Refinement of Time-Difference-Of-Arrival Measurements via Rank Properties in Two-Dimensional Space

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***Abstract*—Two new rank properties for *time difference of arrival* (TDOA) measurements in two-dimensional space are reported in this paper. On the basis of these rank properties, we propose a class of algorithms to refine TDOAs from their observations. Since only the singular value decomposition (SVD) technique is used, these proposed algorithms are very simple. Simulative experiments show that the accuracy of TDOA estimations is significantly improved using the proposed refining algorithms. Moreover, their ability to improve TDOA-based joint source and sensor localization is also proven by simulative experiments.**

***Index Terms*—Time Difference of Arrival, Time of Arrival, Expansion of Time Difference of Arrival**

I. Introduction

Source localization [1], [2], [3], [4] and joint source and sensor localization [5], [6], [7], [8] based on *time difference of arrival* (TDOA) measurements are among the most fundamental problems in signal processing. They have received significant attention during the last three decades because of their wide range of applications. Most of the methods for TDOA-based localization are very sensitive to the accuracy of the TDOA measurements (see [2], [6], [7]). Thus, more accurate TDOA measurements are required.

Typically, TDOA measurements are given by a set of all the TDOA elements, which are obtained from all triples comprising of any two sensors and one source. Given a triple of two sensors and one source, when the source emits a signal and the two sensors receive this signal, the TDOA element of this triple is determined by measuring the peak of the generalized cross-correlation of the two received signals. To improve the accuracy of TDOA measurements, the traditional approach has been to improve the accuracy of estimation of TDOA elements. Another approach has been to use the algebraic properties of TDOAs. In this paper, we focus on the second approach, that is, we find some properties of TDOAs such that TDOA measurements can be refined using the estimated TDOA elements and these properties.

Recently, Velasco et al. [9] proposed a new method of refining TDOA measurements based on the rank and the special singular-value properties of the matrix generated by all TDOA-elements from triples with the same source. This method is based on the redundancy among these TDOA-elements. Concretely, the information of the $N(N-1)/2$ TDOA-elements generated by N sensors and a single source is equivalent to that of some $N-1$ appropriately selected TDOA elements. Hence, the process of transferring the information from all $N(N-1)/2$ TDOA-elements to the equivalent information of $N-1$ TDOA-elements can help reduce the errors in TDOA measurements. Note that the roles of the sources are independent in the process given in [9] for refining TDOAs. Interestingly, we found that the information of the remaining $N-1$ TDOA-elements for each source also includes redundancy if we combine the information for all sources. Propositions 1 and 2 in the next section describe this redundancy. These results are based on the *rank property* of the *times of arrival* (TOAs) which were introduced in [10] and studied carefully in [11]. The rank properties given by Propositions 1 and 2 are applied to $\mathbb{R}^2$ but they can be easily generalized to $\mathbb{R}^3$. However, since the proofs of Propositions 1 and 2 in $\mathbb{R}^3$ have not yet been found, here we only focus on $\mathbb{R}^2$.

Algorithms for our proposed method of refining TDOA measurements are presented in Section III. These algorithms are evaluated by simulative experiments by considering the differences between the results of refinement and the ground truth of TDOA measurements. In addition, we also evaluate these algorithms in terms of TDOA-based joint source and sensor localization via the algorithm given by [12]. The evaluations described in Section IV show that our proposed method significantly refines for TDOA measurements.

II. TDOA-distance matrix and Rank Properties

The focus of this paper, namely, the *TDOA-distance matrix*, is defined as follows.

Definition 1. *A real matrix $\mathbf{\Gamma} = \left(\tau_{mn}\right)_{M\times(N-1)}$ is called a **TDOA-distance matrix** in two dimensions (2D) if there exist M points $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and N points $\mathbf{y}_1, \ldots, \mathbf{y}_N$ in $\mathbb{R}^2$ such that for all $1 \le m \le M$, $1 \le n \le N-1$,*

$$\tau_{mn} = \|\mathbf{x}_m - \mathbf{y}_{n+1}\|_2 - \|\mathbf{x}_m - \mathbf{y}_1\|_2, \tag{1}$$

where $\|\cdot\|_2$ denotes the Euclidean distance.

