Gridless Compressed Sensing for Fully Augmentable Arrays

Wassim Suleiman, Christian Steffens, Alexander Sorg and Marius Pesavento

***Abstract*—Direction-of-arrival (DOA) estimation using nonuniform linear arrays is considered. We focus on the so called "fully augmentable arrays" (FAAs) with full set of covariance lags. In FAAs, the number of covariance lags is usually larger than the number of sensors in the array. Thus, with FAAs more sources than the number of sensors can be identified. Existing DOA estimation algorithms for FAAs are based on the assumption of uncorrelated sources. In this paper, based on compressed sensing, we present a DOA estimation algorithm for FAAs without assuming uncorrelated sources. The proposed algorithm is based on the newly introduced gridless SPARse ROW-norm reconstruction (SPARROW) formulation for the joint sparse reconstruction from multiple measurement vectors. By numerical experiments, we show that the proposed algorithm outperforms the existing algorithms in the presence of correlated signals or small number of snapshots. Moreover, using simulations, the behavior of the Cramér-Rao Bound (CRB) for the case of correlated source is demonstrated and it is shown that, when the number of sources is larger than the number of sensors, the CRB for FAAs approaches zero at infinitely large signal-to-noise-ratio (SNR) only if the sources are fully correlated.**

***Index Terms*—Fully augmentable arrays, Joint Sparsity, Gridless Parameter Estimation, Direction-of-arrival estimation.**

I. Introduction

DOA estimation of narrowband sources has attracted significant attention [1], since it has many applications including sonar, radar, and seismic exploration. Subspace-based methods such as MUSIC [6], Root-MUSIC [2], ESPRIT [3], and RARE [4], and the methods in [5], [7] are computationally efficient direction finding methods that exhibit the super resolution property. The DOA estimation algorithms in [2]–[5] can only identify a number of sources which is smaller than the number of sensors in the array.

When the number of sensors is limited, nonuniform linear arrays become attractive [8], [9]. Nonuniform linear arrays have larger aperture compared to uniform linear arrays (ULAs) with the same number of sensors, which results in a higher DOA resolution. Moreover, a nonuniform linear array is designed such that it contains much more covariance lags than the number of sensors in the array, which increases the number of identifiable sources by such arrays. The set of covariance lags of a FAA has no missing lags, i.e., it has a full co-array. Whereas a partially augementable array (PAA) has some number of missing lags.

DOA estimation using nonuniform linear arrays in the *conventional case*, i.e., when the number of sources is less than the number of sensors can be achieved using the conventional DOA estimation algorithms, e.g., Root-MUSIC [2] and ESPRIT [3]. In the *superior case*, where the number of sources is larger than the number of sensors in the array, conventional DOA estimation algorithms fail to identify the sources. In [10], DOA estimation in the superior case has been achieved by constructing the augmented covariance matrix (ACM) and using it for the MUSIC algorithm instead of the direct data covariance matrix. In [9] and [11], DOA estimation algorithms for FAAs and PAAs are presented, respectively. The algorithms in [9] and [11] are based on the maximum entropy, the ACM, and the Maximum Likelihood estimator. The algorithms in [9]–[11] assume uncorrelated sources and their performance has not been evaluated in the presence of correlated sources.

Recently, compressed sensing (CS) based DOA estimation techniques [12]–[15] have gained more popularity over subspace-based methods, since CS exhibits good estimation performance even in difficult scenarios and more robust for to correlated sources [15], [16].

In this paper, DOA estimation using FAAs is considered. Based on the newly introduced SPARROW formulation for the joint sparse reconstruction from multiple measurement vectors, a DOA estimation algorithm for FAAs is presented. Similar to the SPARROW formulation, our proposed algorithm does not require the sources to be uncorrelated. By simulations, we show that our proposed algorithm outperforms the algorithms in [9], [10] for correlated sources. The CRB is used to assess the performance of the DOA estimation algorithm. For the superior case, we show by simulations that the CRB approaches zero with increasing SNR when the sources are fully correlated, which is in contrast to the uncorrelated sources case, where the CRB does not approach zero for an infinitely large SNR.

