Multivariance Nonlinear System Identification Using Wiener Basis Functions and Perfect Sequences

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***Abstract*—Multivariance identification methods exploit input signals with multiple variances for estimating the Volterra kernels of nonlinear systems. They overcome the problem of the locality of the solution, i.e., the fact that the estimated model well approximates the system only at the same input signal variance of the measurement. The estimation of a kernel for a certain input signal variance requires recomputing all lower order kernels. In this paper, a novel multivariance identification method based on Wiener basis functions is proposed to avoid recomputing the lower order kernels with computational saving. Formulas are provided for evaluating the Volterra kernels from the Wiener multivariance kernels. In order to further improve the nonlinear filter estimation, perfect periodic sequences that guarantee the orthogonality of the Wiener basis functions are used for Wiener kernel identification. Simulations and real measurements show that the proposed approach can accurately model nonlinear devices on a wide range of input signal variances.**

I. Introduction

The Wiener G-functionals [1], [2] were introduced to overcome one of the main limitations of the Volterra filters, whose polynomial terms are never orthogonal. The G-functionals derive from the orthogonalization of the Volterra series for white Gaussian inputs and allow the efficient identification of nonlinear systems with the cross-correlation method [1]. This approach presents many drawbacks when applied to stochastic inputs at the point that it is often considered just a "legacy" method [3, page 77]. Indeed, it usually requires millions of samples to accurately estimate the nonlinear kernels. Moreover, an exact white Gaussian input cannot be generated due to the limitation of the input signal length and to the input amplitude saturation. Furthermore, the central moments of a Gaussian input soon depart from ideal values as the moment order increases unless millions of values are used [4]. Some improvements of the first implementations of the cross-correlation method (e.g., Lee-Schetzen [1]) were provided in [4], [5] to reduce the input non-ideality and errors due to model order truncation that affect the kernels diagonal points [4].

A known drawback of Volterra and Wiener theory [6] is the input amplitude limitations related to convergence issues when higher-order kernels are needed. The non-idealities of input noise make the output mean square error (MSE) a function of the input variance [4]. In particular, an accurate high order kernel estimation requires high input variances to excite high order nonlinearities, but causing high identification errors in lower order kernels. On the contrary, low input variances produce an underestimation in high order kernels. This phenomenon is known as the "locality" of Volterra series identification, i.e., a Volterra series is optimal only for input variances in a neighborhood of that used for identification [7]. An improved cross-correlation method for nonlinear system identification based on multiple-variance white Gaussian noise (WGN) has been proposed in [7]: low input variances are used to model lower order kernels, while the input variance is gradually increased for higher order kernels.

The Wiener basis functions [8] are a set of polynomial basis functions, which are orthogonal for white Gaussian noise inputs. They can arbitrarily well approximate any discrete time, time-invariant, finite memory, continuous, nonlinear system. A linear combination of Wiener basis functions forms a Wiener nonlinear (WN) filter [2], i.e., a double truncated, with respect to order and memory, Wiener series.

Perfect periodic sequences (PPSs) [9], [10] have been proposed for linear [11], [12] and nonlinear [8], [13]–[17] system identification as an alternative to stochastic inputs. A PPS ensures that the cross-correlation between any two different basis functions of the system, estimated over a period, is zero. Therefore, an unknown system can be efficiently identified with the cross-correlation method using a PPS as input signal. PPSs suitable for the identification of WN filters have been recently proposed in [8] to reduce the accuracy problems in estimating the kernels diagonal points thanks to the perfect orthogonality of the Wiener basis functions for PPSs.

In this paper, a multivariance identification method based on PPSs and WN filters, expressed in term of Wiener basis functions, is presented to exploit the ideal properties of PPSs and the computational saving provided by Wiener basis functions.

The paper is organized as follows. First, the Wiener basis functions are described in Section II. The multiple-variance approach and the PPSs derivation are reported in Section III and Section IV. Then, the choice of the multiple variances is discussed in Section V. Finally, experimental results and concluding remarks are shown in Section VI and Section VII.

