrated sensor and ○ denotes un-calibrated sensors).

Consider a planar array with M omnidirectional sensors as shown in Fig.1. For simplicity, we assume that the array and the signal source are coplanar and a signal impinges on the array with DOA θ. Ideally, steering vector $\mathbf{a}(\theta)$ is given by

$$\mathbf{a}(\theta) = [1, e^{j2\pi\lambda^{-1}\mathbf{p}_2^{\mathrm{T}}\mathbf{r}}, \cdots, e^{j2\pi\lambda^{-1}\mathbf{p}_M^{\mathrm{T}}\mathbf{r}}]^{\mathrm{T}} \tag{1}$$

where $\mathbf{p}_m = [x_m, y_m]^{\mathrm{T}}$ and $\mathbf{r} = [\sin\theta, \cos\theta]^{\mathrm{T}}$ are the coordinate of the m the sensor and the unit vector in the direction of θ, respectively. In the presence of phase error, the steering vector can be written as

$$\widetilde{\mathbf{a}}(\theta) = \mathbf{\Phi}\,\mathbf{a}(\theta) \tag{2}$$

where $\mathbf{\Phi} = \mathrm{diag}\{e^{j\varphi_1}, \cdots, e^{j\varphi_M}\}$, $\varphi_m (m = 1, 2, \cdots, M)$ denotes the phase error of the mth sensor. The presence of the mismatch between the actual and nominal array manifolds significantly degraded the performance of some classical subspace-based direction finding method, such as MUSIC[2], ESPRIT[1], and so on. The received vector of array is thus given by

$$\mathbf{x}(t) = \widetilde{\mathbf{a}}(\theta)s(t) + \mathbf{n}(t) = \mathbf{\Phi}\,\mathbf{a}(\theta)s(t) + \mathbf{n}(t) \tag{3}$$

where $s(t)$ contains the complex envelope of the signal, $\mathbf{n}(t)$ is a complex gaussian additive $M \times 1$ noise vector. The snapshot data matrix composed of L snapshots can be written as

$$\mathbf{X} = [\mathbf{x}(1), \mathbf{x}(2), \cdots, \mathbf{x}(L)] = \mathbf{\Phi}\,\mathbf{a}(\theta)\,\mathbf{S} + \mathbf{N} \tag{4}$$

where $\mathbf{S} = [s(1), \cdots, s(L)]$ and $\mathbf{N} = [n(1), \cdots, n(L)]$. We take the sensor locates at origin as the reference one, and assume that one of other sensors whose label is c has been calibrated. In other words, it can be assumed that $\varphi_r = 0$ and φ_c is known, where r is the label of the reference sensor. Thus our objective is to simultaneously estimate the DOA and phase errors from array output $\mathbf{X}$ or covariance matrix $\widehat{\mathbf{R}} = \frac{1}{L}\mathbf{X}\mathbf{X}^{\mathrm{H}}$.

III. Proposed Method

Before presenting the proposed phase error calibration method, we first construct some selection matrices as follow

$$e^{jA} + e^{jB} = 2\mathrm{Re}(e^{j\frac{A-B}{2}})e^{j\frac{A+B}{2}} \tag{5}$$

where A and B are arbitrary real numbers.

A. Phase-error calibration

Let $\mathbf{X}(m,:)$ be the mth row of $\mathbf{X}$ where $m = 1, 2, \cdots, M$, and denote $\mathbf{X}^{i+k}$ as the summation of $\mathbf{X}(i,:)$ and $\mathbf{X}(k,:)$, i.e.,

$$\begin{aligned}\mathbf{X}^{i+k} &= \mathbf{X}(i,:) + \mathbf{X}(k,:) \\ &= \left[e^{j(2\pi\lambda^{-1}\mathbf{p}_i^{\mathrm{T}}\mathbf{r}+\varphi_i)} + e^{j(2\pi\lambda^{-1}\mathbf{p}_k^{\mathrm{T}}\mathbf{r}+\varphi_k)}\right]\mathbf{S} + \mathbf{N}^{i+k}\end{aligned} \tag{6}$$

where $\mathbf{N}^{i+k} = \mathbf{N}(i,:) + \mathbf{N}(k,:)$ is the compound noise. From (5) we can rewrite $\mathbf{X}^{i+k}$ as

$$\mathbf{X}^{i+k} = c_{i+k} e^{j\Psi_{i+k}} \mathbf{S} + \mathbf{N}^{i+k} \tag{7}$$

