Discrete Bessel Functions for Representing the Class of Finite Duration Decaying Sequences

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***Abstract*—Bessel functions have shown to be particularly suitable for representing certain classes of signals, since using these basis functions may results in fewer components than using sinusoids. However, as there are no closed form expressions available for such functions, approximations and numerical methods have been adopted for their computation. In this paper the functions called discrete Bessel functions that are expressed as a finite expansion are defined. It is shown that in a finite interval a finite number of such functions that perfectly match Bessel functions of integer order exist. For finite duration sequences it is proven that the subspace spanned by a set of these functions is able to represent the class of finite duration decaying sequences.**

I. Introduction

Bessel functions of integer orders play an important role in numerous application fields, as they represent the mathematical solutions of radiation, scattering, magnetics problems, to name just a few [1]–[3].

A well known property of Bessel functions is their orthogonality in a continuous-value finite interval, so that they are able to represent a given signal as an infinite Fourier-Bessel series [4]–[7] whose expansion coefficients are determined by an integral relationship. This expansion has the same structure of the Fourier representation for an infinite sequence given by an integral-series couple of equations.

However, there are no closed form expressions available for Bessel functions, hence approximations [8], [9] and numerical methods [10], [11] have been adopted in the past for their computation.

Additionally, for finite-duration sequences, as the DFT represents a finite expansion of such sequences, it is of interest to derive a similar expansion, if exist, in terms of Bessel functions instead of sinusoidal functions. This is justified by the fact that for certain classes of signals [5], [12] using these basis functions may results in fewer components than using sinusoids.

More specifically, since Bessel functions are non-stationary decaying functions, they are particularly suitable to represent the class of finite duration decaying sequences. The aim of this paper is to derive a representation of signals belonging to the class of finite duration decaying sequences, as a finite expansion in terms of Bessel functions.

With reference to this problem, the functions expressed as a finite summation and called discrete Bessel functions (DBFs) are defined.

It is shown that in a finite interval a finite number of DBFs perfectly match Bessel functions of integer order. Then for finite duration sequences it is proven that a subspace spanned by a set of these functions can be derived.

Numerical results show that such a subspace is able to represent the class of finite duration decaying sequences.

II. Mathematical Theory

A. Discrete Bessel Function

Let us refer to the function $e^{iz\sin\varphi}$ with z being a complex variable and φ a real variable, since this function is a periodic function of φ with period 2π it can be expanded as Fourier series

$$e^{iz\sin\varphi} = \sum_{\nu=-\infty}^{+\infty} J_\nu\left(z\right) e^{i\nu\varphi} \ . \tag{1}$$

It is well known [13] that the coefficients $J_\nu\left(z\right)$ of the summation defines the Bessel functions of the first kind and are given by the usual relationship for the Fourier coefficients

$$J_\nu\left(z\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\varphi} e^{-i\nu\varphi} d\varphi \ , \tag{2}$$

where the integer ν represents the order of the function. For the purpose of this work it suffices to consider z to be real, thus hereafter we will apply this restriction.

Figure 1 depicts several Bessel functions of different order for $z \geq 0$. As you can see for every index ν a finite interval $[0, a]$ exists such that, for $0 \leq z \leq a$, $J_\nu\left(z\right)$ is close to zero. In particular the value of a increases as the index ν increases.

On the basis of this property we can assume that for a value $z = a$ and a given error ε, an index N exists such that for $\nu > N$ it results

$$J_\nu\left(z\right) \cong 0 \quad , \qquad 0 \leq z \leq a \ , \tag{3}$$

where $b \cong c$ means $|b - c| < \varepsilon$ for any given ε.

As a consequence of this result, (1) can be rewritten as

$$e^{iz\sin\varphi} \cong \sum_{\nu=-N}^{N} J_\nu\left(z\right) e^{i\nu\varphi} \tag{4}$$

for $0 \leq z \leq a$, $-\pi \leq \varphi \leq \pi$. Now evaluating the relationship (4) at the $2N + 1$ discrete values

$$\varphi_k = k\frac{2\pi}{2N+1} \quad , \qquad k = 0, 1, \ldots, 2N \ , \tag{5}$$