In the field of signal processing, if $\mathbf{x}_m$ is the position of the mth source, $\mathbf{y}_1$ and $\mathbf{y}_{n+1}$ are the positions of the first and $(n+1)$th sensors, respectively, and moreover, if the velocity of the signal is known, the element τ_{mn} of the TDOA-distance matrix $\boldsymbol{\Gamma}$ is determined by the TDOA element of the triple $\{\mathbf{y}_1, \mathbf{y}_{n+1}, \mathbf{x}_m\}$. Since in most applications of localization, the velocity of the signal can be found, the TDOA-distance matrix is used to study TDOA measurements.

Rank properties are presented in a new matrix, named an *expansion of the TDOA-distance matrix*, which is given by the following definition.

Definition 2. *Assuming $\boldsymbol{\Gamma} = (\tau_{mn})_{M\times(N-1)}$ to be a TDOA-distance matrix, an **expansion** of $\boldsymbol{\Gamma}$ is given by*

$$\widetilde{\boldsymbol{\Gamma}} = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1,N-1} \\ \vdots & \ddots & \vdots \\ \tau_{M1} & \cdots & \tau_{M,N-1} \\ \tau_{21}^2 - \tau_{11}^2 & \cdots & \tau_{2,N-1}^2 - \tau_{1,N-1}^2 \\ \tau_{31}^2 - \tau_{11}^2 & \cdots & \tau_{3,N-1}^2 - \tau_{1,N-1}^2 \\ \vdots & \ddots & \vdots \\ \tau_{M1}^2 - \tau_{M-1,1}^2 & \cdots & \tau_{M,N-1}^2 - \tau_{M-1,N-1}^2 \end{pmatrix}. \quad (2)$$

The size of $\widetilde{\boldsymbol{\Gamma}}$ is $\frac{1}{2}M(M+1)\times(N-1)$.

Note that the first M rows of the expansion $\widetilde{\boldsymbol{\Gamma}}$ give the TDOA-distance matrix $\boldsymbol{\Gamma}$. Thus, a refinement of $\widetilde{\boldsymbol{\Gamma}}$ can be a refinement of $\boldsymbol{\Gamma}$. The following two propositions present two rank properties of $\widetilde{\boldsymbol{\Gamma}}$, from which we can derive some ways of refining $\widetilde{\boldsymbol{\Gamma}}$.

Proposition 1. *If $\boldsymbol{\Gamma} = (\tau_{mn})_{M\times(N-1)}$ is a TDOA-distance matrix in 2D, then for all $1 \le m_1 < m_2 < m_3 < m_4 \le M$,*

$$\textit{rank}\begin{pmatrix} \tau_{m_1,1} & \cdots & \tau_{m_1,N-1} \\ \tau_{m_2,1} & \cdots & \tau_{m_2,N-1} \\ \tau_{m_3,1} & \cdots & \tau_{m_3,N-1} \\ \tau_{m_4,1} & \cdots & \tau_{m_4,N-1} \\ \tau_{m_2,1}^2 - \tau_{m_1,1}^2 & \cdots & \tau_{m_2,N-1}^2 - \tau_{m_1,N-1}^2 \\ \tau_{m_3,1}^2 - \tau_{m_1,1}^2 & \cdots & \tau_{m_3,N-1}^2 - \tau_{m_1,N-1}^2 \\ \tau_{m_4,1}^2 - \tau_{m_1,1}^2 & \cdots & \tau_{m_4,N-1}^2 - \tau_{m_1,N-1}^2 \\ \tau_{m_3,1}^2 - \tau_{m_2,1}^2 & \cdots & \tau_{m_3,N-1}^2 - \tau_{m_2,N-1}^2 \\ \tau_{m_4,1}^2 - \tau_{m_2,1}^2 & \cdots & \tau_{m_4,N-1}^2 - \tau_{m_2,N-1}^2 \\ \tau_{m_4,1}^2 - \tau_{m_3,1}^2 & \cdots & \tau_{m_4,N-1}^2 - \tau_{m_3,N-1}^2 \end{pmatrix} \le 6. \quad (3)$$