This paper is organized as follows. The signal model and conventional DOA estimation algorithms for FAAs are reviewed in Section II and Section III, respectively. In Section IV, the SPARROW approach is introduced and in Section V the gridless SPARROW approach is extended to FAAs. The behaviour of the CRB and the performance of the DOA estimation algorithms is demonstrated by simulations in Section VI.

In this paper, the transpose, complex conjugate, and the Hermitian operators are denoted as $(\cdot)^T, (\cdot)^*$, and $(\cdot)^H$, respectively. The imaginary unit, diagonal matrix formed from scalar arguments, trace of a matrix, and the expectation operator are denoted as, $\jmath$, $\mathrm{diag}(\cdot)$, $\mathrm{Tr}(\cdot)$, and $\mathbb{E}(\cdot)$, respectively.

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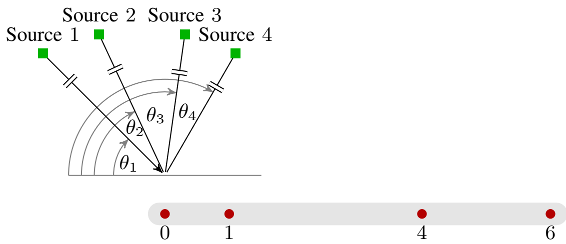

Fig. 1. Nonuniform linear array of $M = 4$ sensors.

II. Signal Model

We consider a nonuniform linear array consisting of M sensors. The set of $M(M-1)/2$ inter-sensor separations has no missing lags, i.e., the array is fully augmentable. Let $d_1 < \cdots < d_M$ denote the position of the sensors in the array with respect to the first (reference) sensor, thus, $d_1 = 0$, and let

$$\mathcal{D} = \{d_1, \dots, d_M\} \tag{1}$$

be the set of sensor positions. The positions $d_1, \dots, d_M$ are measured with half-wavelength of the incident signal carrier. The set of inter-sensor distances is $\mathcal{L} = \{d_j - d_i | i, j = 1, \dots, M, j > i\}$. The smallest inter-sensor distance corresponds to half-wavelength of the incident signal carrier to avoid ambiguities [9]. Let $\breve{M}$ denotes the largest inter-sensor distance, i.e., the array aperture. Then $\breve{M}$ is an integer multiple of the half-wavelength of the signal carrier. The considered nonuniform array possesses the same set of inter-sensor distances as the ULA which consists of $(\breve{M}+1)$ sensors separated by half-wavelength. This ULA is referred to as the co-array. The set of sensor positions of the co-array is $\breve{\mathcal{D}} = \{0, 1, \dots, \breve{M}\}$. In Fig. 1 a FAA of $M = 4$ senors with positions $\mathcal{D} = \{0, 1, 4, 6\}$ is displayed. The co-array corresponding to this FAA consists of $\breve{M}+1 = 7$ sensors with positions $\breve{\mathcal{D}} = \{0, \dots, 6\}$. In the following, the smallest inter-sensor distance is set to half-wavelength and the remaining inter-sensor distances are taken to be an integer multiple of this smallest distance so that the positions contained in the sets $\mathcal{D}$ and $\breve{\mathcal{D}}$ are integer numbers as in the previous example.

Signals of L far-field narrow-band sources impinge onto the array from directions $\boldsymbol{\theta} = [\theta_1, \dots, \theta_L]^T$ relative to the array axis as demonstrated in Fig. 1. The array output at time instant t is modeled as

$$\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{\psi}(t) + \boldsymbol{w}(t), \tag{2}$$

where $\boldsymbol{\psi}(t) \in \mathbb{C}^{L\times 1}$ denotes the signals of the L Gaussian sources and $\boldsymbol{w}(t) \in \mathbb{C}^{M\times 1}$ is the vector representing the additive white Gaussian noise with covariance σ^2. The $M \times L$ steering matrix $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \dots, \boldsymbol{a}(\theta_L)]$ consists of L steering vectors defined as

$$\boldsymbol{a}(\theta_l) = [\exp(\jmath\pi d_l \cos(\theta_l)), \dots, \exp(\jmath\pi d_M \cos(\theta_l))]^T, \tag{3}$$

for $l = 1, \dots, L$, where the sensor positions are defined in (1). Since a FAA is considered, the corresponding co-array steering vector $\breve{\boldsymbol{a}}(\theta_l)$ is written as

$$\breve{\boldsymbol{a}}(\theta_l) = [1, \exp(\jmath\pi \cos(\theta_l)), \dots, \exp(\jmath\pi \breve{M} \cos(\theta_l))]^T. \tag{4}$$