The following notation is used in the paper: $E[\cdot]$ denotes mathematical expectation, $<\cdot>_L$ is the average over a period of L samples, $\mathcal{N}(0, \sigma_x^2)$ is the zero mean and variance σ_x^2 normal distribution.

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II. The Wiener basis functions

The Wiener basis functions [8] are a set of polynomial basis functions that can arbitrarily well approximate any discrete time, time-invariant, finite memory, continuous, nonlinear system,

$$y(n) = f[x(n), x(n-1), \dots, x(n-N+1)], \quad (1)$$

being f a continuous function from $\mathbb{R}^N$ to $\mathbb{R}$. The Wiener basis functions are orthogonal for any white Gaussian input signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$. For $N = 1$, the Wiener basis functions can be obtained by applying the Gram-Schmidt orthogonalization to the polynomial set $\{1,\ x(n),\ x^2(n),\ x^3(n),\ \dots\}$ for white Gaussian inputs, obtaining the set

$$\{1,\ x(n),\ x^2(n) - \sigma_x^2,\ x^3(n) - 3\sigma_x^2 x(n),\ \dots\}. \quad (2)$$

For $N > 1$, the Wiener basis functions are derived by (i) writing the one-dimensional basis functions in (2) for $x(n), x(n-1), \dots, x(n-N+1)$ and (ii) multiplying the terms with different variables in any possible manner, avoiding repetitions. The Wiener basis functions and their linear combinations form an algebra on any compact in $\mathbb{R}^N$ that satisfies the requirements of the Stone-Weierstrass theorem [18] and can arbitrarily well approximate the system in (1).

A WN filter is defined as the linear combination of Wiener basis functions up to a certain order P and memory N. For the sake of simplicity, an order 3 and memory N WN filter will be considered, having the following diagonal form [19]:

$$\begin{aligned} y(n) = k_0 &+ \sum_{t=0}^{N-1} k_{1,t} x(n-t) + \sum_{t=0}^{N-1} k_{2,t,t}[x^2(n-t) - \sigma_x^2] + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} k_{2,t,t+r} x(n-t) x(n-t-r) + \\ &+ \sum_{t=0}^{N-1} k_{3,t,t,t}[x^3(n-t) - 3\sigma_x^2 x(n-i)] + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} k_{3,t,t,t+r}[x^2(n-t) - \sigma_x^2] x(n-t-r) + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} k_{3,t,t+r,t+r} x(n-t)[x^2(n-t-r) - \sigma_x^2] + \\ &+ \sum_{r=1}^{N-2} \sum_{s=r+1}^{N-1} \sum_{t=0}^{N-1-s} k_{3,t,t+r,t+s} x(n-t) x(n-t-r) x(n-t-s). \end{aligned} \quad (3)$$

The set of coefficients $k_{l,\dots}$ of equal order l forms the so-called l-th kernel of the WN filter.

Since the Wiener basis functions are orthogonal for a white Gaussian input signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$, the WN filter coefficients can be estimated with the cross-correlation approach, i.e., computing the cross-correlation between the basis functions and the unknown system output. Assume $w_j(n)$ to be one of the basis functions

$$\begin{gathered} \{1, x(n-t), [x^2(n-t) - \sigma_x], x(n-t)x(n-t-r), \\ [x^3(n-t) - 3\sigma_x^2 x(n)], [x^2(n-t) - \sigma_x] x(n-t-r), \\ x(n-t)[x^2(n-t-r) - \sigma_x], x(n-t)x(n-t-r)x(n-t-s)\} \end{gathered}$$

for any r, s, and t, and k_j the corresponding coefficient. Then,

$$k_j = \frac{E[y(n) w_j(n)]}{E[w_j^2(n)]}, \quad (4)$$

where $y(n)$ is the unknown nonlinear system output. The expectations are usually computed using time averages over large periods of input samples.