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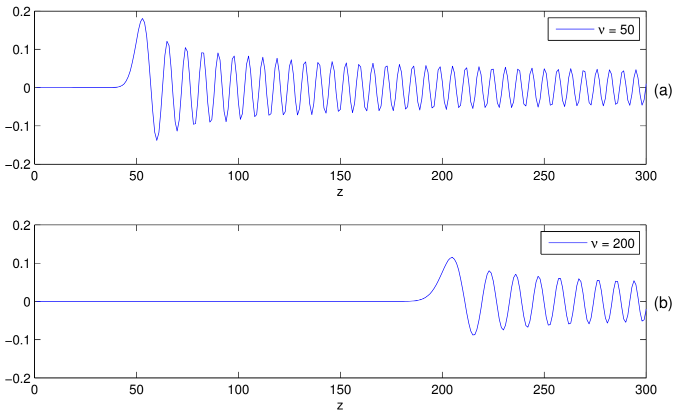

Fig. 1: Two Bessel functions of different orders: *a)* $\nu = 50$, *b)* $\nu = 200$.

yields

$$e^{iz\sin\varphi_k} \cong \sum_{\nu=-N}^{N} J_\nu\left(z\right) e^{i\nu\varphi_k} \ . \quad (6)$$

By multiplying $e^{-i\nu'\varphi_k} / \left(2N+1\right)$ by both sides of (6) and summing the result for $k = 0$ to $k = 2N$, we have

$$\frac{1}{2N+1}\sum_{k=0}^{2N} e^{iz\sin\varphi_k} e^{-i\nu'\varphi_k} \cong \frac{1}{2N+1}\sum_{k=0}^{2N}\left(\sum_{\nu=-N}^{N} J_\nu\left(z\right) e^{i\nu\varphi_k}\right) e^{-i\nu'\varphi_k} = \sum_{\nu=-N}^{N} J_\nu\left(z\right)\left(\frac{1}{2N+1}\sum_{k=0}^{2N} e^{i\left(\nu-\nu'\right)\frac{2k\pi}{2N+1}}\right) = J_{\nu'}\left(z\right) \ , \quad (7)$$

where the last identity is obtained by virtue of the orthogonality property of the complex exponentials

$$\frac{1}{2N+1}\sum_{k=0}^{2N} e^{i\left(\nu-\nu'\right)\frac{2k\pi}{2N+1}} = \begin{cases} 1 & \text{if} \quad \nu = \nu' \\ 0 & \text{if} \quad \nu \neq \nu' \end{cases} \ , \quad (8)$$

and ν, ν' are integers ranging from $-N$ to N.

Finally, we have

$$J_\nu\left(z\right) \cong \frac{1}{2N+1}\sum_{k=0}^{2N} e^{iz\sin\varphi_k} e^{-i\nu\varphi_k}, \quad \varphi_k = k\frac{2\pi}{2N+1} \ , \quad (9)$$

for $0 \leq z \leq a$, $-N \leq \nu \leq N$.

Thus for a finite interval $0 \leq z \leq a$ the couple of equations

$$e^{iz\sin k\frac{2\pi}{2N+1}} = \sum_{\nu=-N}^{N} B_\nu\left(z\right) e^{i\nu k\frac{2\pi}{2N+1}} \ , \quad k = 0, 1, \ldots, 2N \ , \quad (10)$$

and

$$B_\nu\left(z\right) = \frac{1}{2N+1}\sum_{k=0}^{2N} e^{iz\sin k\frac{2\pi}{2N+1}} e^{-i\nu k\frac{2\pi}{2N+1}} \ , \quad -N \leq \nu \leq N \ , \quad (11)$$

represents the finite-dimension version of (1) and (2), where the notation $B_\nu(z)$ has been used in place of $J_\nu(z)$ to denote that $B_\nu(z)$ satisfies the properties

$$B_\nu\left(z\right)\begin{cases} \cong J_\nu\left(z\right) \ , & -N \leq \nu \leq N \\ = 0 \ , & \text{otherwise} \end{cases} \ , \qquad 0 \leq z \leq a \ . \quad (12)$$

The function $B_\nu(z)$ so defined will be called *discrete Bessel function* (DBF) of order ν and the couple (10) and (11) is the DFT representation of $B_\nu(z)$ as it can easily be verified.