Note that $\breve{\boldsymbol{a}}(\theta_l)$ is a Vandermonde vector. The array output for N time instants is stored in the measurement matrix $\boldsymbol{Y} = [\boldsymbol{y}(1), \dots, \boldsymbol{y}(N)]$ which can be written as

$$\boldsymbol{Y} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{\Psi} + \boldsymbol{W}, \tag{5}$$

where $\boldsymbol{\Psi} = [\boldsymbol{\psi}(1), \dots, \boldsymbol{\psi}(N)]$ and $\boldsymbol{W} = [\boldsymbol{w}(1), \dots, \boldsymbol{w}(N)]$.

The true measurement covariance matrix $\boldsymbol{R}$ is defined as

$$\boldsymbol{R} = \mathbb{E}\big(\boldsymbol{y}(t)\boldsymbol{y}^H(t)\big) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^H(\boldsymbol{\theta}) + \sigma^2\boldsymbol{I}_M \tag{6}$$

where $\boldsymbol{I}_M$ is the $M \times M$ identity matrix and

$$\boldsymbol{P} = \mathbb{E}\big(\boldsymbol{\psi}(t)\boldsymbol{\psi}^H(t)\big) \tag{7}$$

is the $L \times L$ source covariance matrix. Note that the matrix $\boldsymbol{P}$ is diagonal only in the case of perfectly uncorrelated sources since the off-diagonal entries of $\boldsymbol{P}$ correspond to the correlations between the sources. In practice, the measurement covariance matrix $\boldsymbol{R}$ in (6) is not available but it can be estimated from N samples of the array output as

$$\hat{\boldsymbol{R}} = \boldsymbol{Y}\boldsymbol{Y}^H/N. \tag{8}$$

Let $\boldsymbol{A}(\tilde{\boldsymbol{\theta}}) \in \mathbb{C}^{M\times K}$ denotes the overcomplete dictionary matrix obtained by sampling the field-of-view in $K \gg L$ spatial directions $\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1, \dots, \tilde{\theta}_K]^T$. For simplicity, the grid $\tilde{\boldsymbol{\theta}}$ is assumed to be very fine such that the true DOAs lie on the grid. The signal model in (5) can be written as a sparse representation $\boldsymbol{A}(\tilde{\boldsymbol{\theta}})\tilde{\boldsymbol{\Psi}} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{\Psi}$, where

$$\tilde{\boldsymbol{\Psi}} = [\tilde{\boldsymbol{\psi}}_1, \dots, \tilde{\boldsymbol{\psi}}_K]^T \in \mathbb{C}^{K\times N} \tag{9}$$

is a sparse representation of the signal waveform matrix $\boldsymbol{\Psi}$, which has non-zero row $\tilde{\boldsymbol{\psi}}_k$ only if the corresponding sampled direction $\tilde{\theta}_k$ corresponds to a true source direction. Using this sparse representation, the DOA estimation problem can be formulated as the well-known convex mixed-norm minimization problem [13], [15]

$$\min_{\tilde{\boldsymbol{\Psi}}} \frac{1}{2} \left\| \boldsymbol{A}(\tilde{\boldsymbol{\theta}})\tilde{\boldsymbol{\Psi}} - \boldsymbol{Y} \right\|_{\mathsf{F}}^2 + \lambda\sqrt{N} \left\| \tilde{\boldsymbol{\Psi}} \right\|_{2,1}, \tag{10}$$