Once the WN filter in (3) has been identified, it can be can converted into a Volterra filter, by equating the polynomial terms of equal degree:

$$\begin{aligned} y(n) = h_0 &+ \sum_{t=0}^{N-1} h_{1,t} x(n-t) + \sum_{t=0}^{N-1} h_{2,t,t} x^2(n-t) + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} h_{2,t,t+r} x(n-t) x(n-t-r) + \\ &+ \sum_{t=0}^{N-1} h_{3,t,t,t} x^3(n-t) + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} h_{3,t,t,t+r} x^2(n-t) x(n-t-r) + \\ &+ \sum_{r=1}^{N-1} \sum_{t=0}^{N-1-r} h_{3,t,t+r,t+r} x(n-t) x^2(n-t-r) + \\ &+ \sum_{r=1}^{N-2} \sum_{s=r+1}^{N-1} \sum_{t=0}^{N-1-s} h_{3,t,t+r,t+s} x(n-t) x(n-t-r) x(n-t-s) \end{aligned} \quad (5)$$

III. Multivariance system identification

Multivariance methods have been proposed to solve the "locality problem" by using a different variance to estimate each of the kernels. The original multivariance approach [7] is based on the cross-correlation between a particular Volterra system, known as the delay system $\prod_{i=1}^{l} x(n-t_i)$, and the system output $E\{y(n) \prod_{i=1}^{l} x(n-t_i)\}$. Note that $E\{y(n)x(n-t)x(n-t-r)x(n-t-s)\} = 3!\sigma_x^6 k_{3,t,t+s,t+r} - \sigma_x^4(k_{1,t}\delta_{t+s,t+r} + k_{1,t+r}\delta_{t,t+s} + k_{1,t+s}\delta_{t,t+r})$.
The unitary impulses, δ_{t_1,t_2}, come from the first order Wiener kernel and are the source of the identification problem of the so-called diagonal points. The corrections needed in diagonal points identification require to recompute the lower odd/even order kernels for each odd/even order kernel to be identified [7]. In contrast, exploiting the Wiener basis functions and the WN filter in (3), the kernels $k_{l,\dots}$ become independent of each other and can be separately estimated using (4).

Clearly, the Wiener basis functions and the WN filter coefficients change with the variance of the input signal. Let us indicate with $k_{l,\dots}^{(i)}$ the coefficients of the kernel of order l estimated with the variance $\sigma_{x,i}$, with $i = 0, 1, \dots$ Given the multivariance Wiener kernels $\{k_0^{(0)}, k_{1,\cdot}^{(1)}, k_{2,\cdot,\cdot}^{(2)}, \dots\}$, we want to estimate the corresponding Volterra kernels, which are independent of the input variance. Equating (3) and (5), it can be noticed that the two largest order kernels of the WN filter are always equal to the corresponding kernels of the Volterra

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filter for any input variance σ_x^2, i.e., for $r = 0,..,N-2$, $s = r,..,N-1$, $t = 0,..,N-1-s$:

$$h_{3,t,t+r,t+s} = k_{3,t,t+r,t+s}^{(3)} \tag{6}$$

and for $r = 0,..,N-1$, $t = 0,..,N-1-r$:

$$h_{2,t,t+r} = k_{2,t,t+r}^{(2)}. \tag{7}$$

Equating the first order terms in (3) and (5) for $\sigma_x = \sigma_{x,1}$, since $k_{3,t,t+r,t+s}^{(1)} = k_{3,t,t+r,t+s}^{(3)}$ for all r, s, and t, with some manipulations it results

$$h_{1,t} = k_{1,t}^{(1)} - \sigma_{x,1}^2 k_{3,t,t,t}^{(3)} - \sigma_{x,1}^2 \sum_{u=0}^{N-1} (k_{3,t,u,u}^{(3)} + k_{3,t,t,u}^{(3)}). \tag{8}$$

Equating the constant terms in (3) and (5) for $\sigma_x = \sigma_{x,0}$, since $k_{2,t,t+r}^{(0)} = k_{2,t,t+r}^{(2)}$ for all r and t, we have

$$h_0 = k_0^{(0)} - \sigma_{x,1}^2 \sum_{t=0}^{N-1} k_{2,t,t}^{(2)}. \tag{9}$$

The procedure is described for WN and Volterra filters of order 3, but it can be applied also for higher order filters.