Proof. Assuming that $\boldsymbol{\Gamma}$ is a TDOA-distance matrix in 2D, there exist M points $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and N points $\mathbf{y}_1, \ldots, \mathbf{y}_N$ in $\mathbb{R}^2$ such that $\tau_{mn} = \|\mathbf{x}_m - \mathbf{y}_{n+1}\|_2 - \|\mathbf{x}_m - \mathbf{y}_1\|_2$ for all m, n. Let us denote $d_{mn} = \|\mathbf{x}_m - \mathbf{y}_n\|_2$ as the *time-of-arrival (TOA) distance* between $\mathbf{x}_m$ and $\mathbf{y}_n$ and $\mathbf{D} = (d_{mn})$ as the *TOA-distance matrix*. For a TOA-distance matrix with a *low-rank property* (see [10]), the rank of $\left(d_{mn}^2 - d_{m1}^2 - d_{1n}^2 + d_{11}^2\right)_{(M-1)\times(N-1)}$ is at most two. Let $\mathbf{H}$ be the matrix given in (3) and $\mathbf{K}$ be the matrix comprising only columns n_1, n_2 and n_3 of $\mathbf{H}$. Let $\mathbf{K}_{i,j,h}$ be the matrix comprising only rows i, j and h of $\mathbf{K}$. On the basis of the low-rank property, in our previous work [12] we showed that for all $1 \le m_1 < m_2 < m_3 < m_4 \le M$ and $1 \le n_1 < n_2 < n_3 \le N-1$,

$$\begin{aligned} &-8|\mathbf{K}_{1,2,3}|z_1z_2z_3 + 8|\mathbf{K}_{1,2,4}|z_1z_2z_4 \\ &-8|\mathbf{K}_{1,3,4}|z_1z_3z_4 + 8|\mathbf{K}_{2,3,4}|z_2z_3z_4 \\ &+4|\mathbf{K}_{1,2,10}|z_1z_2 - 4|\mathbf{K}_{1,3,9}|z_1z_3 + 4|\mathbf{K}_{1,4,8}|z_1z_4 \\ &+4|\mathbf{K}_{2,3,7}|z_2z_3 - 4|\mathbf{K}_{2,4,6}|z_2z_4 + 4|\mathbf{K}_{3,4,5}|z_3z_4 \\ &-2|\mathbf{K}_{1,6,7}|z_1 + 2|\mathbf{K}_{1,5,7}|z_1 - 2|\mathbf{K}_{1,5,6}|z_1 \\ &+2|\mathbf{K}_{2,6,7}|z_2 - 2|\mathbf{K}_{3,5,7}|z_3 + 2|\mathbf{K}_{4,5,6}|z_4 \\ &+|\mathbf{K}_{5,6,7}| \;=\; 0, \end{aligned} \quad (4)$$

where z_1, z_2, z_3, z_4 are $d_{m_11}, d_{m_21}, d_{m_31}, d_{m_41}$, respectively, and $|\cdot|$ denotes the determinant of a matrix. Note that $-8|\mathbf{K}_{1,2,3}|$, $8|\mathbf{K}_{1,2,4}|, \ldots, |\mathbf{K}_{5,6,7}|$ are the elements of $\mathbf{C}_{m_1m_2m_3}(n_1, n_2, n_3)$ given by Table 1 in [12] when m_1, m_2, m_3, m_4 are replaced by $1, m_1+1, m_2+1, m_3+1$, respectively. Interestingly, equation (4) can be expressed by the matrix formula given by (5). Because of the last three rows of the matrix given in (5), the last three row vectors of this matrix are linearly independent of the others. Assuming that z_1, z_2, z_3, z_4 are positive, (5) implies that

