Perfect Periodic Sequences for Nonlinear Wiener Filters

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***Abstract*—A periodic sequence is defined as a perfect periodic sequence for a certain nonlinear filter if the cross-correlation between any two of the filter basis functions, estimated over a period, is zero. Using a perfect periodic sequence as input signal, an unknown nonlinear system can be efficiently identified with the cross-correlation method. Moreover, the basis functions that guarantee the most compact representation according to some information criterion can also be easily estimated. Perfect periodic sequences have already been developed for even mirror Fourier, Legendre and Chebyshev nonlinear filters. In this paper, we show they can be developed also for nonlinear Wiener filters. Their development is non-trivial and differs from that of the other nonlinear filters, since Wiener filters have orthogonal basis functions for white Gaussian input signals. Experimental results highlight the usefulness of the proposed perfect periodic sequences in comparison with the Gaussian input signals commonly used for Wiener filter identification.**

***Index Terms*—Perfect periodic sequences, nonlinear Wiener filters, cross-correlation method**

I. Introduction

The Wiener nonlinear (WN) filters [1], [2], which derive from the truncation of the Wiener series, were introduced to overcome one of the main limitations of Volterra filters. The Volterra filters basis functions are never orthogonal, not even for white input signals. The Wiener series was introduced by applying an orthogonalization procedure for white Gaussian inputs to the Volterra series, deriving the so-called Wiener G-functionals. The WN filters derive from the double truncation with respect to order and memory of the Wiener series. They can be expressed as a linear combination of Wiener basis functions, which are orthogonal for white Gaussian input signals. As a consequence, the WN filter coefficients modeling and unknown nonlinear system can be computed using the cross-correlation method, i.e., calculating the cross-correlation between the basis functions and the unknown system output. The cross-correlation method applied to stochastic inputs presents many drawbacks. First of all, it often requires million of samples to accurately estimate the filter coefficients. Moreover, an exact white Gaussian input cannot be generated, since it is necessary to saturate the maximum sample amplitude. Furthermore, also by using double precision calculation, the central moments of a Gaussian input soon depart from ideal values as the order of the filter increases, unless millions of values are used [3]. So, in the identification with cross-correlation methods like that of Lee-Schetzen [1], it was shown [3] that the input non-ideality affects particularly the diagonal points of the kernels. This problem is further exacerbated in presence of errors due to model order truncation.

Improvements to the Lee-Schetzen's method, in order to overcome the problem of input non-ideality in the identification of diagonal points were provided in [3], [4]. Furthermore a solution to mitigate the identification errors due to model order truncation is proposed in [5].

Recently, other families of polynomial filters, which guarantee the orthogonality of the basis functions for appropriate stochastic inputs, were introduced. The even mirror Fourier nonlinear (EMFN) filters [6] and the Legendre nonlinear (LN) filters [7] have orthogonal basis functions for white uniform input signals in $[-1, +1]$. The EMFN filters are based on trigonometric expansions with even symmetry of the input signal samples, and do not include a linear term among the basis functions. In contrast, the LN basis functions are products of Legendre polynomials of the input signal samples and include a linear term in the first order basis functions. Also the Chebyshev nonlinear (CN) filters [8] have orthogonal basis functions for white input signals with a particular nonuniform distribution in $[-1, +1]$. The CN filters are based on Chebyshev polynomial expansion of the input signal samples and include a linear term.

As an alternative to white random signals, appropriate deterministic input signals have been proposed for system identification. Among them, perfect periodic sequences (PPSs) [9], [10] have been used as inputs for linear system identification [11], [12]. In case of nonlinear filters, a periodic sequence is defined as a PPS if the cross-correlation between any two different basis functions, estimated over a period, is zero. Therefore, using a PPS as input signal it is also possible to efficiently identify an unknown system with the cross-correlation method. Moreover, the basis functions that guarantee the most compact representation of the nonlinear system according to some information criterion can also be easily estimated. In this context, PPSs have already been developed for EMFN [13], [14], LN [15], [16], and CN filters [8]. The PPSs have been obtained by imposing the orthogonality of the basis functions and solving a system of nonlinear equations with an iterative approach.