where $c_{i+k} = 2\mathrm{Re}\left[e^{j\left(\pi\lambda^{-1}(\mathbf{p}_i - \mathbf{p}_k)^{\mathrm{T}}\mathbf{r} + \frac{\varphi_i - \varphi_k}{2}\right)}\right]$ is a real value and $\Psi_{i+k} = \pi\lambda^{-1}(\mathbf{p}_i + \mathbf{p}_k)^{\mathrm{T}}\mathbf{r} + \frac{\varphi_i + \varphi_k}{2}$ which includes the unknown parameters φ_i, φ_k and θ. Ψ_{i+k} can be rewritten as

$$\Psi_{i+k} = \mathbf{b}_{i+k}^{\mathrm{T}} \mathbf{u} \tag{8}$$

where $\mathbf{u} = [\varphi_1, \varphi_2, \cdots, \varphi_M, \sin\theta, \cos\theta]^{\mathrm{T}}$ is a $(M+2) \times 1$ vector and $\mathbf{b}_{i+k}$ is the coefficient vector. Now, let us construct $\mathbf{Y}^{i+k}$ as

$$\begin{aligned} \mathbf{Y}^{i+k} &= \begin{bmatrix} \mathbf{X}(1,:) \\ \mathbf{X}^{i+k} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ c_{i+k} e^{j\Psi_{i+k}} \end{bmatrix} \mathbf{S} + \begin{bmatrix} \mathbf{N}(1,:) \\ \mathbf{N}^{i+k} \end{bmatrix} \end{aligned} \tag{9}$$

According to the subspace principal, we have the following equation

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ c_{i+k} e^{j\Psi_{i+k}} \end{bmatrix}\right) = \mathrm{span}\left(\gamma^{i+k}\right) \tag{10}$$

where γ^{i+k} is the principal eigenvector of $\mathbf{R}^{i+k}$, which is the corresponding covariance matrix of $\mathbf{Y}^{i+k}$. If the first entry of γ^{i+k} is normalized to one, we have

$$\begin{bmatrix} 1 \\ c_{i+k} e^{j\Psi_{i+k}} \end{bmatrix} = \gamma^{i+k} \tag{11}$$

Notice that c_{i+k} may be negative and this would lead to a π-ambiguity in estimation of Ψ_{i+k}. However, if the following inequality hold ture

$$\max(\mathrm{abs}(\Psi_{i+k})) \leq \frac{\pi}{2} \tag{12}$$

we have

$$\Psi_{i+k} = \mathbf{b}_{i+k}^{\mathrm{T}} \mathbf{u} = \angle\left[\gamma^{i+k}(2)\right] \tag{13}$$

Because the value of Ψ_{i+k} obtained from (13) is on the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and hance, it can be estimated correctly.

From the analysis above, we know that for given i and k, equation (13) can provide a set of measurements of φ_i, φ_k and θ with the certain weighted coefficient $\mathbf{b}_{i+k}$. Provided that $\varphi_1 = 0$ and φ_c has been known, (13) can be rearranged as

$$\mathbf{w}_{i+k}^{\mathrm{T}} \mathbf{v} = \angle\left[\gamma^{i+k}(2)\right] - \varphi_c \, \mathbf{b}(c) \tag{14}$$

where $\mathbf{v} = [\varphi_2, \cdots, \varphi_{c-1}, \varphi_{c+1}, \cdots, \varphi_M, \sin\theta, \cos\theta]^{\mathrm{T}}$ is modified parameter vector by discarding φ_1 and φ_c from $\mathbf{u}$. $\mathbf{w}_{i+k}$ is similarly obtained by discarding the corresponding terms from $\mathbf{b}_{i+k}$.

The analysis above shows that we can establish a series of linear equations with respect to the unknown parameters under mild conditions. If we can obtain enough equations, it is possible to determine the unknown parameters. In practical, however, there may be a small portion of i and k satisfying the conditions expressed by (12). This would result in an under-determined problem which gives innumerable solutions. In the next subsection, we introduce a method to increase the number of equations that satisfying the given conditions.

