Statistical models for SAR amplitude data: a unified vision through Mellin transform and Meijer functions

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***Abstract*—In the past years, many distributions have been proposed to model SAR images. In previous works, it has been shown that Mellin transform is a powerful tool to analyse random variable products: when speckle is modelled by a Gamma distribution, and when texture can be modelled by a "classical" distribution, Mellin convolution provides analytical expressions of SAR image distribution so that parameter estimations can be processed [13], [11].**

In this paper we focus on the product of probability density functions, and more specifically on the Inverse Generalized Gaussian distribution [10]. This approach has been validated in SAR image processing by Frery et al. [7]. We show that the Mellin statistics framework can provide some enlightments about this probability density function family, and can clearly link the Mellin convolution pdf family and the product pdf family.

Finally, it will be shown that the Meijer functions give a unified framework for many SAR distributions so that quantitative comparisons between pdf can be achieved.

I. Introduction

Many SAR sensors are regularly acquiring images of the earth for various applications: cartography and mapping, urban area monitoring, damage assesment, 3D reconstruction, change detection, deformation monitoring, etc. Although their all-weather and all-time capacities make these sensors very useful, due to the coherent imaging system they suffer from a strong speckle noise. Since the beginning of SAR imaging systems, many models for amplitude or intensity distributions have been proposed. Recent satellite sensors and airborn campaigns provide very high resolution images making necessary adapted models.

This paper proposes a unified vision through Mellin transform of many distributions that have been proposed in the past few years. There are two main contributions in this paper. First, we give some enlightments on distributions derived from products of distributions and analyze them in the Mellin framework. Then we present the Meijer functions as a generative family allowing a simple manipulation of the SAR distributions.

The paper is organized as follows. First, we briefly recall the log-statistics framework (section II). Then, we compare different ways of defining distributions through Mellin convolution or product model, and show how they have been used for SAR images (section III). Eventually, we introduce Meijer functions and their usefulness for SAR imagery (section IV).

II. The log-statistics

A. Log-statistics framework

SAR images are a typical example of positive data polluted by a multiplicative noise called speckle. With the help of Goodman's approach [4], homogeneous areas can be modelled by the Rayleigh-Nakagami distribution for amplitude data, and by the Gamma distribution for intensity data. Yet textured areas seem to be more difficult to model as the proposed approaches have no physical justification: at present time no consensus can be found for probability density function (pdf) modelling on non homogeneous areas.

As the data are positive, "log-statistics" ([11], [13]) can be used so that very useful tools can process these images. Based on the "second kind characteristic function" defined by a Mellin transform of the pdf instead of a Fourier transform in regular statistics, this approach well matches with SAR data.

Firstly, Rayleigh-Nakagami $\mathcal{RN}\left[\mu, L\right]$ (for amplitude data) and Gamma distribution $\mathcal{G}\left[\mu, L\right]$ (for intensity data) have second kind characteristic functions easy to deal with. Reference [13] provides their pdf's expression associated to their second kind characteristic functions and their log-cumulants.

Moreover, multiplicative noise acts as a Mellin convolution in a same way as additive noise acts as a regular convolution.

For example, if $p(x)$ and $q(x)$ are two pdf, $q(x)$ acting as multiplicative noise on $p(x)$, the resulting pdf can be written as :

$$r(x) = p(x) \hat{\star} q(x) \Leftrightarrow \mathcal{M}\left[r\right](s) = \mathcal{M}\left[p\right](s)\mathcal{M}\left[q\right](s). \tag{1}$$

with $\hat{\star}$ the Mellin convolution and $\mathcal{M}$ the Mellin transform [13]. As for regular statistics, the second characteristic function of second kind is defined as the logarithm of the second kind characteristic function, so that its derivatives can be called "log-cumulants". For our previous example ($r(x) = p(x) \hat{\star} q(x)$), we have:

$$\tilde{\kappa}_{r,n} = \tilde{\kappa}_{p,n} + \tilde{\kappa}_{q,n} \quad \forall n \in \mathbb{N} \tag{2}$$

Globally speaking, a pdf is generally defined by a scale parameter (generally its mean value or its mode) and shape parameters. In the log-statistics approach, only the first log-cumulant depends on the "mean value" parameter, the higher log-cumulants depending only on shape parameters.