where $\lambda > 0$ is the regularization parameter. For increasing λ more emphasis is put on the sparsity of the signal matrix $\tilde{\boldsymbol{\Psi}}$, i.e., the minimizer of (10) will contain less non-zero rows. In (10), $\|\cdot\|_{\mathsf{F}}$ denotes the Frobenius matrix norm and $\|\cdot\|_{2,1}$ is the $\ell_{2,1}$ mixed norm, i.e.,

$$\left\| \tilde{\boldsymbol{\Psi}} \right\|_{2,1} = \sum_{k=1}^{K} \left\| \tilde{\boldsymbol{\psi}}_k \right\|_2. \tag{11}$$

Note that using (10), the DOA estimation problem is reduced to the identification of the non-zero rows in the estimated row-sparse matrix $\hat{\tilde{\boldsymbol{\Psi}}}$. The DOA estimates are the grid points, i.e., the elements of $\tilde{\boldsymbol{\theta}}$, which correspond to the L rows of $\hat{\tilde{\boldsymbol{\Psi}}}$ with the largest norm.

III. Direct Augmentation and Subspace Truncation Approachs

Traditionally DOA estimation algorithms for nonuiform linear arrays distinguish between two DOA estimation cases:

1) The conventional case: where $L < M$, thus conventional subspace-based DOA estimation algorithms, e.g., MUSIC [6] can be directly applied to the array.
2) The superior case: where $M \leq L \leq \breve{M}$. In this case conventional DOA estimation methods fail in identifying the sources. Therefore new DOA estimation methods,

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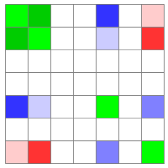

Fig. 2. The co-array covariance matrix corresponding to the FAA with sensor positions $\mathcal{D} = \{0, 1, 4, 6\}$, where the colored entries are obtained directly from the covariance matrix of the FAA.

e.g., direct augmentation [10] and subspace truncation [9] approaches, are proposed to achieve DOA estimation in such a case.

The focus of this paper is the superior case, however, the proposed algorithm is also applicable in the conventional case. In the following, direct augmentation and subspace truncation are reviewed. Since these approaches assume perfectly uncorrelated sources, i.e., a diagonal matrix $\boldsymbol{P}$, in this section the sources are assumed to be perfectly uncorrelated.

In [10], DOA estimation in the superior case is achieved by constructing the $(\breve{M}+1) \times (\breve{M}+1)$ co-array covariance matrix, denoted as $\breve{\boldsymbol{R}}$, from the $M \times M$ array covariance matrix and then applying the MUSIC algorithm to the resulting co-array matrix $\breve{\boldsymbol{R}}$. Note that since the sources are assumed to be uncorrelated, the co-array covariance matrix $\breve{\boldsymbol{R}}$ is Toeplitz, i.e., the entries on the main- and sub- diagonals of $\breve{\boldsymbol{R}}$ are equal and correspond to the entries of $\boldsymbol{R}$ with the same covariance lag. If more than one entry of $\boldsymbol{R}$ corresponds to the same covariance lag, these entries are averaged and the average is used in the co-array covariance matrix $\breve{\boldsymbol{R}}$. This approach is referred to as the direct augmentation approach (DAA) [10]. Fig. 2 demonstrates the 7×7 co-array covariance matrix corresponding to the FAA of Fig. 1. In Fig. 2, the entries with white color (not colored) are the entries which cannot be estimated directly from the array output (missing sensors). The entries which are on the same main- or sub- diagonals have the same colors (ideally these entries should be identical). Note that at each diagonal at least one entry is available, since the array is fully augmentable.

In [9], the DAA is used to compute an initial estimate of the co-array covariance matrix. Then subspace truncation (SST) is applied to the resulting co-array covariance matrix to equalize its noise eigenvalues, while maintaining the Toeplitz structure of the matrix. The resulting estimate of the co-array covariance matrix is used to initialize the Maximum Likelihood estimator, for details refer to [9].

We remark that in the case of correlated source the co-array covariance matrix does not exhibit the Toeplitz property and the aforementioned DOA approaches are not valid.