$$\begin{vmatrix} & \frac{1}{2z_1} & 0 & 0 & 0 \\ & 0 & \frac{1}{2z_2} & 0 & 0 \\ & 0 & 0 & \frac{1}{2z_3} & 0 \\ \mathbf{K}_{1,2,3,4,5,6,7} & 0 & 0 & 0 & \frac{1}{2z_4} \\ & 1 & -1 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 1 & 0 & 0 & -1 \end{vmatrix} = 0, \quad (6)$$

and, therefore,

$$\text{rank}\begin{pmatrix} & \frac{1}{2z_1} & 0 & 0 & 0 \\ & 0 & \frac{1}{2z_2} & 0 & 0 \\ & 0 & 0 & \frac{1}{2z_3} & 0 \\ \mathbf{H}_{1,2,3,4,5,6,7} & 0 & 0 & 0 & \frac{1}{2z_4} \\ & 1 & -1 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 1 & 0 & 0 & -1 \end{pmatrix} \le 6. \quad (7)$$

Thus, the seven vectors of $\mathbf{H}_{1,2,3,4,5,6,7}$ are linearly dependent. Similar conclusions can be derived for any other seven vectors of $\mathbf{H}$. Hence, rank$(\mathbf{H}) \le 6$ and Proposition 1 is proven. □

Proposition 2. *If $\boldsymbol{\Gamma}$ is an $M\times(N-1)$ TDOA-distance matrix ($M \ge 4$), then*

$$\textit{rank}(\widetilde{\boldsymbol{\Gamma}}) \le M + 2, \quad (8)$$

where $\widetilde{\boldsymbol{\Gamma}}$ is an expansion of $\boldsymbol{\Gamma}$.

Proof. Proposition 1 implies that Proposition 2 is correct when $M = 4$. Using the *induction method*, the proof of Proposition 2 is complete if we assume that Proposition 2 is correct for $M-1$ ($M > 4$) and prove that it is also correct for M. Let us denote *vectors of order 1* for the first M rows of $\widetilde{\boldsymbol{\Gamma}}$ as $(\tau_{m1}, \ldots, \tau_{m,N-1})$ (abbreviated to $\widetilde{\boldsymbol{\Gamma}}_m$) and *vectors of order 2* for the last $M(M-1)/2$ rows as $(\tau_{m_i1}^2 - \tau_{m_j1}^2, \ldots, \tau_{m_i,N-1}^2 - \tau_{m_j,N-1}^2)$ (abbreviated to $\widetilde{\boldsymbol{\Gamma}}_{m_im_j}$). A row is said to "*contain*" an index m if it is $\widetilde{\boldsymbol{\Gamma}}_m$, $\widetilde{\boldsymbol{\Gamma}}_{m\star}$ or $\widetilde{\boldsymbol{\Gamma}}_{\star m}$.

We assume that there exist $M+3$ rows of $\widetilde{\boldsymbol{\Gamma}}$ that are linearly independent. For all indexes m, these $M+3$ rows should have at least two rows that contain the index m. Thus, if there exists an index m such that only one of these $M+3$ rows contains it,

ISBN 978-0-9928626-7-1

$$\begin{vmatrix} 2z_1\tau_{m_1n_1} & 2z_1\tau_{m_1n_2} & 2z_1\tau_{m_1n_3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2z_2\tau_{m_2n_1} & 2z_2\tau_{m_2n_2} & 2z_2\tau_{m_2n_3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2z_3\tau_{m_3n_1} & 2z_3\tau_{m_3n_2} & 2z_3\tau_{m_3n_3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2z_4\tau_{m_4n_1} & 2z_4\tau_{m_4n_2} & 2z_4\tau_{m_4n_3} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \tau^2_{m_2n_1}-\tau^2_{m_1n_1} & \tau^2_{m_2n_2}-\tau^2_{m_1n_2} & \tau^2_{m_2n_3}-\tau^2_{m_1n_3} & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \tau^2_{m_3n_1}-\tau^2_{m_1n_1} & \tau^2_{m_3n_2}-\tau^2_{m_1n_2} & \tau^2_{m_3n_3}-\tau^2_{m_1n_3} & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \tau^2_{m_4n_1}-\tau^2_{m_1n_1} & \tau^2_{m_4n_2}-\tau^2_{m_1n_2} & \tau^2_{m_4n_3}-\tau^2_{m_1n_3} & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ \tau^2_{m_3n_1}-\tau^2_{m_2n_1} & \tau^2_{m_3n_2}-\tau^2_{m_2n_2} & \tau^2_{m_3n_3}-\tau^2_{m_2n_3} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \tau^2_{m_4n_1}-\tau^2_{m_2n_1} & \tau^2_{m_4n_2}-\tau^2_{m_2n_2} & \tau^2_{m_4n_3}-\tau^2_{m_2n_3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \tau^2_{m_4n_1}-\tau^2_{m_3n_1} & \tau^2_{m_4n_2}-\tau^2_{m_3n_2} & \tau^2_{m_4n_3}-\tau^2_{m_3n_3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0. \qquad (5)$$