Now, an important property of the Mellin approach can be emphasized. If a variable x is defined by its pdf $p(x)$, we can define a new variable y so that $y = x^\eta$ and we denote by $q(y)$ its pdf. If $\phi_p(s)$ is the second kind characteristic function of $p(x)$, the second kind characteristic function of $q(y)$, $\phi_q(s)$, can be written as :

$$\phi_q(s) = \phi_p\left(\frac{s+\eta-1}{\eta}\right) \tag{3}$$

B. A first famous example: the K distribution

The first famous example of non homogeneous area is the K distribution proposed by Jakeman [9]. By modelling the textured area as a Gamma distribution, and assuming that speckle acts as multiplicative noise, Jakeman obtains this K distribution:

$$\begin{aligned}\mathcal{K}[\mu, L, M] &= \frac{1}{\Gamma(L)\Gamma(M)} \frac{2\,L\,M}{\mu} \left(\frac{L\,M\,x}{\mu}\right)^{\frac{M+L}{2}-1} \\ K_{M-L} &\left[2\left(\frac{L\,M\,u}{\mu}\right)^{\frac{1}{2}}\right] \text{ with } \mu > 0,\ L > 0,\ M > 0\end{aligned} \tag{4}$$

Yet Epstein [6], thirty years before, had proposed an interesting approach with the help of Mellin transform so that the K distribution can be written as :

$$\mathcal{K}[\mu, L, M] = \mathcal{G}\left[\mu, L\right] \hat{\star}\, \mathcal{G}\left[1, M\right] \tag{5}$$

The additive properties of log-cumulants directly allow the expression of its log-cumulants:

$$\begin{aligned}\tilde{\kappa}_1 &= \log(\mu) \\ &\quad + (\Psi(L) - \log(L)) + (\Psi(M) - \log(M)) \\ \tilde{\kappa}_2 &= \Psi(1, L) + \Psi(1, M), \\ \tilde{\kappa}_3 &= \Psi(2, L) + \Psi(2, M).\end{aligned} \tag{6}$$

so that the effects of multiplicative noise can be seen only by adding the log-cumulants.

C. The $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram

In order to easily compare some pdf, and as $\tilde{\kappa}_2$ and $\tilde{\kappa}_3$ depend only on shape parameters, an oversimple comparison of these two parameters can be proposed: by mimicing the famous $\beta_1 - \beta_2$ diagram [8], we can construct the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram. As, by definition, $\tilde{\kappa}_2$ is always positive for pdfs, this parameter is chosen to be the vertical axis, $\tilde{\kappa}_3$ corresponding to the horizontal axis [13].

As we can see on figure 1, the Gamma distribution and the Inverse Gamma distribution play a symetrical role, the log-normal distribution being restricted on the vertical axis. The K distribution is located up to the Gamma distribution and down a caustic defined by a specific K distribution : $\mathcal{K}[\mu, L, L]$. In this diagram,

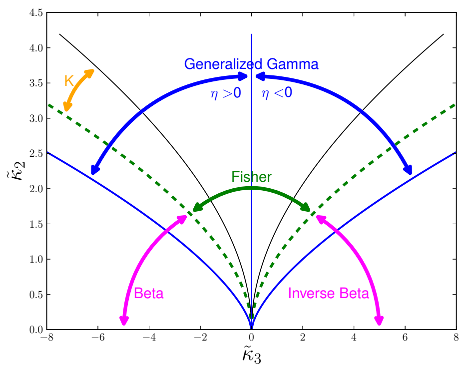

Fig. 1. The $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram : dotted lines correspond to the Gamma distribution (left) and the Inverse Gamma distribution (right). The K distribution is located up to the Gamma distribution dotted line and down the specific K-caustic (thin line). The heavy tailed distributions correspond to the positive $\tilde{\kappa}_3$ quadrant.

it is possible to define the location of a lot of usual distributions. More, the construction of distributions based on a Mellin convolution is directly obtained in the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram by an oversimple sum of their log-cumulants.