IV. The SPARROW Formulation

In [15, Theorem 1] it is proved that the $\ell_{2,1}$ mixed-norm minimization problem (10) can equivalently be formulated as the SPARse ROW-norm reconstruction (SPARROW) problem

$$\min_{\boldsymbol{S}\in\mathbb{D}_+} \operatorname{Tr}\big(\boldsymbol{A}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\boldsymbol{A}^H(\tilde{\boldsymbol{\theta}}) + \lambda\boldsymbol{I}_M\big)^{-1}\hat{\boldsymbol{R}}\big) + \operatorname{Tr}(\boldsymbol{S}), \tag{12}$$

where the sample covariance matrix $\hat{\boldsymbol{R}}$ is defined in (8), λ is the regularization parameter in (10), and $\operatorname{Tr}(\cdot)$ is the trace of a matrix. The optimization variable in (12) $\boldsymbol{S}$ is restricted to the set of diagonal matrices with non-negative entries $\mathbb{D}_+$. It can be shown that the solution of the minimization problem (12)

$$\hat{\boldsymbol{S}} = \operatorname{diag}\big(\hat{s}_1, \ldots, \hat{s}_K\big) \tag{13}$$

contains on its main diagonal the scaled row-norms of the estimated row sparse signal matrix $\hat{\tilde{\boldsymbol{\Psi}}} = [\hat{\tilde{\boldsymbol{\psi}}}_1, \ldots, \hat{\tilde{\boldsymbol{\psi}}}_K]^T$, which is the minimizer of (10) and an estimate of $\tilde{\boldsymbol{\Psi}}$ in (9). In [15], it is proven that the solutions of (10) and (12) can be uniquely obtained from each other. The DOAs can be computed from a minimizer of the SPAROW formulation $\hat{\boldsymbol{S}}$ by finding the L largest peaks of the diagonal entries of $\hat{\boldsymbol{S}}$. The points of the grid $\tilde{\boldsymbol{\theta}}$ which correspond to these peaks are the estimates of the L DOAs.

The SPAROW formulation is attractive since:

1) The SPARROW optimization variables are the elements on the diagonal of the matrix $\boldsymbol{S}$, which is of size K, whereas the optimization variables in the $\ell_{2,1}$ formulation in (10) are the elements of $\tilde{\boldsymbol{\Psi}}$, which is of size $K \times N$.
2) The SPARROW formulation allows gridless DOA estimation when the overcomplete dictionary $\boldsymbol{A}(\tilde{\boldsymbol{\theta}})$ is a Vandermonde matrix [15].
3) The structure of the optimization variable of the SPARROW formulation $\boldsymbol{S} \in \mathbb{D}_+$ does not depend on the source correlation, particularly, even if the sources are correlated, $\boldsymbol{S}$ will still be a diagonal matrix with non-negative entries. This is in contrast to the CS algorithms which are derived from the covariance matrix in (6) and their optimization variable is the source covariance matrix, e.g., refer to [17].

We remark that the covariance based DOA estimation approach in [16] has similarities to the SPARROW formulation and can be used for DOA estimation in nonuniform linear arrays. However, the approach in [16] was derived under the assumption of uncorrelated sources in contrast to the SPARROW formulation where the assumption of uncorrelated sources has not been used. In the following, we rewrite the DOA estimation problem for FAAs using the SPARROW formulation[1].

V. Gridless SPARROW for FAA

In the case that the array manifold exhibits a Vandermonde structure, the SPARROW approach in (12) can be written as a gridless minimization [15]. For FAAs, the co-array steering matrix $\breve{\boldsymbol{A}}(\theta) = [\breve{\boldsymbol{a}}(\theta_1), \ldots, \breve{\boldsymbol{a}}(\theta_K)] \in \mathbb{C}^{(\breve{M}+1)\times L}$, where the vector $\breve{\boldsymbol{a}}$ is defined in (4), is a Vandermonde matrix. We use this property of the FAAs to reformulate the SPARROW algorithm as a gridless estimation problem. Let $\breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}}) = [\breve{\boldsymbol{a}}(\tilde{\theta}_1), \ldots, \breve{\boldsymbol{a}}(\tilde{\theta}_K)] \in \mathbb{C}^{(\breve{M}+1)\times K}$ denote the

[1]The approach in [16] can also be extended to FAAs in the same way as the SPARROW approach which is explained in Section V.