In this paper, we show that PPSs can be developed also for Wiener filters. Since the Wiener basis functions are orthogonal for white Gaussian input signals, the development of the PPSs is non trivial and differs from that of EMFN, LN, and CN

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filters. The perfect orthogonality of the Wiener basis functions for PPS inputs allows to avoid a priori the accuracy problems in the estimation of the kernels diagonal points. In the experimental results the PPSs are used for system identification, that it is a fundamental issue of several applications, such as, nonlinear system equalization and nonlinear effects emulation.

The paper is organized as follows. Section II derives the Wiener basis functions and discuss the cross-correlation method. Section III introduces the PPSs for WN filters. Section IV provides several experimental results demonstrating the advantages of nonlinear system identification based on cross-correlation method and PPSs. Concluding remarks follow in Section V.

The following notation is used throughout the paper: $E[\cdot]$ denotes mathematical expectation, $< \cdot >_L$ denotes average over a period of L samples, $\mathcal{N}(0, \sigma_x^2)$ indicates the zero mean, variance σ_x^2, normal distribution.

II. Wiener Nonlinear filter

In contrast to the classical approach for introducing the WN filter, in this section we first derive the set of Wiener basis functions. Then the WN filter is defined as the linear combination of the Wiener basis functions up to a certain order P and memory N.

We are interested in deriving a set of polynomial basis functions that can arbitrarily well approximate a discrete time, time-invariant, finite memory, continuous, nonlinear system,

$$y(n) = f[x(n), x(n-1), \ldots, x(n-N+1)] \tag{1}$$

with f a continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$. The set of basis functions should be orthogonal for any white Gaussian input signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$.

Let us first consider the one-dimensional case, i.e., $N = 1$. The nonlinear system can be arbitrarily well approximated with the set of polynomial basis functions,

$$\{1,\ \ x(n),\ \ x^2(n),\ \ x^3(n),\ \ \ldots\}, \tag{2}$$

which however are not orthogonal for white Gaussian input signals. A set of orthogonal polynomial basis functions can be obtained by applying the Gram-Schmidt orthogonalization to the set in (2), as follows

$$\{1,\ \ x(n),\ \ x^2(n) - \sigma_x^2,\ \ x^3(n) - 3\sigma_x^2 x(n),\ \ \ldots\}. \tag{3}$$

In order to derive a set of orthogonal basis functions for the N-dimensional case, the same procedure of [6], [7], and [8] is followed. First, the one-dimensional basis functions are written for $x(n), x(n-1), \ldots, x(n-N+1)$,

$$\begin{array}{l} 1,\ \ x(n),\ \ x^2(n) - \sigma_x^2,\ \ \ldots \\ 1,\ \ x(n-1),\ \ x^2(n-1) - \sigma_x^2,\ \ \ldots \\ \qquad\qquad\vdots \\ 1,\ \ x(n-N+1),\ \ x^2(n-N+1) - \sigma_x^2,\ \ \ldots \end{array}$$

Then, the terms with different variable are multiplied in any possible manner, taking care of avoiding repetitions. These basis functions and their linear combinations form an algebra on any compact in $\mathbb{R}^N$ that satisfies the requirements of the Stone-Weierstrass theorem [17]. Thus, the Wiener basis functions can arbitrarily well approximate the system in (1). The Wiener basis functions of order from 0 to 3 and memory N are summarized in Table I. A Wiener filter of order P, memory N, is a linear combination of the Wiener basis functions $w_i(n)$ up to the order P and memory N,

$$\tilde{y}(n) = \sum_i c_i w_i(n). \tag{4}$$

It is easy to verify that by construction the Wiener basis functions are orthogonal for a white Gaussian input signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$, i.e., $E[w_i(n) w_j(n)] = 0$ for any $i \neq j$. Thus, the coefficients c_i in (4) can be estimated with the cross-correlation approach as follows:

$$c_i = \frac{E[y(n) w_i(n)]}{E[w_i^2(n)]}, \tag{5}$$

where $y(n)$ is the unknown nonlinear system output. The most relevant basis functions, i.e., the basis functions that guarantee the most compact representation according to some information criterion can also be estimated. Indeed, exploiting the orthogonality, the mean square error reduction (MSE) provided by a basis function $w_i(n)$ is

$$\delta\text{MSE}_i = \frac{E[y(n) w_i(n)]^2}{E[w_i^2(n)]}. \tag{6}$$

The basis functions can be ranked according to the MSE reduction they produce and the most important basis functions according to some information criterion can be selected. In