B. Method to improve practicality

It is assumed that ϕ_m is zero-mean and distributed uniformly in the range of $[-\Delta, \Delta]$. We have known the the DOA of the signal locates in $[\theta_0 - \delta, \theta_0 + \delta]$, where θ_0 is a coarse estimation of the DOA by using a direction finding method such as Capon beamforming, and δ describes the perturbation of the DOA estimation. A careful examination of Ψ_{i+k} shows that a sufficient condition making (12) hold is given by

$$\begin{aligned} \pi\lambda^{-1}(\mathbf{p}_i + \mathbf{p}_k)^{\mathrm{T}}\mathbf{r} - C_{i+k} &\in [-\Delta_{i+k}, \Delta_{i+k}] \\ \text{for } \theta &\in [\theta_0 - \delta, \theta_0 + \delta] \end{aligned} \tag{15}$$

and

$$\Delta + \Delta_{i+k} \leq \frac{\pi}{2} \tag{16}$$

where C_{i+k} is the phase of compensation term denoted by

$$C_{i+k} = \frac{\varphi_{Cmax}^{i+k} + \varphi_{Cmin}^{i+k}}{2} \tag{17}$$

In (17), φ_{Cmax}^{i+k} and φ_{Cmin}^{i+k} can be determined by

$$\varphi_{Cmax}^{i+k} = \max_{\theta}\left[\pi\lambda^{-1}(\mathbf{p}_i + \mathbf{p}_k)^{\mathrm{T}}\mathbf{r}\right] \tag{18}$$

$$\varphi_{Cmin}^{i+k} = \min_{\theta}\left[\pi\lambda^{-1}(\mathbf{p}_i + \mathbf{p}_k)^{\mathrm{T}}\mathbf{r}\right] \tag{19}$$

where $\theta \in [\theta_0 - \delta, \theta_0 + \delta]$. From (15) to (19), we know that

$$\Delta_{i+k} = \frac{\varphi_{Cmax}^{i+k} - \varphi_{Cmin}^{i+k}}{2} \quad (20)$$

Then the sufficient condition that make (16) hold can be expressed as

$$\Delta + \frac{\varphi_{Cmax}^{i+k} - \varphi_{Cmin}^{i+k}}{2} \leq \frac{\pi}{2}, \text{for } \theta \in [\theta_0 - \delta, \theta_0 + \delta] \quad (21)$$

If equation (16) is satisfied for the given i and k, we can use $e^{-jC_{i+k}}$ to compensate $\mathbf{X}(i,:)$ and $\mathbf{X}(k,:)$, then sum them up to obtain $\widetilde{\mathbf{X}}^{i+k}$. This process can be represented mathematically as follows

$$\begin{aligned} \widetilde{\mathbf{X}}^{i+k} &= \mathbf{X}(i,:)e^{-jC_{i+k}} + \mathbf{X}(k,:)e^{-jC_{i+k}} \\ &= c_{i+k}e^{j\widetilde{\Psi}_{i+k}}\,\mathbf{S} + \widetilde{\mathbf{N}}^{i+k} \end{aligned} \quad (22)$$

where $\widetilde{\Psi}_{i+k} = \Psi_{i+k} - C_{i+k}$. According to the analysis above, it is known that $\max\left(\text{abs}\left(\widetilde{\Psi}_{i+k}\right)\right)$ is no larger than $\frac{\pi}{2}$. Construct $\widetilde{\mathbf{Y}}^{i+k}$ as $\widetilde{\mathbf{Y}}^{i+k} = \begin{bmatrix} \mathbf{X}(1,:) \\ \widetilde{\mathbf{X}}^{i+k} \end{bmatrix}$, and compute $\widetilde{\gamma}^{i+k}$ throuth EVD of $\widetilde{\mathbf{R}}^{i+k}$, which denotes the covariance matrix of $\widetilde{\mathbf{Y}}^{i+k}$, we have the following equation developed from (14)

$$\mathbf{w}_{i+k}^{\mathrm{T}}\mathbf{v} = \angle\left[\widetilde{\gamma}^{i+k}(2)\right] - \varphi_c\mathbf{b}(c) + C_{i+k} \quad (23)$$

C. Least squares problem with a quadratic constraint

Now, it can be found that we could obtain a series of equations about the unknown parameters $\mathbf{v}$, according to (23) under specific condition. Let us define $\mathbf{W}$ as the coefficient matrix that constructed by all $\mathbf{w}_{i+k}^{\mathrm{T}}$ along column direction, and define $\mathbf{d}$ as the data vector piled up with the right part of (23), where i and k are some specific indices that make (16) hold. Then the problem of DOA and phase error estimation can be equivalently described as a LS optimization problem with a quadratic constraint

$$\begin{aligned} &\min_{\mathbf{v}} \ \|\mathbf{W}\,\mathbf{v} - \mathbf{d}\,\|^2 \\ &s.t.\ \|\,\mathbf{Z}\,\mathbf{v}\,\|^2 = 1 \end{aligned} \quad (24)$$

Assume that $\widetilde{\mathbf{v}}$ is a feasible solution of (24), then the constraint matrix $\mathbf{Z} = \left[\mathbf{0}_{2\times(M-2)}\ \ \mathbf{I}_2\right]$ makes $\sin^2\theta + \cos^2\theta = 1$ hold. It is known from [15] that if rank $\left([\mathbf{W}^{\mathrm{T}}\ \mathbf{Z}^{\mathrm{T}}]^{\mathrm{T}}\right) = M$, the problem of (24) can be solved. In this paper, we solve the problem of (24) by two different approaches.