the $M+2$ remaining rows will contain the $M-1$ remaining indexes. And hence by the induction method, these $M+2$ rows are linearly dependent, resulting in a contradiction.

In other words, if the group of all vectors of order 2 in these $M+3$ rows has the sequence of rows $\widetilde{\mathbf{\Gamma}}_{m_1m_2}$, $\widetilde{\mathbf{\Gamma}}_{m_2m_3}, \ldots, \widetilde{\mathbf{\Gamma}}_{m_1m_2}, \widetilde{\mathbf{\Gamma}}_{m_{i-1}m_i}, \widetilde{\mathbf{\Gamma}}_{m_1m_i}$ for any indexes m_1, $m_2, \ldots, m_i$, these vectors of order 2 are linearly dependent, and thus these $M+3$ vectors are not linearly independent. However, the fact that all indexes m appear at least twice in the $M+3$ rows implies that (i) there always exists a sequence of indexes $m_1, m_2, \ldots, m_i$ such that $\widetilde{\mathbf{\Gamma}}_{m_1m_2}, \widetilde{\mathbf{\Gamma}}_{m_2m_3}, \ldots, \widetilde{\mathbf{\Gamma}}_{m_1m_2}, \widetilde{\mathbf{\Gamma}}_{m_{i-1}m_i}, \widetilde{\mathbf{\Gamma}}_{m_1m_i}$ are in these $M+3$ rows or (ii) there exists a sequence of indexes m_1, m_2, m_3, m_4 such that $\widetilde{\mathbf{\Gamma}}_{m_1}, \widetilde{\mathbf{\Gamma}}_{m_2}, \widetilde{\mathbf{\Gamma}}_{m_3}, \widetilde{\mathbf{\Gamma}}_{m_4}, \widetilde{\mathbf{\Gamma}}_{m_2m_1}$, $\widetilde{\mathbf{\Gamma}}_{m_3m_1}$ are in these $M+3$ rows. In both cases, these $M+3$ rows are linearly dependent. Therefore, the assumption that there exist $M+3$ linearly independent rows is incorrect. Thus, $\text{rank}(\widetilde{\mathbf{\Gamma}}) \le M+2$. □

III. Refinement of TDOA-distance matrix

In this section, we discuss how to refine a TDOA-distance matrix from its noisy observation. Given a TDOA-distance matrix $\mathbf{\Gamma}$ and its observation $\mathbf{\Gamma}_{\text{No}}$, i.e.,

$$\mathbf{\Gamma}_{\text{No}} = \mathbf{\Gamma} + \text{Noises}, \qquad (9)$$

a refinement of the TDOA-distance matrix is defined as finding a new matrix $\mathbf{\Gamma}_{\text{Re}}$ such that (i) $\|\mathbf{\Gamma}_{\text{Re}} - \mathbf{\Gamma}\|_F < \|\mathbf{\Gamma}_{\text{No}} - \mathbf{\Gamma}\|_F$, where $\|\cdot\|_F$ denotes the *Frobenius norm*, (ii) $\mathbf{\Gamma}_{\text{Re}}$ satisfies some properties of $\mathbf{\Gamma}$ and (iii) the difference $\|\mathbf{\Gamma}_{\text{Re}} - \mathbf{\Gamma}\|_F$ is as small as possible.