IV. Perfect Periodic Sequences for WN filters

The main limitation of the cross-correlation approach in (4) using Gaussian input signal is the huge number of input samples (in the order of millions or more) necessary to guarantee the approximate orthogonality of the basis function and a reasonable accuracy in the sample estimations. PPS are periodic sequences that guarantee the perfect orthogonality of the basis functions over a period and thus can accurately estimate the coefficients of the filter with the cross-correlation approach replacing the expectations in (4) with time averages over a period. A PPS $x_{\rm p}(n)$ of period L suitable for the identification of the Wiener filters up to an order P and memory N and with Gaussian variance σ_x^2 can be developed following the approach of [8]. The PPS $x_{\rm p}(n)$ can be derived by imposing that all joint moments of the input signal, estimated over a period, involved in the identification of the WN filter, are equal to those of a white Gaussian signal $x(n) \in \mathcal{N}(0, \sigma_x^2)$. Thus, the following system of nonlinear equations is imposed:

$$\begin{aligned} < x_{\rm p}^{k_0}(n) \cdot x_{\rm p}^{k_1}(n-1) \cdot \ldots \cdot x_{\rm p}^{k_{N-1}}(n-N+1) >_L = \\ = \mu_{k_0} \cdot \mu_{k_1} \cdot \ldots \cdot \mu_{k_{N-1}} \end{aligned} \tag{10}$$

for all $k_0, \ldots, k_{N-1} \in \mathbb{N}$ ($k_0 > 0$ and $k_0 + \ldots + k_{N-1} \leq 2P$), and μ_k the k-th moment of the Gaussian process $\mathcal{N}(0, \sigma_x^2)$,

$$\mu_k = E[x^k(n)] = \begin{cases} 0 & \text{for } k \text{ odd,} \\ \sigma_x^k (k-1)!! & \text{for } k \text{ even,} \end{cases} \tag{11}$$

with $q!! = q \cdot (q-2) \cdot (q-4) \cdot \ldots \cdot 1$.

The nonlinear system in (10) has a number of equations $Q = \binom{N+2P-1}{N}$ and for sufficiently large L is an underdetermined system of equations in the variables $x_{\rm p}(n)$. The system in (10) has been solved using the Newton-Raphson method, implemented as in [20, ch. 9.7] starting from a Gaussian distribution of the variables with variance σ_x^2. Different PPSs for WN filters of order 3, signal power $\sigma_x^2 = 1/12$, and memory N ranging from 5 to 20 have been developed and are available for download at [21].

If we scale the PPS by a factor c, any order k joint moment in (10), with $k = k_0 + k_1 + \ldots + k_{N-1}$, is scaled by a factor c^k and the sequence is still a PPS suitable for the identification of WN filters but for Gaussian variance $c^2 \sigma_x^2$. Thus, the PPSs can be used for the multiple-variance identification approach of Section III. It suffices to replace the input signals with PPSs of appropriate variance and to estimate all cross-correlation terms over a PPS period.

When a PPS suitable for WN filter up to order P and memory N is used for identifying a system with order greater than P, the estimation of the Wiener kernels will be affected by an error. Following arguments similar to those in [14], it can be proved that the error affects mainly the highest order kernels and, in general, only mildly low-order kernels. Similarly, when a PPS suitable for WN filter up to order P and memory N is used for identifying a system with memory greater than N, the estimation of the Wiener kernels will also be affected by an error. It can be proved that this error affects mainly the coefficients of kernels associated with the most recent samples, $x(n)$, $x(n-1)$, ..., and, in general, only mildly the coefficients of the basis functions associated with the less recent samples, $x(n-N+1)$, $x(n-N+2)$.