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Vandermonde overcomplete dictionary corresponding to the co-array. Thus the matrix

$$\breve{\boldsymbol{T}} = \breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\breve{\boldsymbol{A}}^H(\tilde{\boldsymbol{\theta}}) \tag{14}$$

is a $(\breve{M}+1)\times(\breve{M}+1)$ Toeplitz matrix and can be used to formulate the gridless SPARROW approach similar to [15]. We introduce the $M \times (\breve{M}+1)$ selection matrix $\boldsymbol{J}$, which is used to retrieve the matrix $\boldsymbol{T} = \boldsymbol{A}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\boldsymbol{A}^H(\tilde{\boldsymbol{\theta}})$ from the Toeplitz matrix $\breve{\boldsymbol{T}}$ as follows:

$$\boldsymbol{J} = [\boldsymbol{e}_{d_1+1}, \ldots, \boldsymbol{e}_{d_M+1}]^T, \tag{15}$$

where $d_1, \ldots, d_M$ are the positions of the sensors of the FAA defined in (1). The vector $\boldsymbol{e}_i$, for $i = 1, \ldots, \breve{M}+1$, denotes the ith column of the $(\breve{M}+1)\times(\breve{M}+1)$ identity matrix. For example, the selection matrix corresponding to the FAA of Fig. 1 is $\boldsymbol{J} = [\boldsymbol{e}_{d_1+1}, \boldsymbol{e}_{d_2+1}, \boldsymbol{e}_{d_3+1}, \boldsymbol{e}_{d_4+1}]^T = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_5, \boldsymbol{e}_7]^T$ where $\boldsymbol{e}_1 = [1,0,0,0,0,0,0]^T$, $\boldsymbol{e}_2 = [0,1,0,0,0,0,0]^T$, and the remaining two vectors are defined similarly. The matrix $\boldsymbol{J}$ selects the rows of the co-array overcomplete dictionary $\breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}})$ which correspond to the FAA, i.e.,

$$\boldsymbol{A}(\tilde{\boldsymbol{\theta}}) = \boldsymbol{J}\breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}}). \tag{16}$$

Thus, the matrix $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = \boldsymbol{A}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\boldsymbol{A}^H(\tilde{\boldsymbol{\theta}}) = \boldsymbol{J}\breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\breve{\boldsymbol{A}}^H(\tilde{\boldsymbol{\theta}})\boldsymbol{J}^T = \boldsymbol{J}\breve{\boldsymbol{T}}\boldsymbol{J}^T, \tag{17}$$

where (14) is used for the last equality. Substituting (17) in the SPARROW formulation (12) yields

$$\min_{\breve{\boldsymbol{T}}\in\mathbb{T},\, \breve{\boldsymbol{T}}\succeq\boldsymbol{0}} \quad \mathrm{Tr}\big((\boldsymbol{J}\breve{\boldsymbol{T}}\boldsymbol{J}^H + \lambda\boldsymbol{I}_M)^{-1}\hat{\boldsymbol{R}}\big) + \frac{1}{\breve{M}+1}\mathrm{Tr}(\breve{\boldsymbol{T}}) \tag{18}$$

where $\mathbb{T}$ is the set of Toeplitz matrices and $\breve{\boldsymbol{T}} \succeq \boldsymbol{0}$ means that the matrix $\breve{\boldsymbol{T}}$ must be positive semidefinite. The second term in (18) is computed as follows:

$$\mathrm{Tr}(\boldsymbol{S}) = \frac{1}{\breve{M}+1}\mathrm{Tr}\big(\breve{\boldsymbol{A}}(\tilde{\boldsymbol{\theta}})\boldsymbol{S}\breve{\boldsymbol{A}}^H(\tilde{\boldsymbol{\theta}})\big) = \frac{1}{\breve{M}+1}\mathrm{Tr}(\breve{\boldsymbol{T}}). \tag{19}$$

Note that the optimization variable in (18) is the matrix $\breve{\boldsymbol{T}}$, which has only $(\breve{M}+1)$ unknown entries due to the fact that it exhibits the Topelitz structure.