B. Properties of $B_\nu(z)$

Here we want to show that the function $B_\nu(z)$ satisfies the usual properties of $J_\nu(z)$. In particular assuming z is a real variable, then $B_\nu(z)$ is also real. To prove this proposition let us consider the conjugate complex of $B_\nu(z)$

$$B_\nu^*\left(z\right) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-iz\sin\frac{2k\pi}{L}} e^{i\nu\frac{2k\pi}{L}} = \frac{1}{L}\sum_{k'=0}^{-(L-1)} e^{iz\sin\frac{2k'\pi}{L}} e^{-i\nu\frac{2k'\pi}{L}} \ , \quad (13)$$

where $L = 2N+1$. By posing $k' = k - L$ we have

$$B_\nu^*\left(z\right) = \frac{1}{L}\sum_{k=1}^{L} e^{iz\sin\frac{2k\pi}{L}} e^{-i\nu\frac{2k\pi}{L}} \quad (14)$$

for the periodicity of the complex exponentials. Additionally, it results

$$\left[e^{iz\sin\frac{2k\pi}{L}} e^{-i\nu\frac{2k\pi}{L}}\right]_{k=L} = \left[e^{iz\sin\frac{2k\pi}{L}} e^{-i\nu\frac{2k\pi}{L}}\right]_{k=0} \ , \quad (15)$$

and finally

$$B_\nu^*\left(z\right) = \frac{1}{L}\sum_{k=0}^{L-1} e^{iz\sin\frac{2k\pi}{L}} e^{-i\nu\frac{2k\pi}{L}} = B_\nu\left(z\right) \ , \quad (16)$$

which proves the assertion.

As a consequence of this property, with z real $B_\nu(z)$ can also be written as

$$B_\nu\left(z\right) = \frac{1}{2N+1}\sum_{k=0}^{2N}\cos\left(z\sin\frac{2k\pi}{2N+1} - \nu\frac{2k\pi}{2N+1}\right) \ , \quad -N \leq \nu \leq N \ . \quad (17)$$

C. Matrix Representation of DBFs

Now let us assume z is a finite duration sequence instead of a continuous-value variable belonging to a finite interval of the real axis. In this case $z(n)$ is a discrete function of n, $n = 0, 1, \ldots, 2N$, and (11) can be written as

$$B_\nu\left(z\left(n\right)\right) = B\left(n, \nu\right) = \sum_{k=0}^{2N} U\left(n, k\right) W\left(k, \nu\right) \ , \quad (18)$$

where

$$U\left(n, k\right) = \frac{e^{iz(n)\sin\frac{2k\pi}{2N+1}}}{\sqrt{2N+1}} \quad (19)$$

and

$$W(k,\nu) = \frac{w_{2N+1}^{k\nu}}{\sqrt{2N+1}} , \quad (20)$$

being

$$w_{2N+1} = e^{-i\frac{2\pi}{2N+1}} . \quad (21)$$

Once the matrices

$$[B]_{n\nu} = B(n,\nu) \quad (22)$$

$$[U]_{nk} = U(n,k) \quad (23)$$

$$[W]_{k\nu} = W(k,\nu) \quad (24)$$

are defined, (18) becomes

$$B = UW . \quad (25)$$

This is the matrix representation of the discrete Bessel functions for a finite-duration sequence

$$z = [z(0), z(1), \ldots, z(2N)]^T . \quad (26)$$

By posing

$$\gamma_k = \sin\frac{2k\pi}{2N+1} \quad (27)$$

we can write

$$U = [u_0, u_1, \ldots, u_{2N}] , \quad (28)$$

where

$$u_k = \frac{e^{i\gamma_k z}}{\sqrt{2N+1}} , \quad k = 0,1,\ldots,2N , \quad (29)$$

are column vectors.

Similarly, we can write

$$W = [w_0, w_1, \ldots, w_{2N}] \quad (30)$$

and

$$B = [b_{-N}, b_{-N+1}, \ldots, b_{N-1}, b_N] , \quad (31)$$

where

$$w_\nu = \frac{1}{\sqrt{2N+1}}\left[1, w_{2N+1}^{\nu}, w_{2N+1}^{2\nu}, \ldots, w_{2N+1}^{2N\nu}\right]^T \quad (32)$$

and

$$b_\nu = B_\nu(z) , \quad -N \le \nu \le N . \quad (33)$$

Discrete Bessel Transform

As it is well known that Bessel functions are orthogonal in a continuous value domain, and thus are able to represent a unitary transform, it is worth to investigate whether a unitary matrix based on the DBF exists for finite duration sequences.