III. SAR images : towards high resolution data through "compound" distributions

At present time, the improvements in SAR image resolution allows the possibility to deal with metric or submetric images (provided for example by Terrasar-X sensor or Cosmo-Skymed one). Comparing with decametric resolutions of sensors like ERS, advances are significant. Indeed, in a decametric pixel enlighted by a centimetric wavelength, a lot of various targets are mixed so that heterogeneous areas can appear as homogeneous ones, yielding Gamma distribution for intensity values. When resolution is improved, for example in the case of metric pixels, targets can be isolated so that the values between pixels vary strongly, yielding heavy tailed distributions.

Figure 2 provide a $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram on homogeneous area, both for decametric sensor (ERS) and for metric one (Cosmo-Skymed) : the scatter plot is rather located around the theoretical value, i.e. the a $\tilde{\kappa}_2 - \tilde{\kappa}_3$ of a Gamma distribution with $L = 1$.

In the case of heterogeneous data, acquired on the same Paris area (figure 3), the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram for decametric data (ERS) is located in the left quarter plane. For metric data (Cosmo-Skymed), the scatter plot occupies the whole part up to the Gamma distribution and the inverse gamma one: more, some plots belong to the heavy tailed distribution area (corresponding to positive values of $\tilde{\kappa}_3$).

To cope with heterogeneous areas for image processing, a parametric approach requires an easy to use probability density function. At present time, three approaches can be proposed : the first one is based

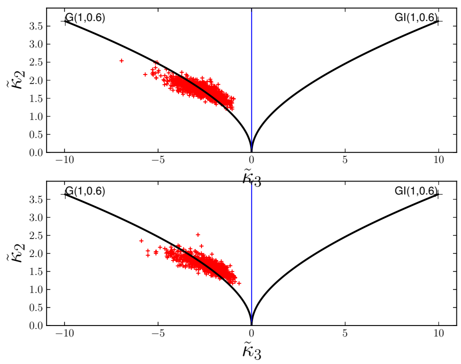

Fig. 2. $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagrams on homogeneous areas for ERS sensor (up) and Cosmo-Skymed sensor (down). Each point in the diagram corresponds to the parameters of the distribution estimated on a 512 samples window.

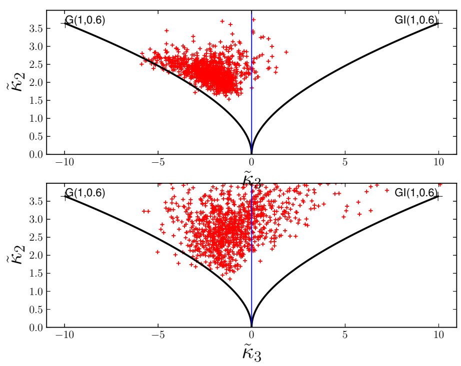

Fig. 3. $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagrams on heterogeneous areas for ERS sensor (up) and Cosmo-Skymed sensor (down). Each point in the diagram corresponds to the parameters of the distribution estimated on a 512 samples window.

on Mellin convolution, the second one consists in changing the power of the variable and the third one is simply a product.