Let $\mathring{\breve{\boldsymbol{T}}}$ denotes a minimizer of (18). Then (14) implies that that the matrix $\mathring{\breve{\boldsymbol{T}}}$ spans the same principal subspace as the matrix $\breve{\boldsymbol{A}}(\boldsymbol{\theta})$. Consequently, the Toeplitz matrix $\breve{\boldsymbol{T}}$ can be used for DOA estimation based on polynomial rooting, e.g., as in the Root-MUSIC algorithm [2], instead of the matrix resulting from the DAA as in [10] and [9]. The benefit of using $\breve{\boldsymbol{T}}$ from (18) rather than the DAA is that $\breve{\boldsymbol{T}}$ is Toeplitz even if the sources are correlated whereas the DAA uses the co-array covariance matrix which does not exhibit the Toeplitz structure when correlated sources are present.

Similar to [15], problem (18) is written as

$$\begin{aligned} \min_{\breve{\boldsymbol{T}}\in\mathbb{T},\, \breve{\boldsymbol{T}}\succeq\boldsymbol{0},\, \boldsymbol{U}_M\succeq\boldsymbol{0}} \quad & \mathrm{Tr}(\boldsymbol{U}_M\hat{\boldsymbol{R}}) + \mathrm{Tr}(\breve{\boldsymbol{T}})/(\breve{M}+1) \\ \text{s.t.} \quad & \begin{bmatrix} \boldsymbol{U}_M & \boldsymbol{I}_M \\ \boldsymbol{I}_M & \boldsymbol{J}\breve{\boldsymbol{T}}\boldsymbol{J}^H + \lambda\boldsymbol{I}_M \end{bmatrix} \succeq \boldsymbol{0}, \end{aligned} \tag{20}$$

where $\boldsymbol{U}_M$ is an $M \times M$ positive semidefinite matrix which is also an optimization variable of (20), refer to [15]. The problem (20) is a positive semidefinite program and can be solved using standard convex optimization tools, e.g., [18].

To summarize, our proposed algorithm consists of two steps. In the first step, the problem in (20) is solved, with $\mathring{\breve{\boldsymbol{T}}}$ denoting a solution. Then in the second step, a polynomial rooting based estimation technique for ULAs, such as the Root-MUSIC algorithm, is applied to $\mathring{\breve{\boldsymbol{T}}}$.

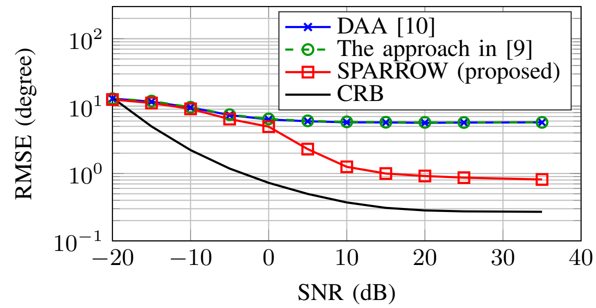

Fig. 3. RMSE versus SNR for $\rho = 0.8$ in the superior case.

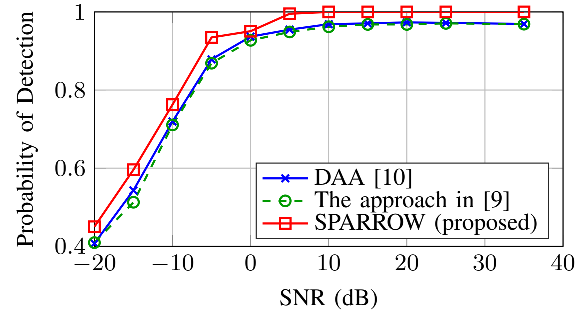

Fig. 4. Probability of detection versus SNR for $\rho = 0.8$ in the superior case.