The first approach is based on generalized singular value decomposition (BSVD). Using the Lagrange multiplier method, we can formulate normal equation as

$$\begin{aligned} &(\mathbf{W}^{\mathrm{T}}\mathbf{W} + \mu\mathbf{Z}^{\mathrm{T}}\mathbf{Z})\,\mathbf{v} = \mathbf{W}^{\mathrm{T}}\mathbf{d} \\ &\|\,\mathbf{Z}\,\mathbf{v}\,\|^2 = 1 \end{aligned} \quad (25)$$

The normal equation may have several solutions $(\mathbf{v}_i, \mu_i)$ and the optimal solution is given as the solution with largest μ. The solution to (24) can be achieved with the BSVD method, cf. [15] for more details.

The second approach is based on semidefinite relaxation (SDR). In order to apply the SDR technique to (24), we introduce an extra variable k and formulate the following optimization problem

$$\begin{aligned} &\min_{\mathbf{v},k} \ \|\,\mathbf{W}\,\mathbf{v} - k\,\mathbf{d}\,\|^2 \\ &s.t.\ \|\,\mathbf{Z}\,\mathbf{v}\,\|^2 = 1, k^2 = 1 \end{aligned} \quad (26)$$

Subsequently, SDR can be applied and the optimal solution $\mathbf{v}^{\star}$ can be obtained by either gaussian randomization or EVD, cf. [16] for more details.

IV. Simulations and Results

In this section, numerical experiments are provided to make comparisons with other techniques. In the next simulations, the phase error $\{\varphi_m\}_{m=1}^{M}$ of sensors are generated by

$$\varphi_m = \sqrt{12}\sigma_\varphi\eta_m \quad (27)$$

where η_m is independent and identically distributed random variable which is distributed uniformly in the range of $[-0.5, 0.5]$, σ_φ is the standard deviation of φ_m.

The sensor configuration is shown in Fig.2, where sensor 1 that locates at origin is taken as the reference one,

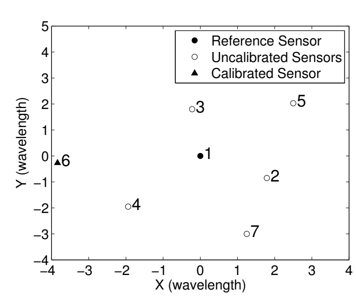

Fig 2: Array configuration.

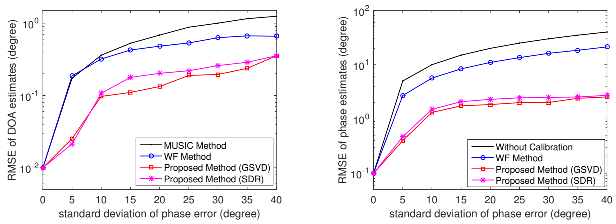

Fig 3: RMSE of estimates versus standard deviation of phase error.

sensor 6 has been calibrated. Each of other sensors has an unknown phase error. Consider one signal impinging on the array from direction at $\theta = 26°$, and we have known it is located at $[\theta_0 - \delta, \theta_0 + \delta]$ with $\theta_0 = 24°$ and $\delta = 8°$. The signal-to-noise ratio (SNR) is 20dB, and number of samples is 512. Based on 2000 experiments, the root mean square error (RMSE) of DOA and phase error estimates versus the standard deviation of the phase error σ_φ are shown in Fig.3. It can be seen from Fig.3 that all methods degrade as σ_φ increases. The WF method performs slightly better than MUSIC algorithm using partly calibrated sensors, and the propose method outperforms the WF method.

V. Conclusion

In this paper, we present a new method for estimating DOA of a single signal and phase errors of array with one calibrated sensor. A new way to construct a series of equations with respect to the unknown parameters is developed. The DOA and phase errors are then determined from a LS minimization problem with a quadratic constraint. The proposed method is computationally attractive. Furthermore, it allows us to calibrate a sensor array with arbitrary geometry by using one calibrated sensor, and does not require two calibrated sensors to be consecutively spaced. The effectiveness of the proposed method is confirmed by various simulation results.

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