Assuming

$$z(n) = n , \quad n = 0,1,\ldots,2N , \quad (34)$$

it is straightforward to show that in this case the columns of U are not orthogonal. For this purpose, let us form the scalar product

$$u_k^T u_j^* = \frac{1}{2N+1}\sum_{n=0}^{2N} e^{i\gamma_k z(n)} e^{-i\gamma_j z(n)} = \frac{1}{2N+1}\sum_{n=0}^{2N} e^{i(\gamma_k-\gamma_j)\frac{2\pi n}{2N+1}} . \quad (35)$$

For $\gamma_k = \gamma_j$ it results $u_k^T u_j^* = 1$, while for $\gamma_k \neq \gamma_j$ we have

$$u_k^T u_j^* = \frac{1}{2N+1}\cdot\frac{1-\left(e^{i(\gamma_k-\gamma_j)\frac{2\pi}{2N+1}}\right)^{2N+1}}{1-e^{i(\gamma_k-\gamma_j)\frac{2\pi}{2N+1}}} = \frac{1}{2N+1}\cdot\frac{1-e^{i(\gamma_k-\gamma_j)2\pi}}{1-e^{i(\gamma_k-\gamma_j)\frac{2\pi}{2N+1}}} \neq 0 \quad (36)$$

as $e^{i(\gamma_k-\gamma_j)\frac{2\pi}{2N+1}} \neq 1$ and $e^{i(\gamma_k-\gamma_j)2\pi} \neq 1$. Then U is not a unitary matrix.

Similarly, we have

$$w_\nu^T w_{\nu'}^* = \frac{1}{2N+1}\sum_{k=0}^{2N} e^{i(\nu-\nu')\frac{2\pi k}{2N+1}} . \quad (37)$$

and from (8) we conclude that W is a unitary matrix.

As a consequence, from (25) it results

$$b_\nu = U w_\nu , \quad \nu = -N,\ldots,N , \quad (38)$$

and by forming the scalar products of two generic columns ν and ν' of B

$$b_\nu{}^T b_{\nu'}^* = w_\nu^T U^T U^* w_{\nu'}^* , \quad (39)$$

we can conclude that as U is not a unitary matrix the columns of B are not orthogonal and B is not unitary.

Nevertheless, due to the orthogonality of W, from (25) it results

$$U = BW^T . \quad (40)$$

The couple of equations (25) and (40) is the corresponding matrix representation of (10) and (11) for a finite-duration sequence $z(n)$, $n = 0,1,\ldots,2N$.

D. Subspace Bessel Representation

The column b_ν of B are not all linearly independent since it is well known that for the Bessel functions of the first kind, the following condition

$$J_{-\nu} = (-1)^\nu J_\nu \quad (41)$$

holds.

As a consequence only a number $M < 2N+1$ of columns are linearly independent, so that they generate a subspace of dimension M.

In order to investigate the capability of such subspace in representing a signal y that belongs to the class of non-stationary decaying signals, an approximation y_M of y belonging to the space spanned by the linear independent columns b_ν of B will be derived.

To this end using the Q–R decomposition the matrix B can be written as

$$B = QR , \quad (42)$$

where

$$Q^T Q = QQ^T = I , \quad (43)$$

being Q a unitary matrix, and R is an upper triangle matrix.

Thus given a generic vector y, the following representation

$$\begin{cases} y = Qk \\ k = Q^T y \end{cases} \quad (44)$$

holds.

In order to derive an approximation y_M of y belonging to the subspace spanned by the M linearly independent columns of B, $B_M = [b_0, b_1, \ldots, b_{M-1}]$, we partition Q as

$$Q = [Q_M \, Q_\eta] \tag{45}$$

where the columns of Q_M correspond to the non zero diagonal elements of R.

Accordingly, y is given by

$$y = Qk = [Q_M Q_\eta] \begin{bmatrix} k_M \\ k_\eta \end{bmatrix} = Q_M k_M + Q_\eta k_\eta \ , \tag{46}$$

$$k = \begin{bmatrix} k_M \\ k_\eta \end{bmatrix} = Q^T y = \begin{bmatrix} Q_M^T y \\ Q_\eta^T y \end{bmatrix} \tag{47}$$

and also

$$B = [B_M B_\eta] = [Q_M Q_\eta] \begin{bmatrix} R_M & R_{M\eta} \\ 0 & R_{\eta\eta} \end{bmatrix} \tag{48}$$

with

$$B_M = Q_M R_M \ , \tag{49}$$

$$Q_M = B_M {R_M}^{-1} \ . \tag{50}$$

Assuming

$$y_M = Q_M k_M \tag{51}$$

is the most significant part of y, we have

$$y_M = Q_M k_M = B_M \left({R_M}^{-1} Q_M^T y \right) \tag{52}$$

or

$$\begin{cases} y_M = B_M k'_M \\ k'_M = R_M^{-1} Q_M^T y \end{cases} . \tag{53}$$

This couple of equations gives the desired representation of the vector y_M, that belongs to the subspace spanned by the columns of B_M and approximates the vector y.