TABLE I
THE WIENER BASIS FUNCTIONS

Order 0
1.
Order 1
$x(n), x(n-1), \ldots, x(n-N+1)$.
Order 2
$x^2(n) - \sigma_x^2, \ldots, x^2(n-N+1) - \sigma_x^2,$
$x(n)x(n-1), \ldots, x(n-N+2)x(n-N+1),$
$\vdots$
$x(n)x(n-N+1)$.
Order 3
$x^3(n) - 3\sigma_x^2 x(n), \ldots, x^3(n-N+1) - 3\sigma_x^2 x(n-N+1),$
$(x^2(n) - \sigma_x^2)x(n-1), \ldots, (x^2(n-N+2) - \sigma_x^2)x(n-N+1),$
$(x^2(n) - \sigma_x^2)x(n-2), \ldots, (x^2(n-N+3) - \sigma_x^2)x(n-N+1),$
$\vdots$
$(x^2(n) - \sigma_x^2)x(n-N+1),$
$x(n)(x^2(n-1) - \sigma_x^2), \ldots, x(n-N+2)(x^2(n-N+1) - \sigma_x^2),$
$x(n)(x^2(n-2) - \sigma_x^2), \ldots, x(n-N+3)(x^2(n-N+1) - \sigma_x^2),$
$\vdots$
$x(n)(x^2(n-N+1) - \sigma_x^2),$
$x(n)x(n-1)x(n-2), \ldots,$
$\vdots$
$x(n)x(n-N+2)x(n-N+1)$.

Section IV, the Bayesian information criterion [18] will be used in the experimental results.

In the next Section, PPSs for WN filters are derived. The PPS input signal guarantees the orthogonality of the basis functions over a period of the sequence, i.e., $< w_i(n)w_j(n) >_L = 0$, for any $i \neq j$. The formulas in (5) and (6) still hold provided the expectation $E[\cdot]$ is replaced by the time average $< \cdot >_L$.

III. PPS FOR NONLINEAR WIENER FILTER

We are interested in developing a PPS $x_{\rm p}(n)$ of period L for a Wiener filter of order P, memory N, and Gaussian input variance σ_x^2. Moreover, we want the PPS to be bounded by 1, i.e., $|x_{\rm p}(n)| < 1$ for all n, so that the input sequence can be faithfully reproduced by digital to analog converters.

The PPS should guarantee the orthogonality of the basis functions over a period. The Wiener basis functions are orthogonal for a white Gaussian input signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$. Let us consider a periodic sequence that over a period provides the same joint moments of a Gaussian process $\mathcal{N}(0, \sigma_x^2)$ up to the order $2P$ and memory N. For the construction rule of the Wiener basis functions, the periodic sequence guarantees the orthogonality of the Wiener basis functions up to the order P and memory N, $< w_i(n)w_j(n) >_L = 0$, for any $i \neq j$, and thus is a PPS.

Differently from [8], [14], [16], in this paper to develop PPSs for WN filters we impose the following system of nonlinear equations:

$$< x_{\rm p}^{k_0}(n) \cdot x_{\rm p}^{k_1}(n-1) \cdot \ldots \cdot x_{\rm p}^{k_{N-1}}(n-N+1) >_L = \\ = \mu_{k_0} \cdot \mu_{k_1} \cdot \ldots \cdot \mu_{k_{N-1}}, \quad (7)$$

for all $k_0, k_1, \ldots, k_{N-1} \in \mathbb{N}$ with $k_0 > 0$ and $k_0 + k_1 + \ldots + k_{N-1} \leq 2P$, and μ_k the k-th moment of the Gaussian process $\mathcal{N}(0, \sigma_x^2)$,

$$\mu_k = E[x^k(n)] = \begin{cases} 0 & \text{for } k \text{ odd}, \\ \sigma_x^k (k-1)!! & \text{for } k \text{ even}, \end{cases} \quad (8)$$

with $q!! = q \cdot (q-2) \cdot (q-4) \cdot \ldots \cdot 1$.