A. The Mellin convolution

Starting from the Gamma distribution and the Inverse Gamma distribution, the Mellin convolution acts as a construction set so that a lot of possible distributions can be proposed. In radar image processing, there are two classical three parameters distributions, the K distribution and the Fisher distribution, defined by Mellin convolution as follows:

$$\mathcal{K}[\mu, L, M] = \mathcal{G}[\mu, L] \,\hat{\star}\, \mathcal{G}[1, M] \quad (7)$$
$$\mathcal{F}[\mu, L, M] = \mathcal{G}[\mu, L] \,\hat{\star}\, \mathcal{GI}[1, M] \quad (8)$$

By construction, second kind characteristic functions are simply obtained by multiplying the second kind characteristic functions :

$$\phi_{\mathcal{K}}(s) = \phi_{\mathcal{G}[\mu,L]}\ \phi_{\mathcal{G}[1,M]} \quad (9)$$
$$\phi_{\mathcal{F}}(s) = \phi_{\mathcal{G}[\mu,L]}\ \phi_{\mathcal{GI}[1,M]} \quad (10)$$

and log-cumulants are simply obtained by adding the log-cumulants. Let us remark that, for Fisher distribution, the limit case $M \to \infty$ yields the Gamma distribution, and the limit case $L \to \infty$ yields the Inverse Gamma distribution.

This approach can be extended at any number of combinations of Gamma pdf and Inverse Gamma pdf. For example, the Delignon's U distribution [3] can be written as:

$$\mathcal{U}[\mu, L, M, N] = \mathcal{G}[1, M] \,\hat{\star}\, \mathcal{GI}[1, N] \,\hat{\star}\, \mathcal{G}[1, L] \quad (11)$$

In this Mellin convolution framework, an easy-to-use formulation can also be obtained with the help of Meijer functions which will be seen in section IV.

B. "Generalized" distributions

Another way to obtain easily a distribution dealing heavy tailed trend consists in taking any η-power of the variable x of a pdf. In the case of Gamma distribution, we obtain the "Generalized Gamma distribution" $\mathcal{GG}[\mu, L, \eta]$ which fullfills an important part of the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram (see figure 1).

$$\mathcal{GG}[\mu, L, \eta](x) = \frac{|\eta|}{\mu} \frac{L^{\frac{1}{\eta}}}{\Gamma(L)} \left(\frac{L^{\frac{1}{\eta}} x}{\mu}\right)^{\eta L - 1} \mathrm{e}^{-\left(\frac{L^{\frac{1}{\eta}} x}{\mu}\right)^{\eta}} \quad (12)$$

Due to the properties of the Mellin transform, starting from the Gamma distribution, second kind characteristic functions and log-cumulants derive directly from those of Gamma distribution (relation 3), yielding:

$$\phi_{\mathcal{GG}}(s) = \phi_{\mathcal{G}}\left(\frac{s + \eta - 1}{\eta}\right) = \mu^{s-1} \frac{\Gamma\left(L + \frac{s-1}{\eta}\right)}{L^{\frac{s-1}{\eta}}\ \Gamma(L)} \quad (13)$$

and

$$\begin{aligned} \tilde{\kappa}_1 &= \log(\mu) + \frac{\Psi(L) - \log(L)}{\eta} \\ \tilde{\kappa}_2 &= \frac{\Psi(1, L)}{\eta^2} \\ \tilde{\kappa}_r &= \frac{\Psi(r, L)}{\eta^r} \end{aligned} \quad (14)$$

iff $\eta \neq 0$. Let us remark that the case $\eta = -1$ corresponds to the Inverse Gamma distribution and the case $\eta = 1$ corresponds to the Gamma distribution.

At present time, this model is currently used in SAR image processing. Yet, in the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram, there is a discontinuity as the 3-parameters Generalized Gamma distribution cannot deal with data so that $\tilde{\kappa}_3 = 0$ (corresponding to $\eta \to 0^+$ or $\eta \to 0^-$).