On the basis of the rank properties given by Propositions 1 and 2, we propose a class of algorithms for a refinement of the TDOA-distance matrix as follows.

Algorithm: *Refinement of TDOA-distance matrix*

Input: $\mathbf{\Gamma}_{\text{No}}$ - An observed TDOA-distance matrix.

κ - number of iterations.

Implementation:

1. M is the initial size of $\mathbf{\Gamma}_{\text{No}}$.

2. $\widetilde{\mathbf{\Gamma}}_{\text{No}}$ is an expansion of $\mathbf{\Gamma}_{\text{No}}$ given by Definition 2.

3. Using singular value decomposition (SVD), determine $\widetilde{\mathbf{\Gamma}}^\circ$ by extracting the $M+2$ largest singular values of $\widetilde{\mathbf{\Gamma}}_{\text{No}}$. (*based on Proposition 2*)

4. For $k = 1, 2, \ldots, \kappa$

For all indexes m_1, m_2, m_3, m_4

a. $\mathbf{K} \leftarrow [\widetilde{\mathbf{\Gamma}}^\circ_{m_1}; \widetilde{\mathbf{\Gamma}}^\circ_{m_2}; \widetilde{\mathbf{\Gamma}}^\circ_{m_3}; \widetilde{\mathbf{\Gamma}}^\circ_{m_4}; \widetilde{\mathbf{\Gamma}}^\circ_{m_1m_2}; \ldots \widetilde{\mathbf{\Gamma}}^\circ_{m_1m_3}; \widetilde{\mathbf{\Gamma}}^\circ_{m_1m_4}; \widetilde{\mathbf{\Gamma}}^\circ_{m_2m_3}; \widetilde{\mathbf{\Gamma}}^\circ_{m_2m_4}; \widetilde{\mathbf{\Gamma}}^\circ_{m_3m_4}]$ (a matrix with ten rows)

b. Using SVD, determine $\mathbf{K}^\circ$ by extracting the six largest singular values of $\mathbf{K}$. (*based on Proposition 1*)

c. $\widetilde{\mathbf{\Gamma}}^\circ_{m_1}, \widetilde{\mathbf{\Gamma}}^\circ_{m_2}, \ldots, \widetilde{\mathbf{\Gamma}}^\circ_{m_2m_4}, \widetilde{\mathbf{\Gamma}}^\circ_{m_3m_4}$ are refined by $\mathbf{K}^\circ_1, \mathbf{K}^\circ_2, \ldots, \mathbf{K}^\circ_9, \mathbf{K}^\circ_{10}$ (rows of $\mathbf{K}^\circ$), respectively.

5. Repeat step 3 with $\widetilde{\mathbf{\Gamma}}_{\text{No}}$ replaced by $\widetilde{\mathbf{\Gamma}}^\circ$. (*based on Proposition 2*)

6. $\mathbf{\Gamma}_{\text{Re}}$ is the first M rows of $\widetilde{\mathbf{\Gamma}}^\circ$.

Output: $\mathbf{\Gamma}_{\text{Re}}$ - A refined TDOA-distance matrix.

In this paper, we consider the proposed algorithm corresponding to $\kappa = 0, 1, 3, 10, 50$ and 200 which are denoted by Alg_0, Alg_1, Alg_3, Alg_{10}, Alg_{50} and Alg_{200}, respectively. Alg_0 evaluates the contribution of Proposition 2, while $\text{Alg}_1, \text{Alg}_3$, Alg_{10}, Alg_{50} and Alg_{200} evaluate the contribution of both Propositions 1 and 2 in refining the TDOA-distance matrix. Moreover, the larger the value of κ, the greater the contribution of Proposition 1.