V. Optimal choice of the multiple variances

An important problem is the choice of the multiple variances $\sigma_{x,i}^2$ used to contrast the locality of the solution. A possible criterion is to estimate each kernel at the input signal variance that minimizes the error in the kernel coefficients identification. For Gaussian inputs, the errors are due to the finite length of the input sequence. For PPS, the errors are caused by the unknown system basis functions having memory or order larger than those considered in the PPS. In both cases, it can be proved that the mean square deviation MSD_i in the identification of the i-th kernel with input variance $\sigma_{x,i}^2$ is:

$$\text{MSD}_i = \eta_{0,i} \sigma_{x,i}^{-2i} + \eta_{1,i} \sigma_{x,i}^{2(1-i)} + \ldots + \eta_{K,i} \sigma_{x,i}^{2(K-i)} + \eta_{\nu,i} \sigma_\nu^2 \sigma_{x,i}^{-2i}, \tag{12}$$

being K the order of the unknown system, possibly larger than the order P of the identification filter, σ_ν^2 the variance of the additive zero mean Gaussian noise, and $\eta_{0,i}, \ldots, \eta_{K,i}, \eta_{\nu,i}$ constant coefficients depending on the unknown system kernels and on the cross-correlation between the i-th order basis functions and some error terms. For space limitation, the proof of (12) is omitted and will be shown in a future work.

Even though the coefficients $\eta_{l,i}$ are rarely known and thus $\sigma_{x,i}^2$ cannot be found solving (12), the equation is very useful to guide the choice of the variances. Indeed, if all the unknown system kernels of order larger than P are negligible, according to (12) the kernel of order P should be identified at the largest possible variance. The kernel of order 0 can be estimated at the lowest possible variance, while the kernel of order 1 should be estimated at the lowest variance for which the effect of noise is negligible. The other kernels should be estimated with variance comprised between that of kernel 1 and kernel P. As a rule

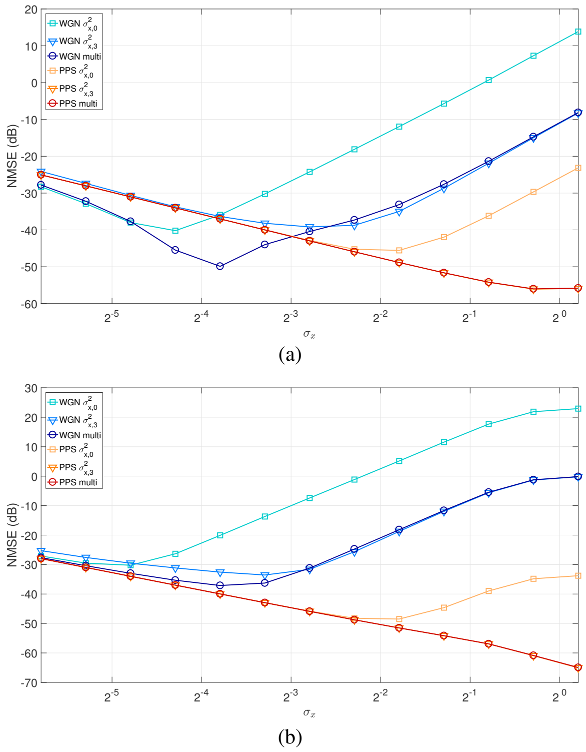

Fig. 1. NMSE obtained in Test 1. (a) Noise. (b) Music.