The nonlinear system in (7) has a number of equations equal to the number of different basis functions in a Volterra filter of order $2P-1$ and memory N (indeed, since $k_0 > 0$, there is always a factor $x_{\rm p}(n)$ in (7)). Thus, the number of equations is $Q = \binom{N+2P-1}{N}$. For sufficiently large L, this is an underdetermined system of equations in the variables $x_{\rm p}(n)$ that may have infinite solutions. Any algorithm capable of solving systems of nonlinear equations can be used for finding a solution to the system in (7). We have applied for this purpose the Newton-Raphson method, which has been implemented as described in [19, ch. 9.7] with the only modification of reflecting the variables $x_{\rm p}(n)$ in $[-1, +1]$ every time they exceeded the range. This modification allows to obtain a sequence bounded by -1 and $+1$, as desired, but the Newton-Raphson method converges to a solution only if the signal power σ_x^2 is sufficiently small. Indeed, the PPSs for Wiener filters have a sample distribution that is similar to the Gaussian. Intuitively, the modified Newton-Raphson method can converge only if the probability of finding samples outside the range $[-1, +1]$ is sufficiently small. In the presented approach the iterations of Newton-Raphson method starts from a random Gaussian distribution of the variables with variance σ_x^2 and the Jacobian matrix is computed analytically. For L ranging between $3Q$ and $4Q$ and for $\sigma_x^2 \leq 1/10$, we have always been able to find a solution for the system in (7). The number of iterations necessary to find a solution depends on the ratio L/Q and on the signal power σ_x^2. Employing a numerical method, only an approximate solution is obtained for the PPS. Nevertheless, the precision of the solution can be arbitrarily improved acting on the stop-condition of the Newton-Raphson method.

The main problem in the derivation of PPSs is the large number of equations Q of the system in (7). Indeed, Q increases exponentially with the order P and geometrically with the memory N. A possibility for reducing Q is that of exploiting sequences with a specific structure. As done in [20] and [7], the number of equations can almost be halved by imposing symmetry (when for every subsequence $a_1, a_2, \ldots, a_N$, there is also the reversed one $a_N, \ldots, a_1$), oddness (when for any subsequence $a_1, \ldots, a_N$, there is also the negated one $-a_1, \ldots, -a_N$), oddness-1 (when for every subsequence $a_1, a_2, a_3, a_4, \ldots, a_N$, there is the subsequence formed by alternatively negating the terms $a_1, -a_2, a_3, -a_4, \ldots, -a_N$). With symmetry, for every couple of symmetric joint moments (e.g., $E[x(n)x^3(n-1)]$ and $E[x^3(n)x(n-1)]$) it suffices to consider only one of them. With oddness, all odd joint moments are a priori zero. With oddness-1, all odd-1 joint moments are a priori zero. We define odd-1 those joint moments that change sign by alternatively negating the sign of the samples. Similarly, odd-2 and odd-4 moments could be considered. We can impose at the same time different structural conditions. The reduction in the number of equations Q obtained with these conditions is often important for being able to find a solution to the system in (7) in acceptable time. Indeed, the Newton-Raphson algorithm has memory and processing time requirements that grow with Q^3.

Different PPSs for WN filters of order 3, signal power $\sigma_x^2 = 1/12$, and memory N ranging from 5 to 20, have been developed and are available for download [21].

IV. EXPERIMENTAL RESULTS

A. First experiment

In the first experiment, a simulated nonlinear system has been considered with a sampling frequency of 44.1 kHz. In particular, it consists of a cascade of a linear filter (i.e., a lowpass filter given by the scaling function of the Daubechies Wavelet of order 10 [5]) and a static nonlinearity that is given by the following function:

$$f(x) = \frac{4.5}{1 + 2e^{-\alpha x}} - \frac{4.5}{3}, \quad (9)$$

where α is the constant used to vary the degree of nonlinearity in the performed experiments.

Figure 1 shows the second, third, and total harmonic distortions on a 1 000 Hz signal at different α. The system of

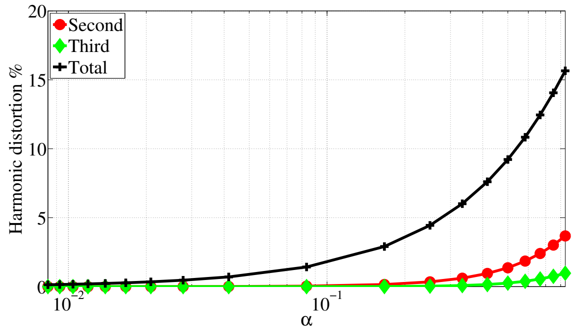

Fig. 1. Second, third and total harmonic distortion.