The residual $Q_\eta k_\eta$ in (46) represents the error in representing y with y_M.

III. Results

As a first result Fig. 2 depicts a comparison between discrete Bessel function $B_\nu(z)$ as given by (11) and continuous-value Bessel function $J_\nu(z)$.

As you can see for both the two values of ν and in the interval $0 \leq z \leq 300$ the two functions match perfectly.

Nevertheless, by choosing different values for ν, N, and the z interval, the results reported in Fig. 3 are obtained. Even though in this case the two functions do not match for all values of z, an interval exists such that they match perfectly for the values of z inside the interval. This means that a behaviour similar to that of Fig. 2 can be achieved by simply restricting the results of Fig. 3 to the interval $0 \leq z \leq 60$.

In order to evaluate the ability of discrete Bessel functions for representing functions belonging to the class of finite duration decaying sequences, we refer to the speech signals.

According to the source-filter model of speech production, a speech signal fragment $x(t)$ corresponding to a voiced sound

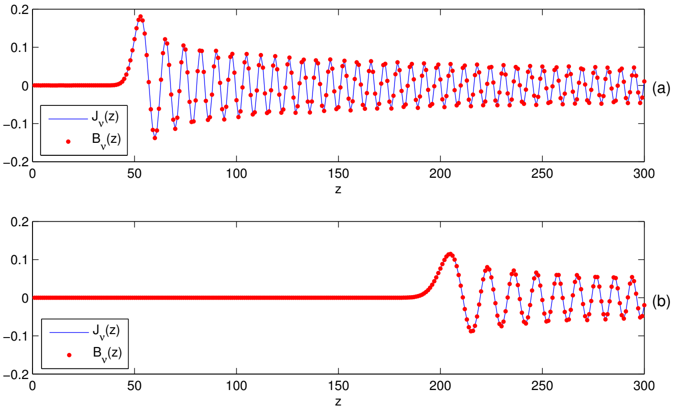

Fig. 2: Comparison of the function $B_\nu(z)$ with $J_\nu(z)$ for different values of ν, *a)* $\nu = 50$, *b)* $\nu = 200$, and $N = 300$.

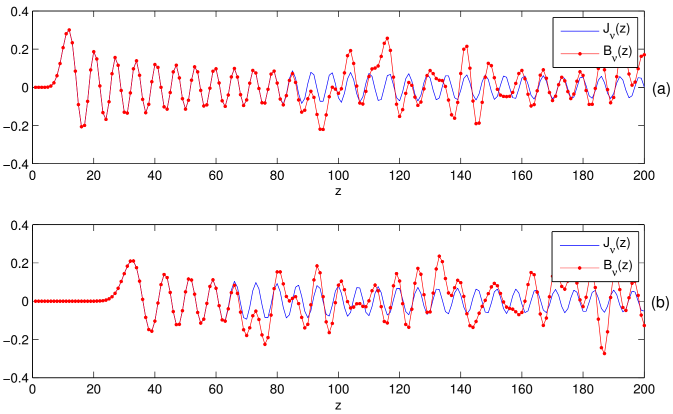

Fig. 3: Comparison of the discrete Bessel function $B_\nu(z)$ with the continuous-value Bessel function $J_\nu(z)$ for different orders *a)* $\nu = 10$, *b)* $\nu = 30$, and $N = 50$.

can be thought as the convolution of an excitation signal $e(t)$ and the vocal tract impulse response $h(t)$, that is

$$x(t) = h(t) * e(t) \ . \tag{54}$$

Homomorphic deconvolution can be successfully applied in separating the two components of a speech waveform [14].

It is well known that the vocal tract impulse response $h(t)$ belongs to the class of finite duration decaying sequences.

Figures 4, 5, and 6 show (solid line) the impulse response of Italian vowels $|a|$, $|o|$, $|u|$ respectively, achieved by homomorphic deconvolution, and for comparison (asterisks) the reconstructed signal accordingly to (53).