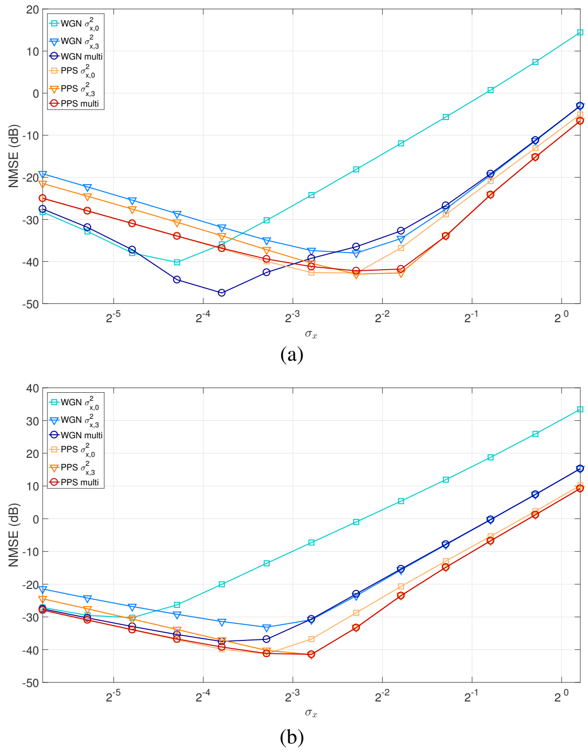

Fig. 2. NMSE obtained in Test 2. (a) Noise. (b) Music.

of thumb, each i-th kernel could be estimated at the variance that maximizes the ratio between the power of the i-th order term and the total output power.

VI. Experimental Results

Several experiments have been carried out to test the exploitation of multiple-variance PPSs in the cross-correlation method, making comparisons with multiple-variance WGN as described in [7]. Moreover, for the sake of completeness, the results achievable with the traditional cross-correlation method based on the same input variance for all kernels are reported.

Different test sessions have been accomplished, considering a simulated nonlinear system as a first step and then, a real-world nonlinear device, modelled using a third-order Volterra series with memory 10 and 25, respectively. The adopted PPS sequence has order $P = 3$, memory $N = 25$, period $L = 1393024$, and variance $\sigma^2_{\text{pps}} = 1/12$, and the WGN sequence has length L. A sampling frequency $f_s = 44.1$ kHz has been adopted. For the multiple-variance cross-correlation method, since in the considered examples the second order kernel is dominant at high input variances, $\sigma^2_{x,0} = \sigma^2_{x,1} = \frac{\sigma^2_{\text{pps}}}{16}$ and $\sigma^2_{x,2} = \sigma^2_{x,3} = \sigma^2_{\text{pps}}$ have been assumed. The same two values for variance have been considered for the traditional cross-correlation method. Therefore, the model identification has been performed considering the following settings: (1) WGN with single variance $\sigma^2_{x,0}$, (2) WGN with single variance $\sigma^2_{x,3}$, (3) WGN with multiple-variance, (4) PPS with single variance $\sigma^2_{x,0}$, (5) PPS with single variance $\sigma^2_{x,3}$, (6) PPS with multiple-variance.

Then, the performance has been evaluated applying WGN and music of length 44 100 samples to the system under test and to the model, assuming several input variances. Results are reported in the following sections in terms of the normalized MSE (NMSE) in the frequency domain between the output of the system under test $y(n)$ and the output of the identified Volterra series $\hat{y}(n)$, according to the following formula:

$$\text{NMSE} = 10\log_{10}\frac{\sum_{n=1}^{N}\left[|Y(f_n)| - \left|\hat{Y}(f_n)\right|\right]^2}{\sum_{n=1}^{N}|Y(f_n)|^2}. \tag{13}$$