IV. Evaluation

A. Refinement of TDOA-distance matrix

The proposed algorithms are evaluated by simulative experiments, which are introduced as follows. The ground truth of an $M \times (N-1)$ TDOA-distance matrix $\mathbf{\Gamma}$ is generated by M points $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and N points $\mathbf{y}_1, \ldots, \mathbf{y}_N$, which are chosen randomly and uniformly in a virtual square of size 5×5 m and Definition 1. The observed TDOA-distance matrix is the original TDOA-distance matrix corrupted by i.i.d. Gaussian

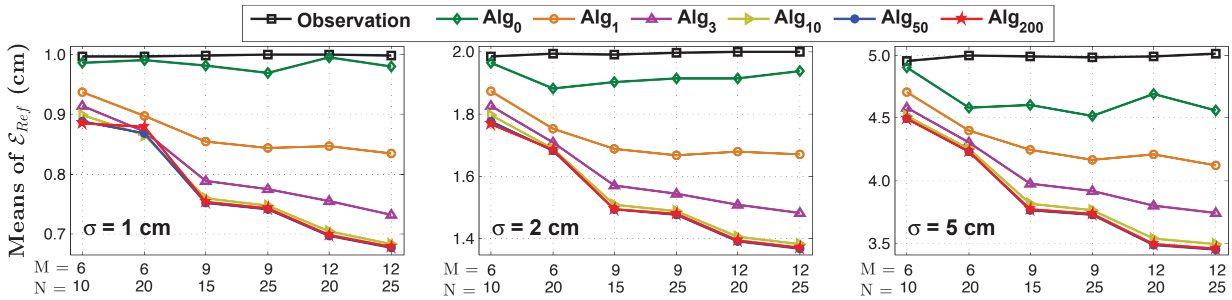

Fig. 1. Evaluation of different methods of refining TDOA-distance matrix for various values of M and N in terms of means of RMSEs given by (10).

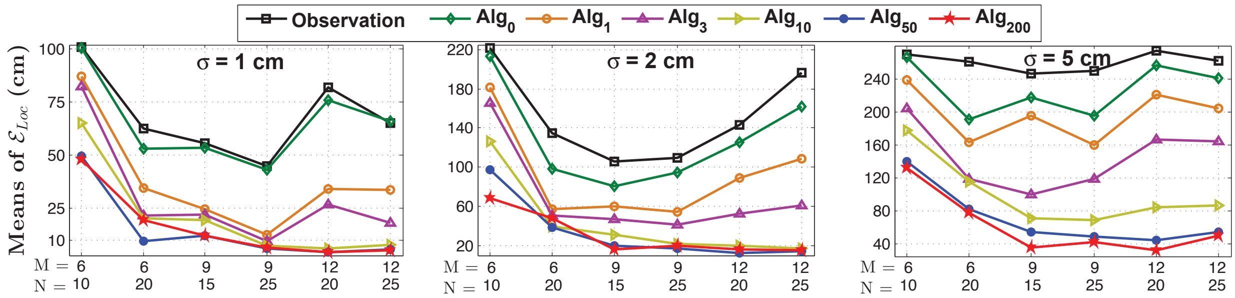

Fig. 2. Evaluation of different methods of TDOA-based joint source and sensor localization for various values of M and N in terms of means of RMSEs given by (11).

noises, i.e., $\mathbf{\Gamma}_{\text{No}} = \mathbf{\Gamma} + \mathcal{N}(0, \sigma^2 \mathbf{I}_{M(N-1)})$, where $\mathbf{I}_{M(N-1)}$ is the identity matrix of size $M(N-1)$ and $\mathcal{N}(0, \sigma^2 \mathbf{I}_{M(N-1)})$ denotes an $(M, N-1)$ Gaussian matrix with zero mean and covariance $\sigma^2 \mathbf{I}_{M(N-1)}$. Some refining algorithms are applied to refine the observed TDOA-distance matrix. The *root-mean-square errors* (RMSEs) of all the elements of the original TDOA-distance matrix,

$$\mathcal{E}_{\text{Ref}}(\mathbf{\Gamma}_{\text{Re}}) = \left[\frac{1}{M(N-1)} \sum_{m=1}^{M} \sum_{n=1}^{N-1} \left(\tau_{mn}^{(\text{Re})} - \tau_{mn} \right)^2 \right]^{\frac{1}{2}}, \quad (10)$$

are used to evaluate the refinements, where $\mathbf{\Gamma}_{\text{Re}}$ is the refined TDOA-distance matrix, and τ_{mn} and τ_{mn}^{Re} are the (m, n)th elements of the original and refined TDOA-distance matrices, respectively.