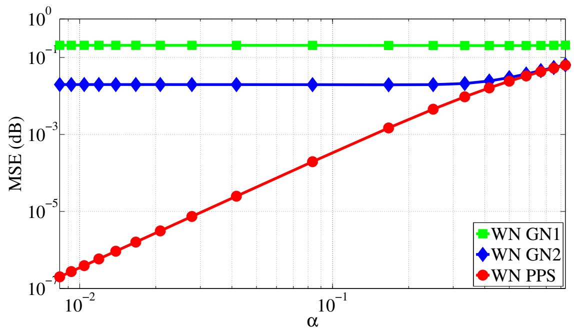

Fig. 2. Mean square error for Wiener filters.

(9) has been identified with Wiener filter exploiting a PPS (WN PPS) suitable for filters of order 3, memory 10, and a period of 12 012 samples, a Gaussian Noise of 12 012 samples (WN GN1) and 1 000 000 samples (WN GN2). The system coefficients have been estimated using the cross-correlation method presented in [3] and the obtained results have been evaluated considering the mean square error between the desired and the estimated output signals. Figure 2 shows that the WN PPS provides the lowest error values in comparison with the Gaussian noise. More in detail, increasing α and thus the effect of the system nonlinearity, the behavior of the identified models tends to approach since the adopted order 3 is insufficient to approximate the nonlinear system characteristics.

B. Second experiment

In the second experiment, we consider the identification of an audiophile vacuum tube preamplifier, Behringer Tube Ultragain Mic 100. Acting on the gain control of the preamplifier, different levels of nonlinear distortion can be imposed. Twenty-two different settings of the gain control have been considered, with increasing value of the nonlinear distortion. Figure 3 shows at the different settings the second, third, and total harmonic distorion on a 200 Hz signal having signal power $< x^2(n) >_L = 1/12$. Different input signals have been fed to the preamplifier and the output signals have been recorded at 8 kHz sampling frequency using a laptop PC. At 8 kHz sampling frequency the preamplifier has a memory lower than 20 sample. Thus, the preamplifier has been identified with PPSs for Wiener, EMFN, LN, and CN filters using sequences

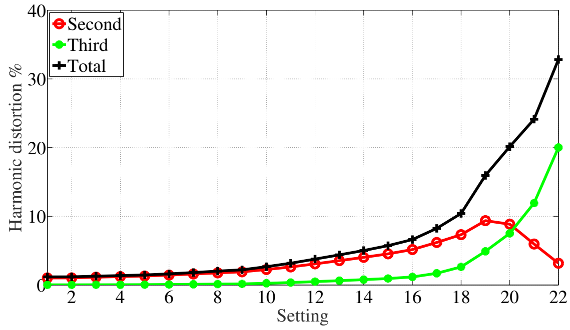

Fig. 3. Second, third and total harmonic distortion.

suitable for filters of order 3, memory 20, and with a period of 357 956 samples. The perfect sequences had the same signal power, i.e., $1/12$. Thus, the peak amplitude of the PPS for WN filter, which has been normalized to 1, is twice the peak amplitude of the PPS for EMFN and LN filters, and it is $\sqrt{6}$-times larger than that of CN filters. The preamplifier has also been identified with a zero mean white Gaussian noise with variance $1/12$. In this case, the 0.055% of the samples that exceeded the range $[-1, +1]$ were truncated. In all conditions, the signal to noise ratio was larger than 46 dB.