A. Simulated system

The nonlinear system under test for the simulated scenario is the Wiener model adopted in [7]. This system is a cascade of a linear filter and a static nonlinearity. The linear part is a low-pass filter given by the scaling function of the Daubechies Wavelet of order ten (D10). The nonlinearity can be described by the following function:

$$g(x) = \frac{4.5}{1+2e^{-x}} - \frac{4.5}{3} \tag{14}$$

where $\widetilde{g}(x) = x + \frac{9}{54}x^2 - \frac{27}{486}x^3$ is the third-order Taylor expansion of $g(x)$. The cascade of $\widetilde{g}(x)$ and the linear part can be described by a third-order Volterra system with known kernels. This system has been modelled in Test 1, thus excluding the truncation error. Then, the cascade of (14) and the low-pass filter has been considered in Test 2. A Gaussian noise with variance $2 \cdot 10^{-6}$ has been added to the output. Once the model has been obtained, inputs with variance σ^2_x in the interval $\left[(1/256, \dots, 1/4, 1/2, 1, 2, 4, \dots, 16)\,\sigma^2_{\text{pps}}\right]$ have

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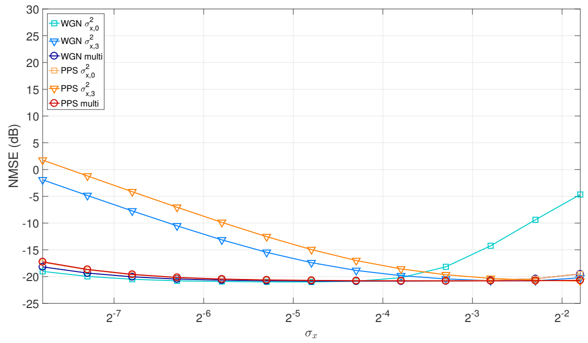
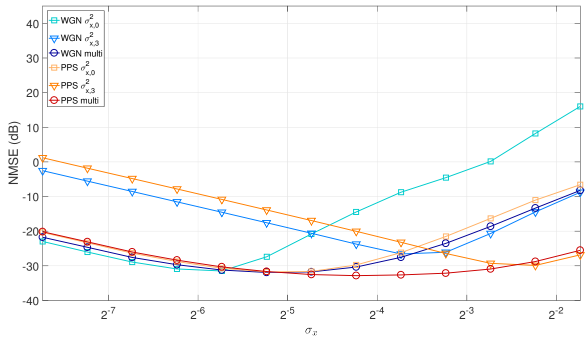
been adopted. Figure 1 shows that the exploitation of multiple-variance PPSs provides two advantages at the same time, i.e., the input region in which the error has acceptable values both for noise and music is widened, thus, overcoming the locality problem, and accuracy is improved with respect to stochastic input. Analogous conclusion can be gathered from Figure 2, where the overall performance is slightly worse than in Figure 1, due to the presence of the truncation error that affects all the obtained models.

B. Real system

The real-world device assumed in Test 3 is the Presonus TubePRE microphone/instrument tube preamplifier. It provides a drive potentiometer controlling the amount of tube saturation, i.e., the amount of applied distortion. The device has been set to provide a second harmonic distortion of 4.2% and third harmonic distortion of 0.5% on a 1 kHz tone signal. Once the model has been obtained, inputs with variance σ_x^2 in the interval $\left[(1/4096, \ldots, 1/4, 1/2, 1)\, \sigma_{\text{pps}}^2\right]$ have been adopted. The results reported in Figure 3 are consistent with those obtained in the simulated test session, where the improvements provided by multiple-variance PPSs can be noted especially for music in Figure 3(b).



(a)



(b)

Fig. 3. NMSE obtained in Test 3. (a) Noise. (b) Music.

VII. Conclusion

A novel method for nonlinear system identification has been presented based on the exploitation of WN filters and multiple-variance PPSs applied to the cross-correlation technique. In this way, the recomputation of all lower order kernels for the estimation of a kernel for a certain input signal variance is avoided. Moreover, both the accuracy and the locality problems can be solved. The performance has been evaluated on both simulated and real-world nonlinear systems, showing the expected effectiveness on a wide range of input variances.

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