Figure 1 presents the means of the RMSEs of 1000 independent simulative experiments given by (10). The RMSEs are computed for the observation $\mathbf{\Gamma}_{\text{No}}$ and the six refinements given by Alg_0, Alg_1, Alg_3, Alg_{10}, Alg_{50} and Alg_{200}. Three different levels of Gaussian noise, $\sigma = 1, 2$ and 5 cm, are assumed. The results in Figure 1 show that the proposed methods are effective for refining the TDOA-distance matrix with the assumption of i.i.d. Gaussian noises, especially when M and N are large. Moreover, it appears that the larger the value of parameter κ, the more effective the proposed methods. Thus, not only Proposition 2 but also Proposition 1 contributes to the refinement of the TDOA-distance matrix.

B. TDOA-based joint source and sensor localization

The most common application of TDOA measurements (or the TDOA-distance matrix) is localization. In this section, we study how the refinement of the TDOA-distance matrix affects the *TDOA-based joint source and sensor localization* introduced in [6], [12] using the proposed methods Alg_0, Alg_1, Alg_3, Alg_{10}, Alg_{50} and Alg_{200}. We apply the algorithm for TDOA-based joint source and sensor localization given in [12] to the observed and refined TDOA-distance matrices. The estimations of positions from the above simulative experiments are compared with the ground-truth positions, $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$, in terms of the following RMSE:

$$\begin{aligned} &\mathcal{E}_{\text{Loc}}\left(\{\mathbf{x}_1^{\text{est}}, \ldots, \mathbf{x}_M^{\text{est}}, \mathbf{y}_1^{\text{est}}, \ldots, \mathbf{y}_N^{\text{est}}\}\right) \\ &= \left[\frac{1}{M} \sum_{m=1}^{M} \left\| \mathbf{x}_m^{\text{est}} - \mathbf{x}_m \right\|_2^2 + \frac{1}{N} \sum_{n=1}^{N} \left\| \mathbf{y}_n^{\text{est}} - \mathbf{y}_n \right\|_2^2 \right]^{\frac{1}{2}}, \end{aligned} \quad (11)$$

where $\mathbf{x}_m^{\text{est}}$ and $\mathbf{y}_n^{\text{est}}$ are the estimated positions.

Figure 2 presents the means of the RMSEs of 200 independent simulative experiments given by (11). The same three different levels of Gaussian noise, $\sigma = 1$, 2 and 5 cm, are also used in this experiment. The results in Figure 2 show that the accuracy of localization is significantly improved from that obtained by the proposed methods. For example, when the standard deviation of Gaussian noise is $\sigma = 1$ cm, the mean of the RMSEs between the estimated positions and the ground-truth positions decreases from 100.9 cm for localization without refinement to 48.0 cm for localization with Alg_{200} when $M = 6$ and $N = 10$, and from 65.1 cm for localization without refinement to 5.4 cm for localization with Alg_{200} when $M = 12$ and $N = 25$.

V. Conclusion

In this paper, we reported two newly found rank properties for the expansion of TDOA-distance matrices. By performing a factorization to satisfy the rank properties, we proposed some algorithms to refine the TDOAs from their observations. These refining algorithms significantly improve the accuracy of both TDOA estimation and TDOA-based joint source and sensor localization. A limitation of this paper is that we have not determined the optimal value of κ, which will be examined in the near future. Extending the two rank properties to $\mathbb{R}^3$ is also planned as future work.

VI. Acknowledgements

This work was supported by a Grant-in-Aid for Scientific Research (A) (Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number 16H01735) and the SECOM Science and Technology Foundation.

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