C. A "product like" empirical model

A third way to obtain an all-purpose distribution consists in a *multiplication* between a Gamma distribution and an inverse Gamma distribution: proposed by Halphen in 1941 [5] and better-known as Generalized

Inverse Gaussian Distribution [10], this pdf can be written as:

$$\mathcal{H}[\alpha, \beta, \varepsilon](x) \;=\; A\, x^{\alpha-1}\, \mathrm{e}^{-\beta x \,-\, \varepsilon x^{-1}} \qquad (15)$$

As α is an intuitive parameter devoted to tune the distribution mode, in this paper we propose to deal with the modified Halphen distribution $\mathcal{HM}[\mu, \beta, \varepsilon](x)$ defined as:

$$\mathcal{HM}[\mu, \beta, \varepsilon](x) \;=\; \frac{1}{2\mu\left(\frac{\varepsilon}{\beta}\right)^{\frac{\beta-\varepsilon}{2}} K_{\beta-\varepsilon}\left(\sqrt{\beta\varepsilon}\right)} \left(\frac{x}{\mu}\right)^{\beta-\varepsilon-1} \mathrm{e}^{-\frac{\beta x}{\mu} \,-\, \frac{\varepsilon\mu}{x}} \qquad (16)$$

At first glance, the Mellin approach is not designed to deal with product of functions. Yet, a theoretical result can be used in this case [2] : if $\varphi(s)$ is the Mellin transform of a function $f(x)$ defined on $\mathbb{R}^+$, and $\psi(s)$ is the Mellin transform of the function $g(x)$ defined on $\mathbb{R}^+$, then the Mellin transform of the function $h(x) = f(x)\, g(x)$ can be written as:

$$\mathcal{M}\left[h(x)\right](s) \;=\; \frac{1}{2j\pi}\int_{c-i\infty}^{c+i\infty} \varphi(s-w)\,\psi(w)\mathrm{d}w \quad (17)$$

By this way, we can obtain the second kind characteristic function of the Halphen distributions. In the case of the modified Halphen distribution, we can write :

$$\phi_{\mathcal{HM}}(s) \;=\; \frac{\mu^{s-1}}{K_{\beta-\varepsilon}\left(2\sqrt{\beta\varepsilon}\right)} \left(\frac{\varepsilon}{\beta}\right)^{\frac{s-1}{2}} K_{s-1+\beta-\varepsilon}\left(2\sqrt{\beta\varepsilon}\right) \qquad (18)$$

As for the Fisher distribution case and the Generalized Gamma distribution case, we have :

- if $\varepsilon \to 0$, then $\mathcal{HM}[\mu, \beta, \varepsilon] \to \mathcal{G}[\mu, \beta]$,
- if $\beta \to 0$, then $\mathcal{HM}[\mu, \beta, \varepsilon] \to \mathcal{GI}[\mu, \varepsilon]$,

so that, in the $\tilde{\kappa}_2 - \tilde{\kappa}_3$ diagram, the modified Halphen distribution fullfills the area bounded by the Gamma distribution (at left) and the inverse Gamma distribution (at right) exactly as the Fisher distribution.

Yet, the Halphen modified distribution differs fundamentally from the Fisher distribution case and the Generalized Gamma distribution because, if $\beta > 0$ and $\varepsilon > 0$, all its positive and negative moments are defined (as for the log-normal distribution). A priori, this distribution does not well match heavy tailed distributions and cannot model correctly high-valued outliers.

D. Application to SAR data: "compound" distributions

Using the previous paragraphs, we can apply these results in the case of SAR images. At present time, the main problem in SAR image processing is to cope with the heterogeneous textures which can be found (see figure 3), leading to the so called "compound" distributions of Goodman (called also "speckled speckle"). As speckle can be seen as multiplicative noise, the Mellin approach has to be privilegiate, so that, for intensity data, by assuming a speckle modelled by a Gamma distribution, and if $g(x)$ is the texture model, the image pdf is modelled by:

$$p(x) \;=\; \mathcal{G}\left[\mu, L\right](x) \,\hat{\star}\, g(x) \qquad (19)$$