As you can see, the approximating signal y_M is able to perfectly follow the original signal y, thus confirming that the subspace spanned by the discrete Bessel functions is able to represent the class of finite duration decaying sequences.

IV. Conclusion

The aim of this paper is to derive a set of functions expressed as a finite expansion that match Bessel functions of integer order. To this end the functions called discrete Bessel

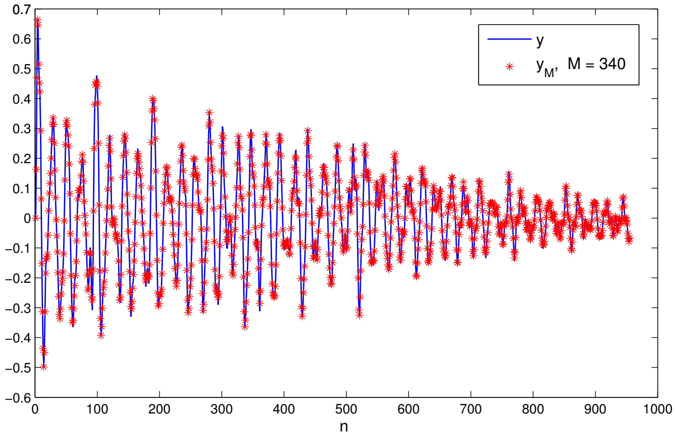

Fig. 4: Impulse response of the Italian vowel $|a|$ of length $2N + 1 = 956$ reconstructed as a linear combination of $M = 340$ discrete Bessel functions. Solid line represents the original signal y, the asterisks represent the approximating signal y_M.

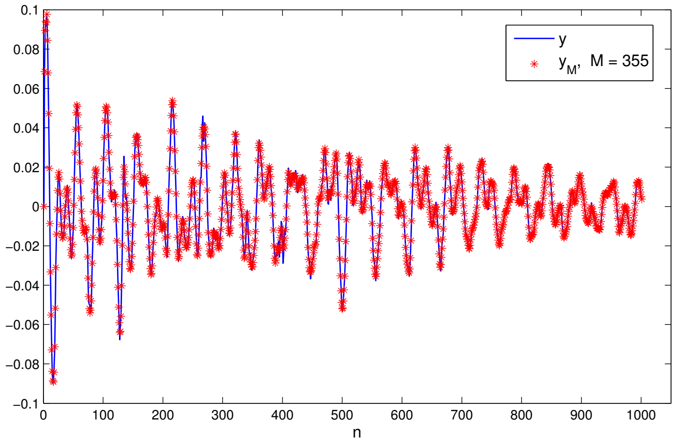

Fig. 5: Impulse response of the Italian vowel $|o|$ of length $2N + 1 = 956$ reconstructed as a linear combination of $M = 355$ discrete Bessel functions. Solid line represents the original signal y, the asterisks represent the approximating signal y_M.

functions are defined. As they are expressed as a finite summation, for finite duration sequences a matrix representation of such functions is given. The matrix so obtained defines a subspace that is able to represent the class of finite duration decaying sequences.

Numerical results show the capability of such subspace in representing a signal that belongs to this class.

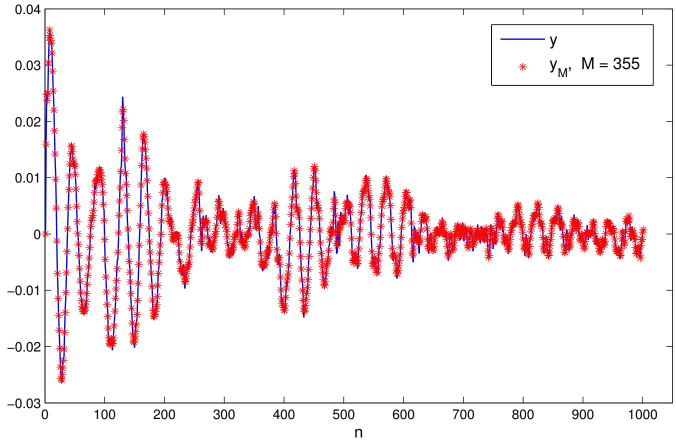

Fig. 6: Impulse response of the Italian vowel $|u|$ of length $2N + 1 = 956$ reconstructed as a linear combination of $M = 355$ discrete Bessel functions. Solid line represents the original signal y, the asterisks represent the approximating signal y_M.

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