VI. Simulations

In the simulations, a FAA with sensors at positions $\mathcal{D} = \{0, 1, 4, 6\}$, as in Fig. 1, is considered. Signals of 4 equal-powered sources impinge onto the FAA from DOAs $45°, 65°, 98°$, and $120°$. A number of $N = 20$ samples of the array output are available. The regularization parameter is chosen as in [19] to be $\lambda = \sqrt{\sigma_n^2 M \log M}$. We compare our results averaged over 500 realizations to the CRB in [20, Equation (2.10)].

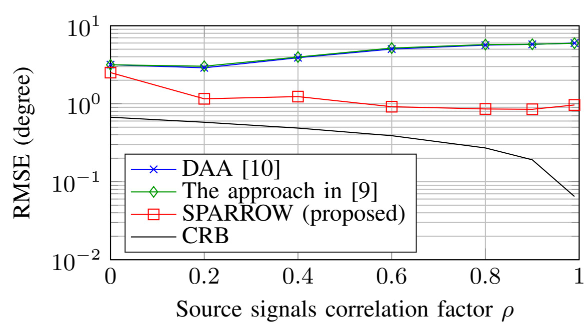

Fig. 5. RMSE versus the source signal correlation ρ in the superior case.

A. DOA Estimation Performance

Fig. 3 shows the root mean square error (RMSE) of DOA estimation against the SNR for our proposed SPARROW

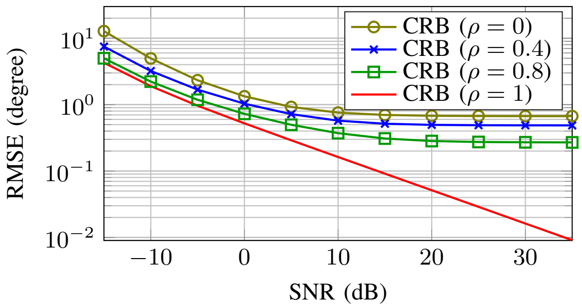

Fig. 6. The CRB for FAA in the superior case for different values of ρ.

based approach, the DAA [10], and the approach in [9]. The correlation factor ρ between all sources is set to 0.8. Observe that while all algorithms do not achieve the CRB, our proposed SPARROW based approach retains the best performance. In Fig. 4, the source resolution probability, as defined in [5], is plotted against the SNR for the same configuration as in Fig. 3. Note that the conventional approaches in [9] and [10] are not always able to identify the sources whereas our proposed approach achieves a resolution probability of 1 at high SNR. In Fig. 5, the RMSE of DOA estimation using the aforementioned algorithms is plotted against the correlation factor ρ. Note that the CRB decreases with increasing ρ. The performance of the proposed approach improves slightly with increasing ρ, however, it does not achieve the CRB. The performances of the conventional methods in [10] and [9] degrade with increasing ρ since these algorithms assume uncorrelated sources.

B. The CRB Behaviour at High SNR

In Fig. 6, the CRB is displayed for different values of the source correlation factor ρ. Observe in Fig. 6 that the CRB does not approach zero for high SNR but it converges at some finite value. Such a behaviour of the CRB has been recognized in [9] for FAAs, in [21] for non-coherent arrays, and in [22] for DOA estimation using fewer receivers than the number of sources. In [22], the authors argue that this behaviour "is typical in scenarios where a signal subspace is nonexistent". Note that for increasing values of ρ the CRB decreases, i.e., DOA estimation performance improves with sources correlation. Moreover, if the sources are fully correlated, i.e., $\rho = 1$, then the CRB approaches zero at infinitely large SNR since the dimension of the signal subspace is reduced in such case.

Conclusion

We have presented a gridless DOA estimation algorithm for FAAs which, in contrast to the conventional algorithms, does not require the assumption of uncorrelated sources. The proposed algorithm is based on the SPARROW formulation for the joint sparse reconstruction from multiple measurement vectors. The performance of our proposed algorithm is shown to be superior to the conventional algorithms when correlated sources are present or a small number of snapshots is available.

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