In the case of amplitude data, the speckle can be modelled by a Rayleigh-Nakagami distribution $\mathcal{RN}\left[\mu, L\right](x)$, and if $h(x)$ is the amplitude texture model, the image pdf is modelled by :

$$q(x) \;=\; \mathcal{RN}\left[\mu, L\right](x) \,\hat{\star}\, h(x) \qquad (20)$$

As we have seen before, using the convolutive approach for intensity data, we have the following three cases :

- if $g(x) = \mathcal{G}\left[1, M\right](x)$, we have obtained the K distribution $\mathcal{K}[\mu, L, M]$,
- if $g(x) = \mathcal{GI}\left[1, M\right](x)$, we have obtained the Fisher distribution $\mathcal{F}[\mu, L, M]$,
- if $g(x) = \mathcal{F}[\mu, M, N]$, we have obtained the U distribution $\mathcal{U}\left[\mu, L, M, N\right]$.

As in the Mellin world, the "generalization" of a pdf is untricky (see the relation 3 for the second kind characteristic function), it is very easy to obtain these three cases for amplitude data, yielding the K amplitude distribution $\mathcal{K}_A[\mu, L, M]$, the Fisher amplitude distribution $\mathcal{F}_A[\mu, L, M]$ and the U amplitude distribution $\mathcal{U}_A\left[\mu, L, M, N\right]$.

Let us remark that, for $\mathcal{U}$ and $\mathcal{U}_A$, we have the obvious result :

$$\left\{ \begin{array}{lcl} \lim_{M\to\infty} \mathcal{U}\left[\mu, L, M, N\right] &=& \mathcal{F}[\mu, L, N] \\ \lim_{M\to\infty} \mathcal{U}_A\left[\mu, L, M, N\right] &=& \mathcal{F}_A[\mu, L, N] \end{array} \right. \qquad (21)$$

Frery [7] proposes to model the texture by a Halphen-like distribution and the multiplicative noise as a Rayleigh-Nakagami distribution. The resulting distribution second kind characteristic function is directly the product of the Halphen modified second kind characteristic function (in the case of amplitude data and using the previous expression of $\phi_{\mathcal{HM}}(s)$) and the Rayleigh-Nakagami distribution second kind characteristic function:

$$\mu^{\frac{s-1}{2}} \frac{1}{K_\alpha\left(2\sqrt{\beta\varepsilon}\right)} \left(\frac{\varepsilon}{\beta}\right)^{\frac{s-1}{4}} \frac{\Gamma\left(\frac{s-1}{2}+L\right)}{L^{\frac{s-1}{2}}\,\Gamma(L)} K_{\frac{s-1}{2}+\alpha}\left(2\sqrt{\beta\varepsilon}\right) \qquad (22)$$

With the help of Mellin transform tables [12], and by using basic Mellin transform properties, we can derive the analytical expression of such a distribution :

$$\frac{2}{K_\alpha\left(2\sqrt{\beta\varepsilon}\right)} \frac{L^L}{\Gamma(L)} \left(\sqrt{\beta}\right)^L \left(\sqrt{\varepsilon}\right)^{-\alpha} x^{2L-1} \left(\varepsilon + Lx^2\right)^{\frac{\alpha-L}{2}} K_{-\alpha+L}\left(2\sqrt{\beta\left(\varepsilon + Lx^2\right)}\right) \qquad (23)$$

which corresponds exactly to the Frery relation (6) in [7].

With a classical analysis, the limit cases of this expression are not at all intuitive. Yet, by analysing the limit cases of the second kind characteristic function (relation 22), we can easily derive these two following results :

- if $\varepsilon \rightarrow 0$, then we obtain a K distribution $\mathcal{K}_A[\mu, \beta, L]$,
- if $\beta \rightarrow 0$, then we obtain a Fisher distribution $\mathcal{F}_A[\mu, \varepsilon, L]$

By comparing with the U distribution previously presented, we obtain exactly the same limits.