The coefficients of the filter have been first estimated with the cross-correlation method. Then, the most relevant basis functions were selected by minimizing the Bayesian information criterion,

$$B(\nu) = L \log_e[\sigma_\epsilon^2(\nu)] + \nu \log_e[L] \tag{10}$$

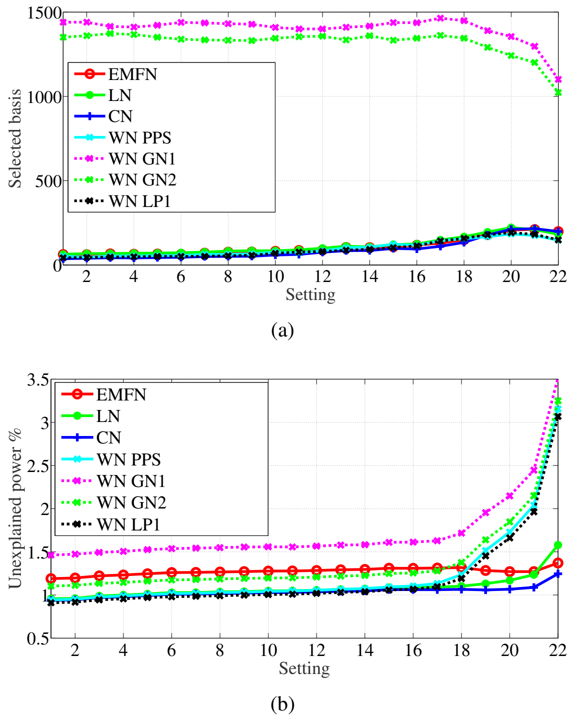

Fig. 4. Number of selected bases (a) and unexplained power (b) for Wiener, EMFN, LN, CN filters.

with $\sigma_\epsilon^2(\nu)$ is the variance of the residual error associated to the first ν most relevant terms of the model and L is the number of data used for the model estimation.

Since the Wiener basis functions are only approximately orthogonal for a white Gaussian noise input, the WN filter has also been identified with an exact least-square algorithm, i.e., the method of Li Peng Irwin [22].

Figure 4 shows the number of selected terms and the percentage of unexplained power (the ratio in percent between the residual MSE and the power of the output signal) in the different identifications. In the figure, the EMFN, LN, and CN filters have been identified with the cross-correlation method over a PPS period. The WN filter has been identified with the cross-correlation method on a PPS period (Wiener PPS), on $357\,956$ Gaussian noise samples (Wiener GN1), on $1\,000\,000$ Gaussian noise samples (Wiener GN2), or with the method of Li Peng Irwin on $357\,956$ Gaussian noise samples (Wiener LPI). The WN filter has been identified on the Gaussian noise also with the method of [5], obtaining MSE results identical to those presented in Figure 4(b) with plots WN GN1 and WN GN2.

Figure 4 clearly shows the very large number of selected basis functions and the worse percentage of unexplained power of the WN filter estimated with the cross-correlation method on Gaussian noise inputs. On the contrary, when the WN filter is identified with a PPS or with the method of Li Peng Irwin, the number of selected basis functions and the percentage of unexplained power are much lower and very similar to values obtained for the other polynomials filters, e.g., LN and CN. The EMFN filter in this experiment provides slightly worse results than the LN, CN and WN filters identified with a PPS, because it lacks a linear term. On the contrary, for saturation distortions higher than those considered here, EMFN filters are able to provide better results than the other filters [8].

It is important to note there is a significant difference in the effort necessary to estimate the WN filter using the cross-correlation method with PPSs and using the method of [22]. Computing the experimental results for the cross-correlation method required a few hours of computer time. In contrast, obtaining the same results with the method in [22] requested days of simulations on the same computer. In fact, indicating with T the number of samples used for the identification, B the number of candidate basis functions, and S the number of selected basis functions, the method of [22] has a computational cost of TBS^2 operations, the cross-correlation method has a cost of TB operations.

V. Conclusions

PPSs for the WN filter have been developed in the paper. Using a PPS input, the Wiener basis functions are orthogonal over a period of the PPS. Thus, the WN filter can be estimated with the cross-correlation method and the most relevant basis functions according to some information criterion can be identified. The PPSs allow to avoid the main problem in the identification of WN filters using the cross-correlation method: the non-ideality of the Gaussian input signals used in common practice, which affect mainly the kernels diagonal points. The perfect orthogonality of the Wiener basis functions for PPS inputs allow to accurately estimate all coefficients of the WN filter. Aliasing errors may be present only in case of an underestimation of the unknown system memory or order, as have already been discussed for PPSs for EMFN [14], LN [16], and CN [8] filters.

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