IV. The Meijer functions approach

In this section we introduce the Meijer functions that can be seen as a "generative" family of SAR distributions. We will see that they have a property allowing an easy handling of such generated distributions.

The Mellin convolution approach allows a rather concise expression for "compound" distributions. In the case when the elementary pdf are either Gamma distribution or Inverse Gamma distribution, the second kind characteristic function can be written as a product of Gamma functions. For example, by combining P Gamma distribution with shape parameter L_p and Q Inverse Gamma distribution with shape parameter M_q, we obtain the second kind characteristic function as:

$$\prod_{p=1}^{P} \frac{\Gamma\left(L_p + s - 1\right)}{\Gamma(L_p)} \prod_{q=1}^{Q} \frac{\Gamma\left(M_q + 2 - s\right)}{\Gamma(M_q)} \tag{24}$$

More, by using Inverse Mellin transform, we obtain the general expression of the resulting pdf:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \prod_{p=1}^{P} \frac{\Gamma\left(L_p + s - 1\right)}{\Gamma(L_p)} \prod_{q=1}^{Q} \frac{\Gamma\left(M_q + 2 - s\right)}{\Gamma(M_q)} \mathrm{d}s \tag{25}$$

and by comparing this expression with the definition of Meijer's functions [1]:

$$\overline{G}_{p,q}^{m,n}\left(x \left| \begin{array}{ccc} a_1, \ldots, a_n & ; & a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m & ; & b_{m+1}, \ldots, b_q \end{array} \right.\right) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)} x^{-s} \, \mathrm{d}s \tag{26}$$

we can assess that the resulting pdf can be written as a Meijer function :

$$\sim \ \overline{G}_{Q,P}^{P,Q}\left(\frac{\prod_p L_p x}{\prod_q M_q \mu} \left| \begin{array}{ccc} -M_1, \ldots, -M_Q; & & . \\ L_1 - 1, \ldots, L_P - 1 & ; & . \end{array} \right.\right) \tag{27}$$

The main interest of this rewriting is the fact that the primitive of a Meijer function is a Meijer function [1]. Several expressions exist for the primitive and we privilegiate this one :

$$\frac{\mathrm{d}}{\mathrm{d}x} \overline{G}_{p+1,q+1}^{m,n+1}\left(x \left| \begin{array}{c} 1, a_1+1, \ldots, a_n+1 \ ; \ a_{n+1}+1, \ldots, a_p+1 \\ b_1+1, \ldots, b_m+1 \ ; \ 0, b_{m+1}+1, \ldots, b_q+1 \end{array} \right.\right) = \overline{G}_{p,q}^{m,n}\left(x \left| \begin{array}{ccc} a_1, \ldots, a_n & ; & a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m & ; & b_{m+1}, \ldots, b_q \end{array} \right.\right) \tag{28}$$

By this way, we derive directly the cumulative function of the pdf (equation 27) :

$$\sim \overline{G}_{Q+1,P+1}^{P,Q+1}\left(\frac{Lx}{M\mu} \left| \begin{array}{ccc} 1, -M_1+1, \ldots, -M_Q-1 & ; & . \\ L_1, \ldots, L_P & ; & 0 \end{array} \right.\right) \tag{29}$$

This link between the pdf and the cumulative function allows an easy handling of such pdf. Indeed it becomes easy to estimate the parameters through the log-cumulants and to draw samples from the distribution. In particular, heterogeneous areas of SAR images can be well modelled with the Meijer functions thanks to their high flexibility.

V. Conclusions

In this paper we have investigated three different ways of generating texture or compound distributions in the case of SAR data: product of random variables, power of random variable, and direct product of pdfs. These frameworks have given birth to different pdfs that are popular in SAR imagery. We have also presented the Meijer functions providing a generative framework through the Mellin transform in a very flexible way.

Further works include experiments of this framework on real SAR data and a theoretical and empirical study of the distances between distributions both for pdf dictionary simplifications and change detection applications.

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